Change of Variables Theorem. In two variables, \[ \iint_D f(x, y) \, dx \, dy = \iint_{D^*} f(T(u, v)) \det \frac{\partial (x, y)}{\partial (u, v)} \, du \, dv \] and in three variables \[ \iiint_D f(x, y, z) \, dx \, dy \, dz = \iiint_{D^*} f(T(u, v, w)) \det \frac{\partial (x, y, z)}{\partial (u, v, w)} \, du \, dv \, dw , \] where \( D \) and \( D^* \) are suitable regions and \( T \) is a suitable transformation such that \( T(D^*) = D \).

Stokes’s Theorem. Let \( S \) be a bounded, piecewise regular, oriented surface in \( \mathbb{R}^3 \) and suppose that \( C = \partial S \) consists of finitely many piecewise \( C^1 \) simple closed curves, oriented consistently with the orientation of \( S \). Suppose that \( F \) is a \( C^1 \) vector field with continuous partial derivatives defined on a domain that includes \( S \). Then \[ \iint_S \nabla \times F \cdot dS = \oint_C F \cdot \mathbf{n} \, ds = \oint_C F \cdot ds . \]

Divergence or Gauss’ Theorem. If \( W \) is a bounded symmetric elementary domain in \( \mathbb{R}^3 \), whose boundary \( S = \partial W \) consists of finitely many piecewise regular closed oriented surfaces, oriented so that the normal vectors point out of \( W \), and \( F \) is a \( C^1 \) vector field defined on \( W \), then \[ \iiint_W \nabla \cdot F \, dV = \iint_S F \cdot \mathbf{n} \, dS = \iint_S \nabla \cdot F \, dS . \]

These theorems, and also Green’s Theorem in the plane, which they generalize, can be applied to regions that have suitable decompositions.

1. (6 points) Let \( A \) be the region in the \( xy \)-plane bounded below by \( y = x^2 \) for \(-2 \leq x \leq 2\), and above by \( y = x^2 + 3 \) for \(-1 \leq x \leq 1 \) and \( y = 4 \) for \(-2 \leq x \leq -1, \ 1 \leq x \leq 2 \). Explain why Green’s Theorem cannot be used directly for this domain, and show how Green’s Theorem can be applied indirectly.

2. (10 points) a. In Stokes’ Theorem what does it mean to say that the surfaces in question are regular?

b. What does it mean to say that the boundary curve(s) is (are) oriented compatibly with the surface(s)?
3. (12 points) Compute $\iint_R (x + y)^2 e^{x-y} \, dA$ where $R$ is the region bounded by the lines $y = 1 - x$, $y = 4 - x$, $y = x + 1$, $y = x - 1$. 
4. (22 points) Compute \( \iiint_W \frac{dV}{(x^2 + y^2 + z^2)^2} \), where W is the region between the spheres \( x^2 + y^2 + z^2 = a^2 \), \( x^2 + y^2 + z^2 = 16 \), and \( 0 < a < 4 \). Use your result to evaluate the same integral for the entire region inside the sphere \( x^2 + y^2 + z^2 \leq 16 \).
5. (15 points) Compute the area of the portion of the plane \(2x + 4y + 6z = 12\) that lies in the first octant \((x \geq 0,\ y \geq 0,\ z \geq 0)\). Suggestion: orient the surface so that \(\mathbf{n}\) points away from the origin.

6. (22 points) Compute \(\int_C \mathbf{G} \cdot d\mathbf{s}\) for \(\mathbf{G} = (3xy^2, -3x^2y)\), where \(C\) is given by the segment from \((1,0)\) to \((2,0)\), the circular arc of \(x^2 + y^2 = 4\) from \((2,0)\) to \((0,2)\), the segment from \((0,2)\) to \((0,1)\), and finally the circular arc of \(x^2 + y^2 = 1\) from \((0,1)\) back to \((1,0)\).
7. (12 points) Let \( x = u^2 + \sin(v) \), \( y = \ln(1 + w^2) + uw \), and \( z = 1/(u^2 + w^2) \).

Compute \( \frac{\partial(x, y, z)}{\partial(u, v, w)} \). If \( (x, y, z) = T(u, v, w) \) is given by the formulas above, what is the volume change factor in the vicinity of the point \( P (u, v, w) = (0, \pi, -1) \)? Is this transformation orientation reversing or preserving near the point \( P \)?

8. (6 points) For the vector field \( \mathbf{F} \) shown below, show a path \( L \) from \( P \) to \( Q \) so that \( \int_C \mathbf{F} \cdot ds > 0 \), and a path \( M \) from \( P \) to \( Q \) so that \( \int_C \mathbf{F} \cdot ds = 0 \). What does this tell you about the vector field \( \mathbf{F} \)?
9. (15 points) Compute \( \int_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} \, dS \) if \( \mathbf{F} = (3z^2, -y^2, x + y) \) and \( S \) is the portion of the surface \( z = 10 - x^2 - y^2 \) that lies above the plane \( z = 1 \). Assume \( \mathbf{n} \) points away from the origin.