

Note! For full credit you must show sufficient work to support your answer. In problems requiring evaluation of an integral, the complete set-up is worth the bulk of the points and the actual integration and evaluation relatively little. There are 120 points. Good luck!

**Change of Variables Theorem.** In two variables,  $\iint_D f(x, y) dx dy = \iint_{D^*} f(T(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$  and in three variables  $\iiint_D f(x, y, z) dx dy dz = \iiint_{D^*} f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$ , where  $D$  and  $D^*$  are suitable regions and  $T$  is a suitable transformation such that  $T(D^*) = D$ .

1. (14 points) Let  $x = u^2v$ ,  $y = \ln(v^2 + 4)$ , and  $z = e^{-u^3w}$ . Compute  $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ .

If  $(x, y, z) = T(u, v, w)$  is given by the formulas above, what is the volume change factor in the vicinity of the point  $(u, v, w) = (-1, 2, 0)$ ?

$$\begin{aligned} J &= \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ 0 & \frac{\partial y}{\partial v} & 0 \\ -3u^2w & \frac{\partial z}{\partial w} & -u^3e^{-u^3w} \end{vmatrix} = \frac{\partial y}{\partial v} (2uv - u^3) e^{-u^3w} \\ &= \frac{-4u^4v^2}{v^2+4} e^{-u^3w} \\ J(-1, 2, 0) &= \frac{-4(1)(4)}{4+4} e^0 = \frac{-16}{8} = -2 \end{aligned}$$

volume multiplier =  $|J| = \boxed{2}$

$$\begin{aligned} \vec{C}'(t) &= (3\cos 3t, 1, -3\sin 3t) \quad \frac{9(\cos^2 3t + \sin^2 3t)}{= 9} \\ ds &= \| \vec{C}'(t) \| dt = \sqrt{9\cos^2 3t + 1 + 9\sin^2 3t} dt = \sqrt{10} dt \end{aligned}$$

2. (10 points) Let  $C$  be the curve given by  $\mathbf{c}(t) = (\sin(3t), t, \cos(3t))$  on the interval  $0 \leq t \leq \pi$ . Compute  $\int_C y^2 ds$ .

$$\begin{aligned} \int_C y^2 ds &= \int_0^\pi t^2 \sqrt{10} dt = \sqrt{10} \frac{t^3}{3} \Big|_0^\pi \\ &= \boxed{\frac{\sqrt{10}}{3} \pi^3} \end{aligned}$$

3. (15 points) Let  $\mathbf{F} = (y^2 \sin(xy^2) + z^3, 2xy \sin(xy^2), 3xz^2 + \sec^2 z)$ . This is a gradient vector field in  $D$ :  $-\infty < x < \infty$ ,  $-\infty < y < \infty$ , and  $-\pi/2 < z < \pi/2$ ; find the potential function  $f(x, y, z)$  so that  $\mathbf{F} = \vec{\nabla}f$  on  $D$ .

$$f = \int (y^2 \sin(xy^2) + z^3) dx = -\cos(xy^2) + A(y, z) \\ + z^3 x$$

$$f = \int 2xy \sin(xy^2) dy = -\cos(xy^2) + B(x, z)$$

$$f = \int (3xz^2 + \sec^2 z) dz = xz^3 + \tan z + C(x, y)$$

$$f(x, y, z) = -\cos(xy^2) + xz^3 + \tan z + E$$

Note: This only makes sense for

$$-\frac{\pi}{2} < z < \frac{\pi}{2},$$

ordinary  
cont.

→ Compute  $\vec{\nabla} \times \vec{F}$ ; see if it is  $\vec{0}$  or not.

$$\vec{\nabla} \times \vec{F} = (-\cos(2yz)(2z) + \cos(2yz)(2y),$$

4. (12 points) Explain why  $\mathbf{F} = (e^{xy}(1+xy), x^2 e^{xy} + \sin(2yz), \sin(2yz))$  can NOT be a gradient vector field  $\vec{\nabla}f$  for any scalar function  $f(x, y, z)$  on  $\mathbb{R}^3$ .

$$0 - 0, [x^2 y(e^{xy}) + 2x e^{xy}] - [e^{xy} x + e^{xy} x (1+xy)] \\ = (2\cos(2yz)(y-z), 0, 0) \neq \vec{0} \text{ in general on } \mathbb{R}^3. \text{ Since } \vec{\nabla} \times \vec{\nabla} f = \vec{0}, \text{ this } \vec{F} \text{ cannot have the form } \vec{\nabla} f.$$

5. (15 points) Find a linear transformation  $(x, y) = T_A(u, v) = (au + bv, cu + dv)$ , i.e., the matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , that carries the unit square  $D^* = [1, 0] \times [0, 1]$  to the parallelogram  $D$  with corners at  $(0, 0), (2, -2), (-3, 1)$ , and  $(-1, -1)$ . What is the area of  $D$ ? Does  $T_A$  preserve or reverse orientation, and how do you know?

Let us make

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\text{Then } A = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\det A = 2 - (-3)(-2) \\ = -4$$

$$\text{So area of } D = |\det A| = 4$$

and  $T_A$  reverses orientation since  $\det A < 0$ . Also note that  $\hat{i}, \hat{j}, \hat{k} = \hat{i} \times \hat{j}$  is a right-handed coord sys, but  $\vec{w}, \vec{z}, \vec{w} \times \vec{z}$  is left-handed.

6. (18 points) Compute  $\iiint_H \frac{1}{(x^2 + y^2 + z^2)^{7/4}} dx dy dz$  over the domain  $H$  (for "hollow shell") between  $x^2 + y^2 + z^2 = 25$  and  $x^2 + y^2 + z^2 = b^2$ , where  $b > 5$ . Use this result to compute the same integral over the

$D = \{(x, y, z) | x^2 + y^2 + z^2 \geq 25\}$ . Convert to spherical coords.

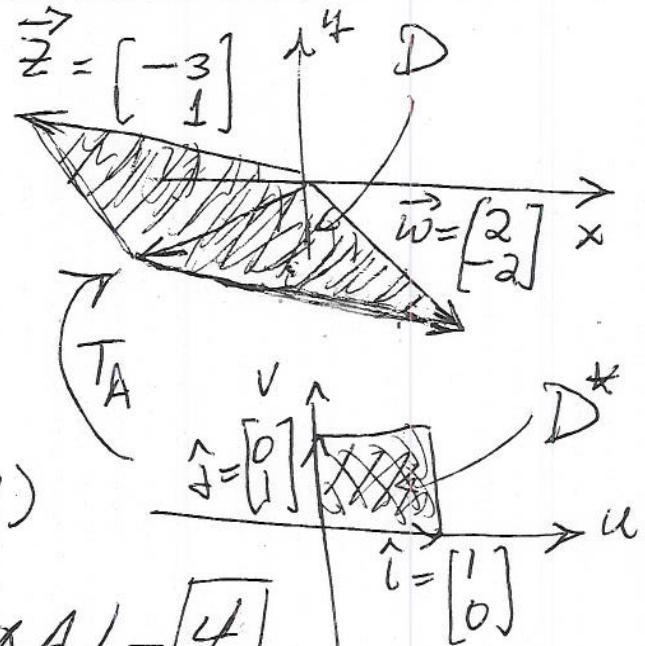
$$\rho^2 = x^2 + y^2 + z^2 \text{ and } (\rho^2)^{7/4} = \rho^{7/2}; \rho^{7/2} = \rho^{3/2}$$

$$\iiint_H \frac{\rho^2 \sin \varphi d\rho d\theta d\varphi}{\rho^{7/2}} = \int_0^{2\pi} \int_0^\pi \int_5^{25} \rho^{-3/2} \sin \varphi d\rho d\theta d\varphi$$

$$= \int_0^{\pi} \int_0^{\pi} \left( -2\rho^{-1/2}/5 \right) \sin \varphi d\theta d\varphi = 2 \left( \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{25}} \right) \int_0^{\pi} (-\cos \varphi) d\varphi$$

$$= 2 \left( \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{25}} \right) (-(-1) + 1)(2\pi) = 8\pi \left( \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{25}} \right)$$

$$\iiint_D \dots = \lim_{R \rightarrow \infty} \iiint_H \dots = \frac{8\pi}{\sqrt{5}}$$



- Panel  
the y-axis  
( $x=0$ )
7. (18 points) Consider the region  $D$  in the first quadrant bounded by the curves  $y = 6 - x^2$  (or  $y + x^2 = 6$ ),  $y = 2 - x^2$ ,  $y = x^2$  (or  $y - x^2 = 0$ ). Compute the integral  $\iint_D x \, dx \, dy$  by making a suitable change of variables. It is not so difficult, but also not so helpful, to solve for  $x$  and  $y$  individually in terms of  $u$  and  $v$ , and it is best to compute  $\frac{\partial(x,y)}{\partial(u,v)}$  indirectly.

Let  $u = y + x^2$  and  $v = y - x^2$ . Then  $y + x^2 = 6$  and  $y + x^2 = 2$  become  $u = 6$ ,  $u = 2$ . Also  $y - x^2 = 0$  becomes  $v = 0$  and  $x = 0$  becomes  $u = v (= y)$ .

So  $D^*$  is this region:

Now  $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 1 \\ -2x & 1 \end{vmatrix}$

 $= 4x$  and hence  $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4x}$  (and  $> 0$  except on the edge of  $D$ )

$$\iint_D x \, dx \, dy = \iint_{D^*} x \cdot \frac{1}{4x} \, du \, dv = \frac{1}{4} \iint_{D^*} du \, dv = \frac{1}{4} \text{Area of } D^*$$

8. (18 points) Let  $D$  be the triangle with vertices  $(0,0)$ ,  $(0,1)$ , and  $(1,0)$ .

Compute  $\iint_D e^{(y-x)/(y+x)} \, dx \, dy$ . Hint: a second substitution in a single variable, say  $w$ , can help with the integration.

$$= \frac{1}{4}(16) = 4.$$

Let  $u = y - x$ ,  $v = y + x$ .

Then  $x + y = 1$  becomes  $v = 1$ ,

$x = 0$  becomes  $u = v (= y)$ ,

and  $y = 0$  becomes  $u = -x = -v$ .

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

and  $|J| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \frac{1}{-2} \right| = \frac{1}{2}$ . Then we get

$$\iint_D e^{(y-x)/(y+x)} \, dx \, dy = \iint_{D^*} e^{\frac{u}{v}} \frac{1}{2} \, du \, dv$$

$$= \iint_{D^*} \int_{-v}^v \frac{1}{2} e^{\frac{u}{v}} \, du \, dv.$$

$$(u=v \Rightarrow w=1) \\ u=-v \Rightarrow w=-1)$$

$$= \int_0^1 \int_{-1}^1 \frac{1}{2} e^w v \, dw \, dv = \int_0^1 \left( e - \frac{1}{e} \right) \frac{1}{2} v \, dv = \boxed{\frac{e - \frac{1}{e}}{4}}$$

let  $w = \frac{u}{v} = \frac{1}{v} u$ ,

$$dw = \frac{1}{v} du, du = v dw$$

$$= \int_0^1 \left( e - \frac{1}{e} \right) \frac{1}{2} v \, dv = \boxed{\frac{e - \frac{1}{e}}{4}}$$