MATH 172 Spring, 2002 Final Exam Name:____________________

There are 150 points. For full credit you must show your work. If you use your calculator for anything more than simple arithmetic, say so! You can leave symbols like $\binom{n}{k}$, ${}_nC_k$, ${}_nP_k$, $n!$, $66 \cdot 65 \cdot 64$, $(.3)(.7)^2$, 5^{10} , etc. in your final answer, unless you are specifically directed otherwise.

1. (6 points) In constructing a table of second differences of a sequence a_n you find that they are all the same number, 14. Also you know that $a_1 = 9$. Which of the following is most likely to be the correct solution to the problem, and why? (Show calculations, or argue by analogy, but give some reason for picking one over the rest.) (a) $a_n = 14n - 5$, (b) $a_n = (7n-4)^2$, (c) $a_n = 2n^7 + 7$, (d) $a_n = 7n^2 + 2n$.

2. (10 points) a. Compute the derivative of $S(t) = 8\cos(\frac{1}{2}t) + t\sin(5t^2)$.

   b. Give a function f with $f(0) = -2$, an amplitude of 2 and a period of 3π .

3. (18 points) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} -4/7 & 6/7 \\ 9/7 & 11/7 \end{bmatrix}$.

a. Compute $\mathbf{w} + 2\mathbf{v}$ and $\mathbf{v} \cdot \mathbf{w}$

b. The matrix A has eigenvalues 2 and -1; each of $\mathbf{v}$ and $\mathbf{w}$ is an eigenvector for A. Which vector goes with which eigenvalue?

c. Compute $A^8 \begin{bmatrix} 1 \\ 10 \end{bmatrix}$. Hint: $\begin{bmatrix} 1 \\ 10 \end{bmatrix} = 3\mathbf{v} + \mathbf{w}$.

4. (12 points) Your favorite bumbling weather man has predicted the next day's weather correctly 84 times out of 150, and incorrectly the rest.

a. What is the probability p that he predicts the weather correctly, and q that he does not?

b. What is the probability that in his next 5 attempts he will get 4 or more predictions correct? You may leave your answer in terms of symbols without doing a numerical calculation.

5. (20 points) The amount of drug in the bloodstream (measured in mg) is given by $a_{n+1} = 0.6a_n + 200$, where n is in days.

a. What term in this model represents the daily dose, and which term represents the amount left from the day before?

b. Suppose this dosing pattern started at a time when there was 100 mg of the drug in the bloodstream. Compute a_n in terms of n in this case. (Hint: part of this involves solving the equation when the daily dose is left out; the other part has to do with a possible equilibrium.)

c. What happens to a_n in the long term as this dosing continues? Explain.

6. (12 points) You are given annual survival probabilities and fecundities (numbers of surviving offspring) for a population with three stages: hatchlings, juveniles, and adults.

   a. Construct the transition matrix A.

$$\begin{array}{ccc} H_t & & H_{t+1} \\ J_t & \xrightarrow{30\%} & J_{t+1} \\ A_t & \xrightarrow[70\%]{} & A_{t+1} \end{array}$$

   b. You find that A has three eigenvalues, of which the dominant one is $\lambda = 1.045$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 6.96 \\ 0.93 \\ 0.54 \end{bmatrix}$. What is the approximate annual growth rate? _______

   c. In the long run what growth stage forms the majority of the population, and how can you tell? What fraction of the population is in each growth stage?

7. (8 points) Compute the sum of each series; or state that no sum exists and why.

   a. $\sum_{k=1}^{\infty}(2/5)(-3/5)^k$

   b. $\sum_{k=1}^{\infty}(2/5)(7/2)^k$

8. (18 points) Make a tree diagram, filling in as many values as you can, for the following problem, and then compute the probabilities. The IRS pulls out 2% of all returns for auditing by an automated computer program. Of those that are audited, 85% turn out to be one where the taxpayer owes more money, and the rest are OK. Of the returns that are not audited, the estimate is that 20% owe money, and the rest are OK.

   a. If a return is pulled at random, what is the probability that it is an underpayment (taxpayer owes money)?

   b. If a return is pulled at random, and it turns out to be an underpayment, what is the probability that it would have been caught by the computer?

9. (4 points) Compute $\binom{14}{9}$ by hand, showing all the arithmetic and cancellations. (You may use your calculator to check!)

10. (16 points) You are auditioning for a play and 35 hopefuls have turned up, each with his or her own personality, strengths and weaknesses. The play has 4 male roles and 5 female roles. Of the candidates 20 are female, and the other 15 are male.

   a. If you care which actor gets which part, in how many ways can you cast the play? (Males have to play male roles, females have to play female roles.)

   b. If all the male parts are interchangeable, and the same for the female parts, in how many ways can you cast the play?

   c. Your producer has decided to break the rules and put the play on with an all female cast or an all male cast. You still care which person gets which part. In how many ways can you cast the play?

11. (8 points) As a mathematically talented squirrel you observe that peanut shells sometimes just contain one peanut, but more often have two peanuts, sometimes three, and very rarely four peanuts. Sometimes a perfectly good looking shell turns out to be empty. The next 70 shells you gather, you record how many peanuts were in each one, and for each number of peanuts you tally up how many shells had that number. The results are shown below. Compute the probabilities of finding each number of peanuts when you open a shell at random. Compute the expected number of peanuts per shell if a shell is opened at random (you can think of this as a payoff, where the payoff is counted in peanuts). On average, how many shells will you have to open in order to get 366 peanuts?

number of nuts	number of shells	probability
0	6	
1	13	
2	40	
3	9	
4	2	

12. (18 points) The density function for an episode of feeding activity away from the nest by a certain bird is given by $p(x) = 0.1e^{-0.1x}$, where x is the time in minutes. Recall that the cumulative distribution function is given by $F(T) = \int_0^T p(x)\,dx$, and the median feeding time is the number T^* so that $F(T^*) = 1/2$.

a. What fraction of the feeding episodes away from the nest last between 1 and 4 minutes?

a. Compute $F(10)$. What does it mean in terms of birds and feeding times? What does $1 - F(10)$ represent?

b. Compute the median feeding time T^*. What does this number represent in real world terms?