MATH 172 Spring, 2005 Exam #2 Name:______________________

As always, for full credit you must show the essential work. Recall that a model of form $u_n = au_{n-1} + b$ has an explicit solution of form $u_n = a^n c + E$, where E is the equilibrium value, which can be precisely calculated, and c can be determined by using a specific u_n value. You are expected to know how to solve more basic discrete models.

1. (7 points) Given below is the transition matrix for a weather model with three states: sunny (S), cloudy (C) and rainy (R).

$$A = \begin{array}{cccc} \text{tomorrow}\downarrow \backslash\text{today}\rightarrow & S & C & R \\ S & 1/2 & 0 & 1/8 \\ C & 1/4 & 1/4 & 1/8 \\ R & 1/4 & 3/4 & 3/4 \end{array} \qquad \mathbf{v} = \begin{bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{bmatrix}.$$

If it is cloudy today, what is the probability that it is rainy tomorrow? _______ .
If it is sunny today, what is the probability that it is cloudy the day after tomorrow? _______ (Hint: don't do more work that is absolutely necessary!)
One can show that $A\mathbf{v} = \mathbf{v}$; what is the significance of this observation?

2. (10 points) Compute the sum of each series; or state that no sum exists, and why or why not.

a. $\sum_{k=0}^{\infty}(-2/3)(15/6)^k$

b. $\sum_{j=0}^{\infty}(15/6)(-2/3)^j$

3. (20 points) a. Formulate a model (dependent variable u, independent variable n) in which 80% of drug in the bloodstream from one day to the next is used up, but the remainder is reinforced by a maintenance dose of 48 mg/day.

   b. What is the theoretical steady state (equilibrium) amount of the drug in the bloodstream?

   c. Determine the explicit solution formula for u_n if the initial dose is 20 mg. Then use this formula to describe the presence of the drug in the bloodstream over the long term.

4. (22 points) For each model equation, and initial condition, **first give the explicit solution**. Then answer the other questions.

   a. Model equation $u_n = (0.7)u_{n-1}$ with initial condition $u_0 = 50$; then describe the long term behavior of u_n.

   b. Model equation $z_n = z_{n-1} + 1.3$ with initial condition $z_0 = 2.7$.

c. Model equation $v_n = (-1.3)v_{n-1} + 138$ with $v_0 = 80$. Basing your answer on the explicit solution, what is the long term behavior of v_n, and does the equilibrium appear to be stable? Explain.

5. (7 points) In a 2-variable system you find that eventually $u_n \approx 1.02u_{n-1}$ and $v_n \approx 1.08v_{n-1}$. Is the total population $T_n = u_n + v_n$ growing or declining? Someone claims that the total population must be growing at $(2+8)/2 = 5\,\%$ a year–is this correct? Is there eventually a stable distribution in which each of u_n and v_n maintains less than 100% of the population? Briefly explain your answers.

6. (20 points) A matrix M has eigenvectors $\mathbf{e}_1 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. These go with eigenvalues $\lambda_1 = 1.04$ and $\lambda_2 = 0.8$, respectively. We have $\mathbf{u}_0 = \mathbf{e}_1 + 2\mathbf{e}_2$.

a. Compute $M(\mathbf{u}_0)$. You may leave the symbols $\mathbf{e}_1$ and $\mathbf{e}_2$ in your answer.

b. Compute $M^2(\mathbf{u}_0)$. You may leave the symbols $\mathbf{e}_1$ and $\mathbf{e}_2$ in your answer. This is a computation of u_k for which value of k?

c. Rewrite $M^n(\mathbf{u}_0)$ in the form $a\,\mathbf{e}_1 + b\,\mathbf{e}_2$. This computation is used to compute _______ . What happens to a and to b as n gets larger and larger?

d. If M is a population projection matrix (Leslie matrix), what is the growth rate of the population? What is the stable age distribution?

7. (6 points) The dynamical system given by $\begin{cases} u_n = 2u_{n-1} - 5v_{n-1} + 20 \\ v_n = u_{n-1} - 2v_{n-1} + 10 \end{cases}$ has an equilibrium of $(5, 5)$. Determine the behavior of this system starting from $n = 0$ and going until you see a pattern (you need to pick values for u_0 and v_0), and describe this pattern. You don't need to write down all the values that you compute, but a selection from your table, or a graph should be provided as evidence.

8. (8 points) You are given annual survival probabilities and fecundities (numbers of surviving offspring) for a population of frogs with three stages: tadpoles, juveniles, and adult frogs.

   a. Construct the transition matrix A based on the figures given below from one census to the next.

$$\begin{array}{ccc} T_{n-1} & & T_n \\ & & \\ J_{n-1} & \xrightarrow{10\%} & J_n \\ & & \\ F_{n-1} & \xrightarrow[55\%]{} & F_n \end{array}$$

   b. If there are currently 100 tadpoles, 20 juveniles, and 20 adult frogs, how many of each will there be at the next census?