There are 120 points, which will be scaled down to a percentage (out of 100). For full credit you must show your work. All answers should be correctly rounded to 3 decimal places, or fewer if that is what is provided in the data.

1. (10 points) If \( Q(n) = 5n - n^2 \), compute \( \Delta Q \).

\[
\Delta Q = Q(n+1) - Q(n) = 5(n+1) - (n+1)^2 - (5n - n^2) \\
= 5n + 5 - n^2 - 2n - 1 - 5n + n^2 \\
= -2n + 4
\]

2. (10 points) Papua New Guinea currently has a population of 56.7 hundred thousand people. The net growth rate is 1.253 hundred thousand people per year. Compute the intrinsic or per capita growth rate \( r \). Write a continuous model equation for this process. Give the explicit solution and use your result to predict the population a decade from now.

\[
\frac{dP}{dt} = rP \\
1.253 = r(56.7) \\
r = 0.0221
\]

\[
P(t) = P(0)e^{rt} \\
P(10) = 56.7e^{0.0221(10)} \\
P(10) = 70.72 \text{ thousand people}
\]

3. (15 points) A population \( B(t) \) of bacteria is growing continuously over time so that the per capita rate of increase is 6% /day.

a. Write the continuous model equation that describes this situation.

\[
\frac{dB}{dt} = 0.06B
\]

b. If \( B(0) = 5 \text{ g} \), write the solution equation for this model equation. (We don't count bacteria, we weigh them.) This initial population of 5 g grows to what size in 3 weeks?

\[
B(t) = B(0)e^{0.06t} = 5e^{0.06t} \\
3 \text{ weeks } = 21 \text{ days} \\
B(21) = 5e^{0.06(21)} = 17.627 \text{ g}
\]
4. (20 points) A population $R$ of rats kept in a lab depends on $t$. The experimenter is testing the effect of a poison in the food. Observation takes place at regular weekly intervals. At the start of the experiment ($t = 0$), the population is 400. Week by week, the population shrinks by 9%.
   
   a. Write the difference equation for this discrete model.
   
   $$\Delta R = R_{t+1} - R_t = -0.09 R_t$$
   
   b. Write the updating equation for this model.
   
   $$R_{t+1} = R_t - 0.09 R_t = 0.91 R_t$$
   
   c. Write the explicit solution for this model, and use this to compute the population after 6 months (26 weeks).
   
   $$R_t = (0.91)^t R_0 = (0.91)^t (400)$$
   
   $$R_{26} = 34,446$$ (or 34 rats)
   
   d. At what time is the population 1/2 of the original?
   
   $$200 = (0.91)^t (400)$$ or $$\frac{1}{2} R_0 = (0.91)^t$$
   
   $$t = \frac{\ln(\frac{1}{2})}{\ln(0.91)} = 7.350$$ weeks

5. (10 points) Due to a parasite infection that increases mortality and decreases reproduction, a host population is decreasing at a continuous rate of 4% yr$^{-1}$. Infected organisms move into the habitat at a rate 100/yr. Write the continuous model equation to describe this process, and compute the equilibrium value, if there is one.

   $\frac{dH}{dt} = -0.04 H + 100$

   Equilibrium when $\frac{dH}{dt} = 0$

   $$0 = -0.04 H + 100$$

   $$H = 2500$$

   Extra:

   Note that if $H > 2500$, then $\frac{dH}{dt} < 0$ so $H(t)$ decreases; if $H < 2500$, then $\frac{dH}{dt} > 0$ and $H(t)$ increases
6. (25 points) A drug is given in pill form (160 mg) once a day. We know that 80% of drug in the bloodstream is used up from one day to the next, but the remainder is reinforced by a maintenance dose of one pill a day. Let $A_t$ denote the amount of the drug in mg after $t$ days. The initial dose, given late in the day, is one half pill (80 mg).

a. Formulate a discrete model to describe this process over time. Compute $A_1$ and $A_2$.

\[
A_{t+1} = A_t - 0.8A_t + 160 = 0.2A_t + 160 \quad \text{(20% remains, half dose)}
\]

\[
A_1 = (0.2)A_0 + 160 = 176 \text{ mg}
\]

\[
A_2 = (0.2)A_1 + 160 = 195.2
\]

b. What is the steady state (equilibrium) amount of the drug in the bloodstream?

Look for $A_{t+1} = A_t$ or solve for $A_t$.

\[
A_t = 0.2A_t + 160 \quad \text{or} \quad 0.8A_t = 160 \quad \text{or} \quad A_t = 200
\]

(c) In the calculator version of this model $A_t$ is replaced by $u(n)$. It has $n_{\text{Min}} = 0$, $u(n_{\text{Min}}) = 80$, and $u(n) = (0.8)u(n-1) + (160)$.

d. Compute $A_3 = u_3$, $A_5 = u_5$, $A_{10} = u_{10}$ by hand or by calculator or both. What is the long term trend?

- $A_3 = 199.04$
- $A_5 = 199.96$
- $A_{10} = 200$

level of drug approached steady-state of 200 mg.

d. (Bonus) Is the equilibrium value you found in part (b) stable or unstable?

Explain verbally and / or graphically.

The equilibrium appears to be stable because if we start with $A_0 < 200$, then $A_t \to 200$ as $t$ gets larger. On the other hand if we start with, say, $A_5 = 240$ (or any $A_0 > 200$), then $A_t \to 200$ coming down from above.

* as seen above  ** done on calculator
7. (20 points) An algal population $A(t)$, measured in tons, grows in some lakes and rivers that accumulate nutrients from agricultural runoff. It is governed by the continuous model \[ \frac{dA}{dt} = 0.0003A(200 - A) = 0.06A(1 - \frac{A}{200}) \text{ tons/year}. \]

The initial population is $A(0) = 50$ tons. Since we are measuring in tons, it is reasonable that you may get decimal answers; round off to three decimal places.

a. Apply stepwise estimation using Euler’s method to estimate the population after 12 years. Remember that $\Delta A \approx \frac{dA}{dt} \Delta t$. Use 3 steps, so each time step is $\Delta t = \frac{4}{3}$ to approximate $A(12)$, and show your work! Here is the outline of a table to help you get started.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t$</th>
<th>$A(i)$</th>
<th>$A'(t)$</th>
<th>$\Delta A \approx A'(i) \Delta t$</th>
<th>$A(t + \Delta t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>50</td>
<td>2.25</td>
<td>\frac{9}{59}</td>
<td>59</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>59</td>
<td>2.496</td>
<td>9.982</td>
<td>68.983</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>68.983</td>
<td>2.711</td>
<td>10.856</td>
<td>79.839</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>79.839</td>
<td></td>
<td></td>
<td>$A(12) \approx 79.839$ tons</td>
</tr>
</tbody>
</table>

Extra: If you look at $A(t)$ or $\frac{dA}{dt}$, you’ll see that it is positive as long as $A < 200$.

b. If we want to approximate $A(12)$ again, but use $\Delta t = 1/4 = 0.25$ year (that is, 3 months), how many steps are required to get to 12 years?

$$0.25 = \Delta t = \frac{12}{N}$$

$$N = \# \text{ steps} = 48$$

Exam stats: 31 exams, mean = 61, median = 59, SD = $\pm 25$, low = 10, high = 107.