MATH 172  Fall, 2011   Exam #3  Name: 

There are 100 points. For full credit you must show your work. Recall that the geometric series \( \sum_{n=0}^{\infty} a r^n \) has a sum \( S_\infty = a/(1 - r) \) under a certain condition on \( r \), which you should verify, and fails to exist otherwise.

1. (12 points) Compute the equilibrium point \( (u^*, v^*) \) of the discrete system

\[
\begin{align*}
  u_n &= 2u_{n-1} - v_{n-1} - 1 \\
  v_n &= u_{n-1} - 3v_{n-1} + 8
\end{align*}
\]

\[
\begin{align*}
  u^* &= 2u^* - v^* - 1 \\
  v^* &= u^* - 3v^* + 8 \\
  1 &= u^* - v^* \\
  -8 &= u^* - 4v^*
\end{align*}
\]

\[
\begin{align*}
  8 &= -u^* + 4v^* \\
  9 &= 3v^* \\
  v^* &= 3 \\
  u^* &= 4
\end{align*}
\]

\( (4,3) \)

2. (8 points) A reproductive female in the oldest stage of development produces 5 offspring on average each year. Her annual survival rate is 80%. What is her expected lifetime production of offspring?

\[
5 + 5(0.8) + 5(0.8)^2 + \ldots = \sum_{n=0}^{\infty} 5(0.8)^n
\]

\[
a = 5 \\
10.8/1 < 1
\]

\[
5 \\
0.8
\]

\[
= \frac{5}{0.8} = 6.25
\]

3. (15 points) Find the sum if it does exist, or state that there is no sum, and why.

a. \( \sum_{j=1}^{\infty} (2)(\frac{3}{j}) \) = 

\[
2(\frac{3}{1}) + 2(\frac{3}{2})^2 + 2(\frac{3}{3})^3 + \ldots
\]

\[
a = 2(\frac{3}{1}) \\
r = \frac{3}{j} \quad 1/3 < 1
\]

\[
= \frac{2}{1 - 3/7} = \frac{2}{4/7} = \frac{6}{4} = \frac{3}{2}
\]

b. \( \sum_{n=0}^{\infty} \frac{2}{5} (-\frac{4}{3})^n \) = 

\[
r = \frac{-4}{3} \quad \left| -\frac{4}{3} \right| > 1
\]

There is no sum.
4. (15 points) Consider the following continuous model of a predator-prey system.

\[
\frac{dV}{dt} = 0.7V \left(1 - \frac{V}{350}\right) - 0.02VP = V \left(0.7 \left(1 - \frac{V}{350}\right) - 0.02P\right)
\]
\[
\frac{dP}{dt} = -0.9P + 0.003VP = P(-0.9 + 0.003V)
\]

a. What kind of growth does the victim population exhibit if there are no predators (i.e., \(P = 0\))? 

\[
\frac{dV}{dt} = 0.7V \left(1 - \frac{V}{350}\right) \quad \text{logistic growth}
\]

\[r = 0.7, \quad K = 350\]

b. Why is \((V^*, P^*) = (350, 0)\) an equilibrium (mathematically), and how do you interpret this biologically?

Because both \(\frac{dV}{dt}\) and \(\frac{dP}{dt}\) = 0. This is the case with no predators and prey (victims) at their carrying capacity.

c. Compute the equilibrium \((V^*, P^*)\) other than \((0,0)\) and \((350,0)\) for this system.

\[-0.9 + 0.003V = 0\]

\[V^* = \frac{0.9}{0.003} = 300\]

\[0.7 \left(1 - \frac{300}{350}\right) - 0.02P = 0\]

\[P = \frac{0.1}{0.02} = 5 \quad (300,5)\]
5. (15 points) A not very infectious illness, but one that is occasionally fatal, is spreading through a susceptible population $S(t)$. Each week there is a mass action interaction with transmission coefficient 0.02 of the susceptible population $S(t)$ with the ill and infectious population $I(t)$ in which susceptible individuals become ill and infectious. At the same time 45% of the infectious population recovers; the recovered population is denoted $R(t)$. Just 0.5% of the infected population dies. Most of the recovered population, in fact 80%, is immune to reinfection, but the rest do become susceptible again. Write a system of continuous model equations for this process (the unit of measurement is thousands of people per week).

\[
\frac{dS}{dt} = -0.02SI + 0.2R \\
\frac{dI}{dt} = 0.02SI - 0.45I - 0.005I \\
\frac{dR}{dt} = 0.45I - 0.2R
\]

6. (15 points) A victim population has a growth rate curve (light line) as shown. Superimposed on this graph is a heavy curve indicating the loss rate due to moderate predation by a predator that exhibits a type III functional response.

a. Find the numerical values of the the non-zero equilibrium values for $N^*$ on the graph, determine if each is stable or unstable, and indicate by arrows how $N$ will change if it falls just slightly off each equilibrium value.

b. What is the maximum sustainable predation level, and what is $N$ for this level?

\[N^* = 5,500; \quad 10,000; \quad 18,100\]

stable; unstable; stable
7. (12 points) A metapopulation has a patch to patch colonization rate of 12% and a local extinction rate of 3%; that is \( \frac{dp}{dt} = 0.12p(1-p) - 0.03p \). Find the equilibrium value other than \( p = 0 \) for the percent of occupied patches. Use a graphical analysis to determine if this equilibrium is stable or not.

\[
0 = \frac{dp}{dt} = 0.12p(1-p) - 0.03p
\]

\[p = 0 \quad \text{or} \quad \frac{0.12(1-p) - 0.03}{0.12} = 0.25\]

\[0.03p = 1 - p = \frac{0.03}{0.12} = 0.25\]

\[p^* = 0.75\]

8. (8 points) We have a successional model of species A, B, C, D. Time is measured in decades.

\[
\begin{bmatrix}
0.2 & 0.1 & 0.1 & 0.1 \\
0.7 & 0.4 & 0.2 & 0.1 \\
0.1 & 0.3 & 0.4 & 0.3 \\
0.0 & 0.2 & 0.3 & 0.5
\end{bmatrix}
\]

a. If the current situation is 50% A and 50% B, what is the distribution in one decade?

\[
\begin{bmatrix}
0.5 \\
0.5 \\
0 \\
0
\end{bmatrix}
\]

15% A 20% C 55% B 10% D

b. The dominant eigenvalue for this matrix is 1, which has eigenvector \( \begin{bmatrix} 0.111 \\ 0.282 \\ 0.309 \\ 0.298 \end{bmatrix} \). What information does this give you? Is there a climax species, that is, one that dominates the habitat in the long term?

In the long term, there is 11.1% A, 28.2% B, 30.9% C, 29.8% D. No single species dominates the habitat, so no climax species.