MATH 172X Spring, 2001 Final Exam Name:____________________

There are 150 points. For full credit you must show your work. If you use your calculator for anything more than simple arithmetic, say so! You can leave symbols like $\binom{n}{k}$, ${}_nC_k$, ${}_nP_k$, $n!$, $66 \cdot 65 \cdot 64$, $(.3)(.7)^2$, 5^{10}, etc. in your final answer, unless you are specifically directed otherwise.

1. (6 points) In constructing a table of second differences of a sequence a_n you find that they are all the same number, 8. Also you know that $a_0 = 1$. Which of the following is most likely to be the correct solution to the problem, and why? (Show calculations, or argue by analogy, but give some reason for picking one over the rest.) (a) $a_n = n^8 + 1$, (b) $a_n = 3n^2 + 2n + 1$, (c) $a_n = 4n^2 + 1$.

2. (12 points) a. Formulate a model in which the change in the amount of drug in the bloodstream Δa_n from one day to the next is proportional by a factor of -0.3 to a_n, the amount at the beginning of the day, plus a maintenance dose of 6 mg/day. Then compute the long term steady state ($\Delta a_n = 0$) value of the drug in the bloodstream.

   b. If there is no maintenance dose, but instead a large initial dose of 100 mg, express a_{n+1} in terms of a_n, and then in terms of n and a_0.

3. (10 points) A population changes according to a logistic model $P_{n+1} = P_n + 0.8P_n(1 - (P_n/100))$. What are the steady states (equilibrium values)?

b. If $P_0 = 50$, compute P_1 and P_2. Then sketch a graph showing how P changes with time.

4. (20 points) We have a fox-rabbit model in which from year to year

$$\begin{aligned} R_{n+1} &= 1.2R_n - 0.2F_n \\ F_{n+1} &= 0.1R_n + 0.9F_n \end{aligned}$$

The matrix of coefficients is $A = \begin{bmatrix} 1.2 & -0.2 \\ 0.1 & 0.9 \end{bmatrix}$, which has eigenvectors $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and eigenvalues $\lambda = 1.1$, $\mu = 1$.

a. Give a formula for the fox population F_k in terms of k and the initial population F_0 if there are no rabbits. What happens to the fox population F_k in the long run?

b. Compute $\mathbf{v} - 2\mathbf{w}$.

c. Which eigenvalue goes with which eigenvector? Explain.

d. We know that this model has a general solution $\begin{bmatrix} R_k \\ F_k \end{bmatrix} = A^k \begin{bmatrix} R_0 \\ F_0 \end{bmatrix}$. Using an initial population vector $\begin{bmatrix} R_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \end{bmatrix} = 8\mathbf{v} + 4\mathbf{w}$, and the eigenvalue-eigenvector relationships that you found earlier, describe the long term behavior of $\begin{bmatrix} R_k \\ F_k \end{bmatrix}$. How are the populations rising or falling on an annual basis?

5. (16 points) a. What is the amplitude and period of $y = 3\sin(5x)$?

c. Compute the derivative of $P(t) = 8\cos(\frac{1}{2}t) + \sin(t^3 + 1)$.

d. Compute the antiderivative $\int 2x^3 \sin(x^4 + 1)\, dx$.

6. (4 points) Compute $\binom{12}{7}$ by hand, showing all the arithmetic and cancellations. (You may use your calculator to check!)

7. (18 points) On a typical MWF morning, I find it hard to get to my 9:05 class. The chance that I get breakfast is 0.4; the chance that I get a cappucino (coffee) is 0.7; the chance that I get both is 0.2.

   a. What is the probability that I get breakfast or coffee or both?

   b. If I do not get coffee, which has probability _______ , then the probability that I am grumpy is 0.8. If I do get coffee, then the probabilty that I am grumpy is 0.1. What is the probability that I am grumpy?

   c. You observe that I am grumpy one morning. What is the probability that I did not have coffee?

8. (16 points) The parts of this problem do NOT depend on one another. The top shelf of your bookcase holds 20 books. You have 4 books on mathematical biology (M), 6 on ecology (E), 5 on cell biology and genetics (C), and 5 on evolution and the Darwinian theory of natural selection (D).

   a. Scanning the shelf from left to right, how many possible arrangements of these books are there?

   b. If all you care about the 20 books is what topic is where, how many possible arrangements of these books are there? (Hint: think about positions on the shelf.)

c. Your actual collection is much larger: 6 M's, 10 E's, 12 C's, and 9 D's. In how many ways can you select from your entire collection to fill the top shelf as stated in the beginning of the problem?

d. Your collection is described in part c. You load the top shelf either by using the M's and C's, or by using the D's and E's. In either case you leave room on the far right end for future acquisitions. How many possible arrangements are there if you DO care about the order of the books?

9. (8 points) Champion bowler Jessica bowled a strike 850 times, a spare 200 times, and neither the remaining times out of 1000 times up.

a. What is the probability p that she bowls a strike in one time up? What is the probability q that she bowls a spare in one time up? What is the probability r that she does neither?

b. What is the probability that in 12 times up she would bowl exactly 7 strikes and 3 spares?

10. (8 points) Compute the sum of each series; or state that no sum exists.

a. $\sum_{k=1}^{\infty}(4/3)(3/5)^k$

b. $\sum_{k=1}^{\infty}(3/4)(5/3)^k$

11. (10 points) Twenty (20) bean pods are broken open and the number of beans in each is recorded in the table below. Compute the probabilities of finding each number of beans. Then compute the expected number of beans per pod if a large number of pods are opened. On average, how many pods will you have to pick in order to get 130 beans?

number of beans	observed frequency	probability
2	5	
3	8	
4	4	
5	3	

12. (12 points) The density function for time gaps between foraging attempts by a certain species of bird is given by $p(x) = 0.1e^{-0.1x}$, where x is the time in minutes.

a. Compute the median time gap.

b. If the cumulative distribution function is $F(x)$, what is the value of $F'(3)$?

c. Compute the mean time gap $\mu = \int_0^\infty (0.1)xe^{-0.1x}\,dx$.

13. (10 points) An equilateral triangle has three rotational symmetries: R, R^2, and $R^3 = I$. By what angle does R turn the triangle? There are also three reflection symmetries: D_1, D_2, and D_3. Indicate these on the diagram. Compute $R \circ D_1$ and $D_1 \circ R$.