Instructor's Maple Manual
to accompany
David Lay's
Linear Algebra and Its Applications,
3rd Edition

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Foreword

This manual is for any instructor who is using Maple and *Linear Algebra and Its Applications* for the first time. It will greatly simplify your task of combining Maple with the text, because it is written by a colleague who has already tried out the materials with considerable success. This manual carefully describes everything you need to know about planning and conducting the course.

The author, Professor Douglas Meade, has more than a decade of Maple experience in both research and teaching. He has been working with the text and the accompanying Maple materials since 1998, and he has used the text in two different linear algebra courses, one with a weekly computer lab. Based on his classroom experience, Professor Meade has modified and refined the projects in this manual, to more closely fit the current presentations in the text. The original authors of the projects were Professors Jane Day, of San Jose State University, and Renate McLaughlin, of the University of Michigan-Flint. I am grateful for Professor Meade's contributions to the course, and I am confident that you and your students will appreciate his work.

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Introduction

This Instructor's Maple Manual provides a wide variety of information, support, and supplemental materials for an instructor teaching a first course in linear algebra from *Linear Algebra and Its Applications* using the Maple computer algebra system.

In addition to general suggestions for incorporating Maple into your course, the manual includes sixteen projects. The projects tend to be shorter than the Application Projects and Case Studies that are on the text's website. Each project described here may be downloaded as a PDF file from the Web. Since many students may have no prior exposure to Maple, most of the worksheets contain the Maple code needed to complete a project. The goal of the projects is to teach linear algebra, not Maple. However, Maple skills acquired here will be directly applicable to other areas of mathematics, science, and engineering.

While the broad selection of projects is very appealing, the textbook remains the cornerstone of the course. The exercises are carefully selected to complement and supplement the exposition. The matrices and vectors — numeric or symbolic — for nearly 1000 exercises can be accessed in Maple with a few keystrokes. Also, special `laylinalg` commands implement the matrix operations exactly as they are described in the text. This allows students using Maple for their homework to focus on the mathematics without worrying (as much) about correctly copying the problem from the book or making algebra errors. When my students realize these benefits, their use of Maple on homework skyrockets.

The *Study Guide* provides detailed information on the use of the commands in the `laylinalg` package. While the main portion of the *Study Guide* is directed towards MATLAB, the Maple implementation is designed to directly parallel the MATLAB usage. The Maple appendix demonstrates the use of Maple commands at a pace appropriate for the skills and theory being presented in the text. The tight coordination between the text and supplements is a great benefit. If you have not already done so, take a few minutes to look at the *Study Guide* — in particular, the Maple appendix.

Some students are apprehensive about the additional work that is required to complete the projects, particularly if the benefits are not well defined. To reinforce the importance of the projects, I strive to select relevant projects and assign them at appropriate times in the course, provide reasonable time for the completion of the work, discuss the projects in class, grade and return the completed projects promptly, and include the project grades in the overall course grade. To further reinforce the importance of the projects, my students also know that each exam will include at least one question based on the projects.

I hope you and your students find the information in this Manual useful. Comments, corrections, and any other feedback — including questions — are greatly appreciated.

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1 GETTING STARTED

PREPARE

The number of options for using technology in support of a course in linear algebra continues to grow. Commercial software packages MATLAB, Maple, Mathematica, and MathCad, and calculators such as the TI-86, TI-92, and HP-48G can all be used together with *Linear Algebra and Its Applications*, Third Edition. The **[M]** exercises in Lay's text are written in a software-independent style and can be solved with any of these tools. This manual, however, is written specifically for instructors interested in using Maple.

This manual also assumes that you have the *Study Guide for Linear Algebra and Its Applications*, Third Edition and the `laylinalg` package, a collection of Maple procedures and data for nearly 1000 exercises in the text. The *Study Guide* contains a built-in guide to using technology with Lay's text. Although the main sections of the *Study Guide* discuss MATLAB, there are appendices for Maple, Mathematica, and several graphing calculators. The Maple appendix provides explicit instructions on the use of Maple for homework problems. Your students will seldom, if ever, need more documentation for this course than the appendix in the *Study Guide* and Maple's online help facility.

If you have not used Maple before, you will want to take some time to familiarize yourself with its basic operations. Your students will assume you are an expert and, although that may not be the case, it will be helpful to have enough hands-on experience to be able to provide some suggestions — and to know your own limitations. The Maple worksheets are user friendly and work much like a word processor with the added dimension of command execution. As a quick start, you may want to work through some of the computer projects at the end of this Manual and the **[M]** exercises in the text. If you are interested in learning more about Maple, you may want to locate a tutorial. Then try your hand at a variety of exercises such as those mentioned above.

STUDENT EDITION OF MAPLE AND THE STUDY GUIDE

Students should have a personal copy of the *Study Guide*. If students will be encouraged — or required — to have a personal copy of Maple, the Student Edition of Maple 8 can be ordered from Waterloo Maple, Inc. For current information on the Student Edition of Maple, go to the Waterloo Maple homepage, `http://www.maplesoft.com/`, and search for "student edition".

Students should have these materials the first day of class. Bookstore managers often order insufficient quantities of supplements, unless you make it a required item. Having the *Study Guide* available for students will save you the trouble of explaining the basic operations needed for the homework exercises. Be sure to place your order in time to have the *Study Guide* in stock before the start of the course.

The *Student Edition of Maple* is inexpensive ($129 as of June 2002) and includes a good User's Guide. It is functionally identical to professional Maple. Installation is fairly straightforward, but on that rare occasion when something does happen, contact Waterloo Maple, Inc. or visit their website (`http://www.maplesoft.com/`). Your local system administrators should be consulted with network and local security issues.

THE laylinalg PACKAGE

The `laylinalg` package is a Maple library that contains additional Maple commands and data for almost 1000 exercises in the text (both regular and **[M]** exercises). This package and installation instructions are accessed through the WWW at the URL `http://www.laylinalgebra.com/`.

Having the data for the majority of the exercises in the text readily available saves each user from typing them in and facilitates the use of Maple when working homework problems. The special functions facilitate solution using the methods discussed in the text and *Study Guide*. It can also be effective to demonstrate the use of these tools as a part of your lectures. of various exercises in the text, and some of them are good for classroom demonstrations. A complete list of Maple commands provided in the `laylinalg` package can be found in Section 3 and are described in more detail in the *Study Guide*.

If you and the students will be accessing the `laylinalg` package via a local network, this should be installed (and tested) before the first day of classes. This prevents most unexpected last minute surprises, improves the effectiveness of the first demonstrations, and encourages students to utilize this resource from throughout the course.

THINGS TO ANNOUNCE AND FIND OUT

At the first class inform the students that, if they plan to use Maple outside your labs, they will need a copy of the `laylinalg` package. It is anticipated that most students will download the `laylinalg` package through `http://www.laylinalgebra.com/`. However, since some students might not have access to the WWW, you might want to create a number of diskettes containing the `laylinalg` package that students can borrow. (Make an archive first and have the students copy just one file.)

It is a good idea to distribute a survey at your first class meeting to find out your students' majors and the previous experience they have with computers, calculators, and various types of software. This will give you an idea of what kind of group you have and which students might be able to help others. You might pair the students off, so that each novice is introduced to someone with experience. All of this information can help you decide on appropriate goals and topics for computer assignments and about practical matters such as whether to ask students to use a text editor, print graphics displays, or work with a partner.

ASSIGNMENTS

The first time you plan to assign computer exercises, you might want to proceed with some caution. Keep a close watch on students' difficulties with each assignment. Various things can require more attention than you might expect, especially at the beginning of the course. For instance, you may find that access to the computer labs is inadequate, maintenance is not as quick as you would like, or students who buy the software have trouble installing it. Equipment has a knack for picking the worst and most unlikely times to act up. Assume "If it can go wrong, it will." and then be pleasantly surprised if things go smoothly.

Your first assignment should not be about math. Your students will panic when something does not work perfectly for them. Have them practice using the computers, saving and transporting their work, and then submitting the assignment in the form you choose — printed and handed in, saved to a diskette that is submitted, or via e-mail. If you would like them to do some math for their first assignment, do not grade it based on the solutions. You might use the opportunity to send back comments on how you would like solutions presented. Then talk about how it went. Remember that the students are resources for each other. Let them address other students' problems. It is too much work to be the sole source for technological support.

It is best to work through each computer assignment yourself before assigning it. While this might be overkill for the end-of-section exercises, it cannot be overemphasized for the Case Studies and Projects. This is the only way you can be certain you will know how this experience will fit with your lectures, whether you should tell students to watch out for anything, how much time to allow, and how much homework is reasonable to assign at the same time. However, do not expect to see your solution from the students. They will employ any number of methods for solving the problems. The main idea is for them to communicate their solution to you. You will have to decide for yourself how much computer work you would like to incorporate into your class. Some classes run entirely on the computer, while others assign a project every two or three weeks. Keep the lines of communication open between you and your students. They will let you know when you are stepping over the line. Try to differentiate between technical problems and content problems. Sometimes the difficulties arise only from inexperience using Maple.

One trap to watch out for is using Maple for answer checking. While it can surely do that, and students should be encouraged to do so if they like, using Maple simply as an answer checker is a waste of time as the focus of computer assignments. Maple places a lot of computing power in the hands of the students. It is a great opportunity to attack real-world situations. Both hand and computer skills are important. Collect and grade kinds of problems.

If you are able, find a student grader with computer algebra system experience. He or she can handle a large portion of the routine problems, leaving you time to spend on more conceptual exercises. Having an assistant in the lab during some hours is also a huge benefit, especially during the first weeks of class.

ALLOW TIME TO ADAPT, THINK, BE CREATIVE

It would be very good to have some release time the first time you incorporate the use of technology in a course. Most likely, you will find yourself spending more and more time creating the kinds of exercises you wish you could have assigned in the past. Maple will free a lot of your creativity, and making changes in a course always takes a lot of effort. Your students will have different interests and questions. Do not be afraid to modify the pace, style, or even topics of the course as it proceeds.

Along the same lines, it is recommended that you teach the course more than once. This provides opportunities to improve your materials and methods and increases the return on your initial preparation for the course.

The very existence of powerful and accessible matrix computation tools raises questions about what topics to emphasize, what skills are most important, and what style of teaching is best. These issues never disappear. Pay attention to the things that excite the students. Allow yourself the flexibility to change your perceptions of what they can learn and what the instructor's role should be. There is considerable research suggesting that people have individual learning styles and learn best when they are in control of how information is processed. This has often been stated as "students need to be involved in their learning". A computer algebra system (CAS) places powerful experimentation and investigative tools in the students' hands. You might provide more conceptual advice and allow the students more freedom to discover what you mean. Discussions are a wonderful venue for assessing and guiding students. Once everyone is comfortable with the tools, you may find that students would like to tackle more realistic situations. The CAS has a way of hiding the complexity and students are able to "see" why math is useful.

COMMON DIFFICULTIES

Many of your students will be very computer literate, but there may be a few with no prior computer experience. The Maple interface and mechanics will be alien to these students. Editing, saving, printing, and executing will be new skills. These skills are not pertinent to linear algebra, but they are central to operating the software. You can avoid such problems by having all results written by hand and rarely printing graphics. However, this keeps many students in the dark and allows anxiety to build. The first rule of my computer-based courses is making everyone feel comfortable. Have the experienced students assist with the introduction. They are a resource for you and would like to help. Many times the novices feel nervous when the instructor is standing over their shoulder but will accept instruction from another student. It is up to you to decide how you would like homework handed in, but do not restrict the students to a minimal set of skills. Answer all of their questions and solicit help from the rest of the students. The class runs much more smoothly when everyone feels free to experiment and to ask questions.

THINGS KEEP CHANGING

At some point, a student may report different numerical results than you anticipate. Do not worry. Different floating-point processors carry different numbers of significant digits. Most CAS will let you change this setting. View these episodes as opportunities to discuss what is going on. After all, this is the type of problem your students will run into frequently when they are on the job. It is not something to fear. On the other hand, if some calculations look way off, ask students to repeat them. Many times, the problem will be things such as the linear history kept by the CAS on executions. Students jump around their worksheets and just need to go back and re-execute things in order. It is not difficult to create accuracy problems for the computer. Understand that they are there. Do not avoid them. This might even make a nice assignment.

2 THE COMPUTING ENVIRONMENT

A COMPUTING LABS

Your department may have several public labs equipped with PCs. They could be networked including a printer somewhere. Depending on the setup and your license agreement, you may want to install Maple on each individual machine or on the server from which the computers are updated each day. If students are required to have an account to access e-mail and the Internet, you probably will want to locate the system administrators and inform them of your plans. They can help you a lot!

There are a couple of ways to get started. You may want to offer an “Introduction to Maple” workshop early in the course when the lab is not crowded. You may have the students work through some simple examples. Most likely, the students will be up and running before you know it and a few simple suggestions will encourage and enable them to complete the assignment on their own. Everyone will benefit from a lab monitor. Having a student familiar with Maple and linear algebra keeping regular hours in the lab is an inexpensive way to provide help. If you do decide on a lab assistant, meet together before you start the lab and go over your intentions. Ask the assistant to be in the lab the first few times with you. There will be a lot of questions and running around.

B HOW STUDENTS DO THEIR COMPUTING

The simplest setup would be for everyone to use the same hardware and to be in the lab at the same time. Get that picture out of your head. Some students will use the lab. An increasing number of students will have computers at home and will want to work there also. When they do use the lab, their attendance will be scattered throughout the day. Some students may find other campus labs more convenient for them. Except for the `laylinalg` package, keep your attention on the lab you have made available and leave them to figure out the rest — they will.

Every so often a student wants to use different software, or a calculator with matrix functions. If you are comfortable with this, give them your blessing, but tell them that they need to communicate solutions to you. Do not spend much time acclimating yourself to their system. Undoubtedly, there are other students in the class who are familiar with the technology and can help. Let everyone know it is OK to ask. This type of independence can only add to your discussions. It can be very useful to see what other machines give for results. Independent learning is your real goal.

3 CLASSROOM DEMONSTRATIONS

A SOFTWARE

In addition to the `laylinalg` package, the built-in `linalg`, `plots`, and `DEtools` packages are the only other Maple libraries needed for this course. While there appears to be some duplication between the `linalg` and `laylinalg` packages, e.g., `gausselim` and `gauss`, the `laylinalg` commands have been implemented so that the results returned are the same as would be obtained when the algorithms in the text are applied by hand. In the case of `gausselim` and `gauss`, the `linalg` command (`gausselim`) uses pivoting and the `laylinalg` command (`gauss`) does not. The results are, of course, equivalent, but verifying this is not the primary emphasis. A complete list of commands defined in the `laylinalg` package can be found in the following table. (Consult the *Study Guide* or Maple's online help worksheets for the specific syntax and examples.)

Function	Description
`bgauss`	(Backward Gauss) Uses the leftmost nonzero entry in a selected row as the pivot, and creates zeros in the pivot column *above* the pivot entry
`gauss`	Uses the leftmost nonzero entry in a selected row as the pivot, and uses row replacement to create zeros in the pivot column, below the pivot entry or in specified rows
`gs`	Performs the Gram-Schmidt process on the columns of a matrix
`nulbasis`	Computes a basis for the nullspace of a given matrix
`polyroots`	Computes floating-point estimates of the eigenvalues of a matrix
`proj`	Computes the orthogonal projection onto a subspace
`randomint`	Creates a matrix with random integer entries
`randomstoc`	Creates a random stochastic matrix
`replace`	Performs one row replacement operation
`scale`	Scales one row of a matrix
`swap`	Interchanges two rows of a matrix

Many of the exercises contain specific numeric or symbolic matrices and vectors. For example, the 3×4 augmented matrix in Exercise 25, Section 1.1, can be entered into a Maple worksheet by loading the `laylinalg` package then issuing the Maple command: `c1s1( 25 );`. Electronic access to the exercise data frees the student from the burden of manually entering (and checking) the matrices and vectors required to solve a problem.

A Maple session can be saved as a worksheet. Maple worksheets are platform independent. That is, regardless of the computer on which the worksheet was created (Windows, Macintosh, Linux, UNIX, ...) it can be loaded into any current version of Maple (and, usually, older and newer versions of Maple). Note that when a Maple worksheet is loaded, it is necessary to re-execute the commands in the worksheet (even if the results from earlier use are still visible).

The library files that define the `laylinalg` package are not platform independent. If you experience any troubles installing this package, be sure you have the files for the correct platform. The only other incompatibility problem is between versions of Maple. As of June 2002, the current version is Maple 8. You should have no difficulty opening earlier worksheets, but older versions of Maple may not be able to load current worksheets or understand the current library files.

B EQUIPMENT

For demonstrations it is necessary to enlarge the contents of the computer screen for the entire class. The most convenient arrangement is a projection unit hooked to a computer. Depending on the size of your classroom, an LCD panel on an overhead projector may be equally adequate. Unless you have a high-powered panel, the room may need to be quite dark to provide a satisfactory display. A laptop computer is also nice since it is very portable and is not an obstacle for students to maneuver around.

C SECURITY

Most labs fasten the computers and monitors to the tables. The lab is usually kept locked except when a lab monitor is present. If it is a departmental lab, you may be required to keep it locked from the outside even when you are using it.

D ADDITIONAL RESOURCES

D.1 Lay Website: `http://www.laylinalgebra.com/`

The Case Study for each chapter and 21 Application Projects tied to specific sections of the text (in addition to the `laylinalg` package) can be downloaded from `http://www.laylinalgebra.com/`. Additional information that can be accessed through this website includes review sheets, sample exam questions, and links to additional information about the projects. A full listing of the Case Studies and Application Projects can be found in Section 5 of this manual. Instructions for downloading and transferring the descriptions and related Maple worksheets can be found on WWW at `http://www.laylinalgebra.com/`.

This website has a wealth of resources for students and faculty, described in the Preface of the Text. When you are planning your Maple assignments, you should review the Case Studies and Application Projects that are available there. The Case Studies amplify the vignettes that introduce each chapter, and the Application Projects cover a wide variety of interesting topics. Many of these resources involve real world data, and they may be downloaded as Maple worksheets. They have been class tested and are an excellent source of out-of-class assignments. See Section E (page 16) in this manual for a synopsis of each Case Study and Application Project.

D.2 Linear Algebra Module Project: `http://www.aw.com/lamp/`

The Linear Algebra Modules Project (LAMP) [3] contains additional projects that are appropriate for use with *Linear Algebra and Its Applications*. The LAMP materials are organized into clusters of individual modules to facilitate its adaptation to many different curricula, types of institutions, types of students, and styles of teaching and learning. The URL for the LAMP homepage is `http://www.aw.com/lamp/`.

4 PURPOSES FOR COMPUTER EXERCISES

A POSSIBLE GOALS

As you consider your students' interests and begin to appreciate the potential of computer exercises, decide what goals are most appropriate for your class. Here are some possibilities:

- To teach applications
- To reinforce understanding of concepts
- To develop some computational wisdom
- To think and solve
- To explore and conjecture
- To learn something new
- To write programs to solve problems, learn algorithms, etc.
- To reduce tedious hand calculation
- To practice routine calculations

B YOUR GOALS

Your goals may or may not intersect with the list above. Technology can open many new doors simultaneously; it is not a bad idea to sit down and identify the one or two that are most important to you. Do not shoot for too many at once. I believe it is much better to accomplish a few goals than to only partially achieve many goals.

Many of your students will have difficulty sorting through the abstract concepts that linear algebra presents. It helps if they can connect new ideas with familiar concepts and skills with things they are able to see. Technology allows students to form concrete examples of the abstraction as often as they need to do so. Encourage them to practice with it.

Computational wisdom refers to the fact that your students are likely to be doing complex calculations when they enter the workforce. They need experience with real-world data and this may be one of the few classes able to introduce them to some of the potential pitfalls. There is no reason to form each exercise to demonstrate possible trouble, but an honest approach to the situation will be appreciated. You might wish to emphasize things such as:

- The matrix algorithms that work well on computers are more sophisticated than those presented in basic linear algebra texts. Professionally written software is almost always used in the workplace.
- Some problems are inherently difficult to solve accurately even with the best algorithms. So, one should never ignore warnings about poor conditioning.
- Professionally written software gives good answers to most problems, but there is almost always some error.

C OTHER POSSIBILITIES

David Lay and I both believe that both applications and theory are very important. He covers more of the text in lectures and makes fewer explicit computer assignments than I do. You will have to decide how much material you would like the students to learn from their computer work. It is all a matter of taste and style; you are encouraged to develop a course that is appropriate for you and your students. (See also [1].)

As you make changes, talk with faculty from other departments whose students take — or should take — linear algebra. Let them know what you are trying to do and ask for ideas. Colleagues from physics, engineering, biology, meteorology, and many other disciplines, have a special interest in their students' knowledge of linear algebra and often have good suggestions. We all would like our math departments to teach linear algebra so effectively that other departments will want their students to take it from us.

D SAMPLE EXAM QUESTIONS

One way to emphasize the importance of the computational work done in the projects is to include questions on the exam that are based on the results obtained in the laboratory. If you permit students to complete the projects in groups, such questions are also effective for determining which members of the group actually participated in the project. Here are some sample questions seen on linear algebra tests where computers were used during instruction — but not during the test.

1. The matrix $\mathrm{R} = \begin{bmatrix} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the reduced echelon form for $\mathrm{A} = \begin{bmatrix} 1 & 2 & -3 & -2 \\ -1 & -2 & 0 & 8 \\ 2 & 4 & -5 & -6 \end{bmatrix}$.
   Write a general solution to $\mathrm{A}\mathbf{x} = \mathbf{0}$. Consider the matrix transformation $\mathbf{x} \to \mathrm{A}\mathbf{x}$; is it 1-1? onto? Find a basis for the column space of A and a basis for the null space of A.

2. Let $\mathrm{A} = \begin{bmatrix} 1 & -3 & -2 \\ 2 & -1 & 3 \\ -1 & 4 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ -22 \\ 32 \end{bmatrix}$.

   (a) Does the system $\mathrm{A}\mathbf{x} = \mathbf{b}$ have a solution? Is the solution unique? Explain your answers.

   (b) Is $\mathbf{b}$ in the span of the columns of A?

   (c) Do the columns of A span R^3? Explain your answer.

3. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and the matrix $\mathrm{A} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ ]$.

   (a) Does the homogeneous system $\mathrm{A}\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Why or why not?

   (b) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly dependent or linearly independent? Explain your answer.

4. Let A be a 2×3 matrix.

   (a) If the system $\mathrm{A}\mathbf{x} = \mathbf{b}$ is consistent, is the solution unique? Why or why not?

   (b) How many pivot columns must A have to guarantee that the system $\mathrm{A}\mathbf{x} = \mathbf{b}$ is consistent for any choice of $\mathbf{b}$ in R^2? Explain your answer.

(c) Suppose A has 1 pivot column. Is the system A**x** = **b** consistent for any choice of **b** in R^2? Explain your answer. In particular, if your answer is "No", explain how you know there is a **b** for which A**x** = **b** is inconsistent.

5. There is a real 3×3 matrix A for which the general solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ is $\mathbf{x} = c\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. What is the general solution to $A\mathbf{x} = \mathbf{0}$?

6. If the columns of an $n \times n$ matrix A are linearly independent, does A^{-1} exist? If this inverse exists, are its columns linearly independent? Explain.

7. Suppose A is an invertible matrix. Is AA^T invertible? Why or why not?

8. Assume $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is linearly independent in R^5. Is $R^5 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$? Why or why not?

9. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a spanning set for a vector space V. Suppose also that the dimension of V is 3. Can S be a basis for V? Must S be contained in a basis for V? Will S contain a basis for V? Explain each answer.

10. Is it possible that all solutions of a homogeneous system of six linear equations in eight unknowns are linear combinations of two fixed non-zero solutions? Explain.

11. A certain population of owls feeds almost exclusively on wood rats. The following matrix describes the evolution of the owl and rat populations from one year to the next:

$$\begin{array}{cc} \text{O} & \text{R} \end{array}$$
$$\begin{bmatrix} 0.5 & 0.05 \\ 0.5 & 0.95 \end{bmatrix} \begin{array}{c} \text{O} \\ \text{R} \end{array}$$

Do not calculate. Answer each of the following in words. Include any equations that are needed; explain what each symbol means.

(a) Suppose you want to find the number of owls and rats five years from now. What would you calculate, why would it work, and how would you interpret the results to provide the populations five years in the future?

(b) How could you use eigenvalues and eigenvectors to describe the long-term behavior of these two populations? Include any equations you need to discuss, and say what all your symbols mean.

12. If the stock market went up today, historical data shows that tomorrow it has a 65% chance of going up, a 10% chance of staying unchanged, and a 25% chance of going down. If the market is unchanged today, tomorrow it has a 20% chance of being unchanged, a 40% chance of going up, and a 40% chance of going down. If the market goes down today, tomorrow it has a 25% chance of going up, a 10% chance of being unchanged, and a 65% chance of going down.

(a) If the market went up today, what is the probability that the market is unchanged the day after tomorrow?

(b) Over the course of many days, what percentage of the time do you expect the market to go up? Explain your answer.

13. We stored a certain 3×3 matrix A in Maple, did several row operations to A, and ended up with

$$\begin{bmatrix} 1.000 & 1.000 & -1.000 \\ 0 & 0.4141 & 1.000 \\ 0 & 0 & -0.000 \end{bmatrix}$$

Based on the information given, do you think A is invertible? Explain your answer. If you are not sure, describe additional information that you need and how you would obtain that information.

14. Let A be a square matrix. Give an example to show that A and 2A do not usually have the same eigenvalues. Are the eigenvectors of A and 2A always the same? Explain.

15. Let A be the $n \times n$ matrix in which every entry is 1. Justify your answers to the following statements and questions.

    (a) There are two distinct eigenvalues of A. What are they? Why?

    (b) What is dim(Nul(A))? Why?

    (c) What is the characteristic polynomial of A? Why?

16. Suppose A is a 7×7 matrix with four distinct eigenvalues.

    (a) Is A diagonalizable? Why or why not? (Do you have enough information to answer this question?)

    (b) Suppose the information above is appended with the fact that one of the eigenvalues has a three-dimensional eigenspace. Is A diagonalizable? Why or why not? (Do you have enough information to answer this question?)

17. Let $A = \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$. Verify that the columns of V are eigenvectors for A. Show your work. Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$.

18. Consider $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 3 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 7 \\ -3 \\ 5 \\ 3 \end{bmatrix}$.

    (a) Show that $A\mathbf{x} = \mathbf{b}$ has no solution.

    (b) Find a least squares solution to $A\mathbf{x} = \mathbf{b}$.

19. Suppose that two partitioned matrices satisfy $\begin{bmatrix} I & X \\ 0 & Y \end{bmatrix} \begin{bmatrix} I & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$, where X, Y, B, D, and the identity matrix (I) are all $n \times n$. Find formulas for X and Y in terms of B and D.

    Explicitly state any additional assumptions that are needed on the matrices B and D.

20. The diary of a Maple session with most Maple commands omitted is reprinted below. Fill in each blank with the appropriate Maple command that produces the listed output.

```
> restart;
> with( linalg ):
> with( laylinalg ):
>
> M1 := matrix( [[ 0,  3, -6,  6,  4, -5 ],
>               [ 3, -7,  8, -5,  8,  9 ],
>               [ 3, -9, 12, -9,  6, 15 ]] );

                  [0     3    -6     6    4    -5]
                  [                              ]
            M1 := [3    -7     8    -5    8     9]
                  [                              ]
                  [3    -9    12    -9    6    15]

> M2 := __________________________________ ;

                  [3    -9    12    -9    6    15]
                  [                              ]
            M2 := [3    -7     8    -5    8     9]
                  [                              ]
                  [0     3    -6     6    4    -5]

> M3 := __________________________________ ;

                  [3    -9    12    -9    6    15]
                  [                              ]
            M3 := [0     2    -4     4    2    -6]
                  [                              ]
                  [0     3    -6     6    4    -5]

> M4 := __________________________________ ;

                  [3    -9    12    -9    6    15]
                  [                              ]
            M4 := [0     1    -2     2    1    -3]
                  [                              ]
                  [0     0     0     0    1     4]

> M5 := __________________________________ ;

                  [3    -9    12    -9    6    15]
                  [                              ]
            M5 := [0     1    -2     2    1    -3]
                  [                              ]
                  [0     0     0     0    1     4]
```

5 COMPUTER PROJECTS

A GENERAL INFORMATION

The projects in this manual are based on material in *Linear Algebra and Its Applications* and on contributions from various workshops. They enrich and expand the text material and are independent of each other. They can be used as assignments or as extra credit. You may copy and use them as written, or adapt them to suit your own situation or interests. Time depends mostly on how much independent reading students must do. So, their work will go faster if you lecture a little on the material before they begin a project, but any of these can be "read and do" assignments.

Think about how strict you want deadlines to be and allow time for lab availability. Remember, the equipment does not always behave, and misjudging difficulty is par for the course. In addition, some students may have access to the computer lab only on certain days, etc.

B PARTNERS

Consider having your students find partners for the computer work. After a couple of days, you may have to step in and help pair those who have not found partners yet. The computer projects generally go much more smoothly when students work together. Have the partners turn in one project with both names. This will reduce the workload on you, as well as give the students experience working with others. Tell them not to hesitate to inform you if their partner is not keeping their end of the bargain. Of course, if a student has a strong preference for working alone, you might want to allow that.

C GRADING

You will have to decide on weighting for computer projects, but a good guideline is about double that of a homework assignment that requires some writing. Consider more weight for longer projects. If you are assigning a computer project once a week, then this will also need to be considered when determining final grades.

In the *Instructor's HP-48G Manual*, Tom Polaski provides a copy of the grading sheet he uses for his classes. A copy of this sheet, translated for use with Maple, can be found in Figure 1.

Instructions for Students

- You are expected to use Maple to complete this project. The general objective of this project is to explore topics discussed in class, to deepen your understanding of the computations involved in linear algebra, and to gain an appreciation for the diverse applications of linear algebra. In some cases you will be expected to read some background materials to learn about topics not directly discussed in class. You will also develop your mathematical writing skills.

- The due date for the project will be announced. Points will be deducted for late work. Work turned in one day late will receive 0 points in the "on time" row of the grading sheet, work turned in two days late will receive -1 points in the "on time" row. No work will be accepted more than two days after the due date.

- You may work with other students and in groups to develop the mathematics, but your final work must be individual and not copied. Specific questions concerning the projects are best answered by the instructor.

- A copy of the following grading sheet with the score clearly indicated will be returned to you when your project has been graded. The scores indicate how the goals are translated into point values.

Grading Sheet for Maple Projects

NAME: ______________________

Points Possible	Grading Criterion	Points Awarded
11	Correct mathematics	
3	Appropriate mathematical notation	
2	Prose is clearly written	
1	Prose is included appropriately	
1	Spelling and grammar	
1	Neat and organized presentation	
1	Report turned in on time (see above)	
20	Total Points	

<u>Additional Comments</u>

Figure 1: Sample cover page for student projects includes some general comments and guidelines and a grading sheet that shows the point distribution. Note that these instructions allow for group work but require individual final reports.

D OVERVIEW OF MAPLE PROJECTS

Project 1 — Introduction to Maple

This project is intended for students with no prior Maple experience. The purpose of this project is to help students become comfortable with the Maple worksheet interface, online help, and the `laylinalg` package. The mathematical content is minimal.

Project 2 — Introduction to Linear Algebra with Maple

This project introduces the `linalg` and `laylinalg` packages. Some of the questions have the students evaluate a given expression by hand and with Maple. Students also have an opportunity to explore properties of matrix algebra. The prerequisite is Section 1.5.

Project 3 — Exchange Economy and Homogeneous Systems

This project is self-contained, but is related to Example 1 in Section 1.6. Students can read this background information on their own. Note the discussion of floating-point and exact arithmetic. Students need to be aware of the potential problems that can arise with floating-point arithmetic. This issue deserves some attention in class – but not too much.

Project 4 — Rank and Linear Independence

A computational definition of the rank of a matrix is provided at the beginning of this project. Otherwise, this project requires only Section 1.7. Students practice applying this definition and its relation to linear independence.

Project 5 — Population Migration

Students like linear dynamical systems and plotting. This project is essentially Exercise 11 in Section 1.10 with some graphical additions. Use Example 3 in Section 1.10 to introduce the project. If you want to go further, ask the students what they expect to happen if this pattern of migration continues indefinitely. Does everyone move to the suburbs? (Why?)

Project 6 — Initial Analysis of the Spotted Owl

This is very similar to the Population Migration project. Students will need to read the Opening Example for Chapter 5, but otherwise can be assigned along with Section 1.10. Be sure to mention that the full analysis of this problem will be completed after eigenvalues have been introduced. (See also Maple Project: *Eigenvalue Analysis of the Spotted Owl* and Case Study 5: *Dynamical Systems and Spotted Owls*.)

Project 7 — The Adjacency Matrix of a Graph

This project is a detailed examination of matrix multiplication (Section 2.1) and its applicability in graph theory. While the mathematics is simple, the application is interesting. Towards the end of the project, students are asked to create a definition — a new, but useful, type of question for most students. A first attempt at a definition is: "A dangerous worker is any worker with the highest level one contact." A difficulty with this definition is that contact level relates two workers. An improved definition is "A dangerous worker is any worker with the highest total level one contact with all other workers." With this definition, workers 1, 4, and 6 are most dangerous (with total level one contact equal to 7). Some students may notice that worker 6 is the only one to be in level two contact with all other workers. Mentioning this may help other students pay more attention to details. (See also the Application Project for Section 2.1: *Adjacency Matrices.*)

Project 8 — An Economy with an Open Sector

This project begins by solving Exercise 13 in Section 2.6. It then asks students to look at the structure of the problem in more detail. Question 3 is designed to test students' understanding of Theorem 11 (Section 2.6). This connection can be provided as a hint, but resist the urge to give away too much information. (See also Case Study 1: *Linear Models in Economics.*)

Project 9 — Curve Fitting

This project describes how linear algebra can be used to find an interpolating polynomial to data. Vandermonde matrices are mentioned, but only to guarantee that the system is invertible. (Vandermonde matrices appear in Exercise 11 in the Supplementary Exercises for Chapter 2). Students like this project because they get to generate their own data. The question about relative maxima and minima should not be ignored — it is a good calculus review. As far as this project is concerned, the `interp` command should be used only to check the answers obtained earlier in the project. (See also the Application Projects for Section 1.2: *Interpolating Polynomials* and *Splines* and for Section 4.1: *Hill Substitution Ciphers.*)

Project 10 — Temperature Distributions

Students use a characterization of the steady-state temperature (average of the temperatures at the four neighboring nodes) to see how steady-state temperatures can be computed using linear algebra. Other temperature distribution problems in the text can be found in Exercises 33 and 34 for Section 1.1 and Exercise 31 for Section 2.5. (See also the Application Project for Section 2.5: *Equilibrium Temperature Distributions.*)

Project 11 — Manipulating Matrices with Maple

The project introduces the family of square matrices $M_n = [m_{ij}]$, where $m_{ij} = \max(i, j)$. Based on selected (small) examples, students are asked to formulate conjectures for the formulas for the determinant (Section 3.2) and inverse (Section 2.2) of M_n for all positive integers n. This is much simpler than it sounds; students have a lot of fun with this one.

Project 12 — Markov Chains and Long-Range Predictions

Markov chains are introduced in Section 4.9, but this project can be assigned after completing Section 2.1. Only terminology is needed from Section 4.9. With appropriate guidance, this project can be a good motivation for eigenvalues. Be sure to allow ample time to complete this project.

Project 13 — Real and Complex Eigenvalues

This project can be a substitute for a formal discussion of Section 5.5. In particular, this project suggests the fact that complex eigenvalues and eigenvalues appear in conjugate pairs. This project can be done very quickly; only Question 2 requires much time.

Project 14 — Eigenvalue Analysis of the Spotted Owl

This is the continuation of the Maple Project: *Initial Analysis of the Spotted Owl*. The first three questions are fairly self-explanatory. Eigenvalues and diagonalization are emphasized. The plots help bring everything together. The extra credit is much more involved. The symbolic capabilities of Maple can be used, but most of the algebra is simple enough to do by hand. The logical connection between the various facts is what students find difficult. Explain the hints as needed. Be sure to allow sufficient time for everyone to at least attempt the extra credit. (The extra credit problem was created by Andre Weideman and is used with his permission.)

Project 15 — The Cayley-Hamilton Theorem

The Cayley-Hamilton Theorem is stated. The students are asked to verify the theorem for randomly-selected matrices of various sizes. The reference for this project is Exercises 5–7 in the Supplementary Exercises for Chapter 5). This project can be assigned anytime after Section 5.2 has been discussed.

Project 16 — Pseudo-Inverse of a Matrix

This project can be used as an application of the Invertible Matrix Theorem and as an introduction to the Moore-Penrose inverse (in Section 7.4). The pseudo-inverse is shown to be theoretically useful in least-squares problems (as presented in Section 6.5). This is demonstrated numerically and graphically for a specific example.

E OVERVIEW OF CASE STUDIES AND APPLICATION PROJECTS

The Case Studies and Application Projects are available to students and instructors on the WWW at `http://www.laylinalgebra.com/`. Each Case Study and each Application Project is available in a separate Maple worksheet. Each worksheet contains all necessary data. Solutions for the exercises are available for instructors.

E.1 Case Studies

Chapter 1: Linear Models in Economics

This case study examines Leontief's "exchange model" and shows systems of linear equations can model an economy. Real economic data is used.

Chapter 2: Computer Graphics in Automotive Design

This case study explores the effective two-dimensional rendering of a three-dimensional image. Perspective projections, rotations, and zooming are discussed and applied to wireframe data derived from a 1983 Toyota Corolla.

Chapter 3: Determinants in Analytic Geometry

This case study examines how determinants may be used to find the equations for lines, circles, conic sections, planes, spheres, and quadric surfaces.

Chapter 4: Space Flight and Control Systems

This case study studies a mathematical model for engineering control systems. The notion of rank is used to determine whether a system is controllable, and a system of linear equations is solved to determine which inputs into the system yield a desired output.

Chapter 5: Dynamical Systems and Spotted Owls

This case study examines how eigenvalues and eigenvectors can be used to study the change in a population over time. Real data from populations of spotted owls, blue whales, and plants (speckled alders) is studied, and the notion of a sustainable harvest is introduced.

Chapter 6: Least-Squares Solutions

This case study uses the method of least squares to fit linear, polynomial, and sinusoidal curves to real data. This data includes performance in the Olympic men's 400-meter run, climatic data from Charlotte, NC, and tidal data from the Cape Hatteras pier.

Chapter 7: The Singular Value Decomposition and Image Processing

This case study examines how the singular value decomposition of a matrix may be used to reduce the amount of data needed to store a reasonable image of a graphical object. Two types of images are considered: three-dimensional surfaces and black-and-white two-dimensional pictures.

E.2 Application Projects

Section 1.2: Interpolating Polynomials

This set of exercises shows how a system of linear equations may be used to fit a polynomial through a set of data points. Polynomial curves are fit to actual acceleration data obtained from *Car and Driver* magazine.

Section 1.2 Splines

This set of exercises shows how a system of linear equations may be used to fit a piecewise-polynomial curve through a set of data points. Cubic splines are fit to actual acceleration data obtained from *Car and Driver* magazine.

Section 1.10 Diet Problems

This set of exercises provides examples of vector equations that result from balancing nutrients in a diet. Real data from the USDA website are used.

Section 1.10 Traffic Flow Problems

This set of exercises shows how a system of linear equations may be used to model the flow of traffic through a network. Real data from the Seattle Transportation Management Division and the Charlotte–Mecklenburg (NC) Utilities Department are used in this exploration.

Section 1.10 Loop Currents

This set of exercises provides further and larger examples involving loop currents, and reinforces the text's development of this topic.

Section 2.1 Adjacency Matrices

This set of exercises studies the adjacency matrix of a graph. The real route maps of several airlines help to motivate graphic-theoretic questions which may be answered with adjacency matrices.

Section 2.1 Dominance Matrices

This set of exercises applies matrices to questions concerning competition between individuals and groups. The problem of ordering teams within a football conference is discussed, and real data from several collegiate and professional football conferences are used.

Section 2.1 Other Matrix Products

This set of exercises introduces and explores the properties of two matrix products: the Jordan product and the commutator product.

Section 2.3 Condition Numbers

This set of exercises motivates the definition of the condition number of a matrix, and explores how its value affects the accuracy of solutions to a system of linear equations.

Section 2.5 The LU and QR Factorizations

This set of exercises shows how to use an LU factorization to perform a QR factorization. (The QR factorization is introduced in Exercise 24 of this Section 2.5.)

Section 2.5 Equilibrium Temperature Distributions

This set of exercises discusses the problem of determining the equilibrium temperature of a thin plate. An appropriate system of equations is derived, and is solved both by finding a matrix inverse and by an LU factorization.

Section 2.6 The Leontief Input-Output Model

This set of exercises provides three real data examples of the Leontief Input-Output Model discussed in the text. American economic data from the 1940's and the 1990's is studied.

Section 3.3 The Jacobian and Change of Variables

The Jacobian is derived and applied to the change of variables in double and triple integrals.
This set of exercises is intended for students who have completed a course in multivariate calculus.

Section 4.1 Hill Substitution Ciphers

This set of exercises studies how matrices may be used to encode and decode messages. Matrix arithmetic modulo 26 is used.

Section 4.6 Error-Detecting and Error-Correcting Codes

This set of exercises studies the construction of methods for detecting and correcting errors made in the transmission of encoded messages. The United States Postal Service bar code is studied as an error-detecting code, and the error-correcting Hamming (7,4) code is also studied.

Section 5.3 The Fibonacci Sequence and Generalizations

This set of exercises introduces the Fibonacci sequence and Lucas sequences. Eigenvalues, eigenvectors and diagonalization are used to derive general formulas for an arbitrary element in these sequences.

Section 5.4 Integration by Parts

This set of exercises shows how the matrix of a linear transformation relative to a cleverly chosen basis may be used to find antiderivatives usually found using integration by parts.

Section 6.4 The QR Method for Finding Eigenvalues

This set of exercises shows how the QR factorization of a matrix may be used to calculate its eigenvalues. Two methods for performing this action are considered and compared.

Section 6.4 Finding Roots of Polynomials with Eigenvalues

This set of exercises describes how the real roots of a polynomial can be found by finding the eigenvalues of its companion matrix. The QR method is then employed to find these eigenvalues.

Section 7.2 Conic Sections and Quadric Surfaces

This set of exercises shows how quadratic forms and the Principal Axes Theorem may be used to classify conic sections and quadric surfaces.

Section 7.2 Extrema for Functions of Several Variables

Quadratic forms are used to investigate relative maximum and minimum values of functions of several variables. Results are derived in terms of the eigenvalues of the Hessian matrix.
This set of exercises is intended for students who have completed a course in multivariate calculus.

6 REFERENCES

References

[1] D. Carlson, C. R. Johnson, D .C. Lay, A. D. Porter, "The Linear Algebra Curriculum Study Group Recommendations for the First Course in Liner Algebra", *College Math. Journal* (24), 1993, pp. 41–46.

[2] D. Carlson, C. R. Johnson, D. C. Lay, A. D. Porter, A. Watkins, and W. Watkins, *Resources for Teaching Linear Algebra*, MAA Notes #42 Mathematical Association of America, Washington, D.C., 1997.

[3] E. A. Herman, M. Pepe, J. R. King, and R. T. Moore, *Linear Algebra: Modules for Interactive Learning with Maple 6*, Addison Wesley Longman, 2000. [URL: `http://www.awl.com/lamp/`]

[4] S. Leon, E. Herman, and R. Faulkenberry, eds., *ATLAST Computer Exercises for Linear Algebra*, Prentice–Hall, 1997. [URL: `http://www.umassd.edu/SpecialPrograms/Atlast/ATLASTBook.html`]

7 MAPLE PROJECTS

The following sixteen (16) projects are provided for you to use as is or to modify to fit the specific needs and objectives for your course. Each project lists prerequisite sections of the text and the Maple commands needed to complete the project. A summary of each project can be found in Section 5.D of this document.

Each project has been designed to provide students with additional practice applying the concepts introduced in the text while exposing them to new applications or theoretical results. In general, students should be encouraged to provide responses to the questions in the space provided. However, I do permit students to submit *well-labeled* Maple worksheets.

Purpose	To begin to learn about Maple as a mathematical tool for linear algebra and Maple's online help system.
Prerequisites	Elementary Algebra
Maple commands used	`help` and `?`

Notes:

- It often helps to do computer work with a partner. Help each other locate and fix typographical errors, discuss Maple's response, and ask and answer each other questions. If further experiments are needed before answering the questions, feel free to do so.

1. (a) Start Maple, then for each of the following expressions, enter the expression in a separate input region, execute it (by pressing the Enter key), and record Maple's response to your command. If the response is too long, you may summarize Maple's response. Be sure your description contains enough information for you to understand the result when you look at these answers later in the course.

`3 * 4;`

`3 * x;`

`a * b ^ 2;`

`(a * b) ^ 2;`

(b) Explain the different results for `a * b ^ 2;` and `(a * b) ^2;`.

(c) What is wrong with the following command: `3 x^2 + 2 x - 5;`?

(d) How would you enter the polynomial $3x^2 + 2x - 5$ in Maple?

2. (a) This set of commands will introduce you to Maple's online help facility. There is a help worksheet for every Maple command (including all commands in the `linalg` and `laylinalg` packages). Each help worksheet contains a full description of the commands arguments and output and includes several examples to illustrate its usage. Read the computer responses to the following instructions, but you need not recored the output. If you want to take notes for your own purposes, use a separate sheet of paper. Do not be concerned if a lot of the information in the help worksheets does not make sense (yet).

   i. `help( linalg );`
   ii. `?matrix`
   iii. `?laylinalg`
   iv. `?replace`
   v. `?scale`
   vi. `?swap`

   Notes:
   - When you are done with a Help window, click its close box in the upper right corner.
   - The `?` "command" is one of the few Maple commands that is not terminated with a semicolon or colon.
   - Type `?help` for some additional information about Maple's online help facility.

   (b) Give a brief description of the Maple online help facility and how to access it.

   (c) Explain how the Maple help system can be accessed via the **Help** pull-down menu.

   (d) What command needs to be executed before using a command from the `laylinalg` package?

   (e) Give the command that lists all problems in Section 1.1 for which there is Maple data. List the problems in Section 1.2 for which data is provided. (If necessary, consult the information in the online help for `laylinalg`.)

Purpose	To learn about the basic linear algebra commands in the `linalg` and `laylinalg` packages.
Prerequisites	Section 1.5
Maple commands used	`evalm`, `restart`, and `with`; `matrix` and `vector` from the `linalg` package.

Prior to doing any linear algebra with Maple, you should load the `linalg` package. The `laylinalg` package contains several additional commands and data for a large number of exercises in the textbook; it should also be loaded in each Maple worksheet that you create for this course. It is also highly recommended that you begin every worksheet with the `restart;` command. This command removes all assignments and results from Maple's memory and restarts your Maple session. Thus, the first commands entered in every Maple worksheet should be:

```
restart;
with( linalg );
with( laylinalg );
```

Notes:

- To remove an assignment to a variable, say, `x`, and make `x` into a variable again, use the command: `unassign( 'x' );`.
- Be sure you use the single quote ('), not the back quote (`).

1. (a) For each of the following Maple commands, enter and execute the command, then record the results in the boxes provided. Remember that each command must end with a semicolon or colon and that pressing the **Enter** key executes all commands in the current execution group. (For more about execution groups and the worksheet interface, see `?worksheet`.)

`restart;`

`with(linalg);`

`with( laylinalg );`

`M := matrix( [ [1, 2, 3], [4, 5, 6] ] );`

```
M;
```

```
evalm(M);
```

```
N := matrix( 2, 3, [1, 2, 3, 4, 5, 6] );
```

```
evalm(N);
```

```
N[2,3] := 7;
```

```
evalm(N);
```

```
v := vector( [1, 2, 3, 1] );
```

```
v[4];
```

```
v;
```

```
evalm(v);
```

Purpose	Find the equilibrium price for an exchange model economy by solving a homogeneous system.
Prerequisites	Section 1.6
Maple commands used	`convert` and `map`; `augment`, `diag`, and `rref` from the `linalg` package; `bgauss`, `econdat`, `gauss`, and `scale` from the `laylinalg` package.

1. Let $T = \begin{bmatrix} .20 & .17 & .25 & .20 & .10 \\ .25 & .20 & .10 & .30 & 0 \\ .05 & .20 & .10 & .15 & .10 \\ .10 & .28 & .40 & .20 & 0 \\ .40 & .15 & .15 & .15 & .80 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$.

   Consider the system of linear equations $T\mathbf{x} = \mathbf{x}$.

   (a) *Without using Maple*, write out the five equations in this system.

   (b) Collect terms in your equations to get a homogeneous linear system, and write out the five new equations:

2. Let B**x** = **0** denote the homogeneous system you obtained in Question 1(b), and calculate the reduced echelon form of A = [B **0**]. Record the reduced form in the table provided at the end of this question.

   The following Maple commands load the matrix T from the `laylinalg` package, create the matrix B by subtracting the 5×5 identity matrix from T (see Section 2.1), and create the augmented matrix A:

```
econdat( );                          # load T from the laylinalg package
B := evalm( diag( 1 $ 5 ) - T );     # matrix for homogeneous system: B=I-T
A := augment( B, vector(5,0) );      # augmented matrix A = [B 0]
```

   (a) Use `bgauss`, `gauss`, and `scale` to obtain the reduced echelon form of the augmented matrix.

   Notes:

   - When you finish the forward elimination the (5,5) entry in the reduced matrix should be very small. In fact, it is so small that you will not be surprised to learn that this entry should be zero. In order to successfully complete this project you need to either remember that this number is really zero or physically put a zero in this location in the matrix. Recall that to change the $(5, 5)$ entry of a matrix M, use the command:

```
M[5,5] := 0;   # replace one entry of a matrix
```

   (b) The problems associated with floating-point arithmetic can be avoided by converting all floating-point numbers in the original matrix to fractions (rational numbers). The simplest way to do this is with the command:

```
Arat := map( convert, A, rational );   # convert floating-point to fractions
```

A: original	A: ref (floating point)	A: ref (rational)

3. Read about Leontief Economic Models in Section 1.6 of the text. Now consider an exchange model economy which has five sectors: Chemicals, Metals, Fuels, Power, and Agriculture; and assume the matrix T in Question 1 above gives an exchange table for this economy as follows:

$$\mathrm{T} \;=\; \begin{array}{c} \begin{matrix} \mathrm{C} & \mathrm{M} & \mathrm{F} & \mathrm{P} & \mathrm{A} \end{matrix} \\ \begin{bmatrix} .20 & .17 & .25 & .20 & .10 \\ .25 & .20 & .10 & .30 & 0 \\ .05 & .20 & .10 & .15 & .10 \\ .10 & .28 & .40 & .20 & 0 \\ .40 & .15 & .15 & .15 & .80 \end{bmatrix} \end{array} \begin{matrix} \mathrm{C} \\ \mathrm{M} \\ \mathrm{F} \\ \mathrm{P} \\ \mathrm{A} \end{matrix}$$

(a) Verify that each column of T sums to one. This indicates that all output of each sector is distributed among the five sectors, as should be the case in an exchange economy. The economy is in equilibrium when the system of equations $\mathrm{T}\mathbf{x} = \mathbf{x}$ is satisfied. As you saw in Question 1, this is equivalent to saying that the system $\mathrm{B}\mathbf{x} = \mathbf{0}$, with $\mathrm{B} = \mathrm{I} - \mathrm{T}$, is satisfied.

(b) Let x_C represent the value of the output of Chemicals, x_M the value of the output of Metals, etc. Using the reduced echelon form of $[\mathrm{B}\ \mathbf{0}]$ from Question 2, write the general solution for $\mathrm{T}\mathbf{x} = \mathbf{x}$:

$$\begin{bmatrix} x_C \\ x_M \\ x_F \\ x_P \\ x_A \end{bmatrix} =$$

(c) Suppose that the economy described above is in equilibrium and $x_A = 100$ million dollars. Calculate the values of the outputs of the other sectors and record this particular solution for the system $\mathrm{T}\mathbf{x} = \mathbf{x}$:

$$\begin{bmatrix} x_C \\ x_M \\ x_F \\ x_P \\ x_A \end{bmatrix} =$$

(d) Consider the matrices T and B created above. As previously observed, each column of T sums to one. Consider how you obtained B from T and explain why each column of B must sum to zero.

Extra Credit Let B be any matrix of any shape, with the property that each column of B sums to zero. Explain why the reduced echelon form of B must have a row of zeros.
Use appropriate linear algebra terminology whenever possible. Do not attempt to cram your explanation on the bottom of this page; use the back of this page or attach a separate page.

No content or questions on this page. Use this space for additional work if needed.

Purpose	To define rank and learn its connection with linear independence.
Prerequisites	Section 1.7
Maple commands used	`rank`, `rref`, and `transpose` from the `linalg` package; `indat` and `randomint` from the `laylinalg` package

Definition: The *rank* of a matrix A is defined to be the number of pivot columns in A.

One way to find rank is to calculate the reduced echelon form and then count the number of pivot columns. A quicker way is to use Maple's `rank` function.

Example: Suppose the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 9 & 12 & 15 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ is defined in your Maple worksheet. When the command `rref( A );` is executed, the result is $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There are two pivot columns in the reduced matrix so the rank of A is 2. To verify this conclusion, check the result when the command `rank( A );` is executed.

Notes:

- Recall that `D` is a reserved name in Maple. To avoid problems, the matrix D is stored under the name `DD`.

1. Use the two methods described above to find the rank of each of the following four matrices. For each matrix, record the reduced echelon form, circle each pivot column, and record the rank in the table on the next page. To get the matrices, use the `laylinalg` command `indat( );`.

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 1 & 2 & 1 & 1 \\ 3 & 6 & -5 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 5 & 7 & 9 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 2 & 0 & 4 & 7 & 6 \\ 1 & 3 & -1 & 10 & 13 & 21 \\ 1 & 4 & -2 & 20 & 23 & 56 \\ 1 & 5 & -3 & 35 & 38 & 126 \\ 1 & 6 & -4 & 56 & 59 & 252 \end{bmatrix}$$

Matrix	B	C	D, i.e., `DD`
Reduced echelon form, with pivot columns circled			
Rank			

Matrix	E
Reduced echelon form, with pivot columns circled	
Rank	

2. Read the discussion of linear independence in Section 1.7. Write the definition of a linearly independent set of vectors.

3. Let A = $[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_k]$ be a matrix whose columns are $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$. Explain why the following are logically equivalent:

   - The set of vectors $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ is linearly independent.
   - The rank of A is k.

4. Explain why the set of columns of A could <u>not</u> be linearly independent if A has more columns than rows.

5. Examine the matrices B, C, D, and E in Question 1. For which of these matrices is the set of its columns a linearly independent set?
   This question should be answered without the use of Maple.

6. Verify each of the following conclusions by hand or with Maple. Explain your method in the space provided.

(a) Let $\mathbf{v}_1 = (2,3,5,1)$, $\mathbf{v}_2 = (1,1,-2,9)$, and $\mathbf{v}_3 = (3,4,0,0)$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent since the rank of $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 5 & -2 & 0 \\ 1 & 9 & 0 \end{bmatrix}$ is 3.

(b) The set of vectors $\left\{ \begin{bmatrix} 1 \\ 4 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 12 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 15 \\ 1 \end{bmatrix} \right\}$ is not a linearly independent set. The matrix with these columns has rank 2.

(c) The set $\left\{ \begin{bmatrix} 1 \\ 4 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 9 \\ 11 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 12 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 15 \\ 1 \end{bmatrix} \right\}$ is linearly independent because the rank of $\begin{bmatrix} 1 & 4 & 6 & 1 \\ 2 & 5 & 9 & 11 \\ 4 & 6 & 12 & 1 \\ 4 & 6 & 15 & 1 \end{bmatrix}$ is 4.

Purpose	To study the population movement described in Exercise 11, Section 1.10 in more detail.
Prerequisites	Section 1.10
Maple commands used	`for .. do .. end do`, `plot`, and `zip`; `augment` from the `linalg` package; `c1s10` from the `laylinalg` package; `display` from the `plots` package.

In preparation for this project, read Exercise 11 from Section 1.10. This describes a simple migration model which assumes people just move around and the total population of the US remains constant. If M is the migration matrix and **x** is a vector whose components are the number of people in each area this year, then M**x** is the number in each area next year.

1. To obtain the data for this exercise, load the `laylinalg` package into your current Maple session with the command `with( laylinalg );` then load the specific matrices for Exercise 11 in Section 1.9 with the command `c1s10(11);`.

   (a) Record the values of `M` and `x0`.

   (b) Describe the calculations needed to produce the entries in M from the information in this exercise.

2. Study the migration model described above. Use the following Maple commands to calculate the population in California and in the rest of the US for the years 1990 - 2003 and store that data as the columns of a matrix P.

```
x := x0/10.^6 :                  # rescale population data to millions
P := evalm( x ):                 # put 1990 data in first column of P
for i from 1991 to 2003 do       # for each year ...
  x := evalm( M &* x );          # – update populations
  P := augment( P, x );          # – add new column to P
end do:
evalm( P );                      # let's see the final result
```

Record the data from P in the table below. Round each number to 5 digits.

Population (in millions) assuming no external migration

Year	1990	1991	1992	1993	1994	1995	1996
California							
Rest of US							

Year	1997	1998	1999	2000	2001	2002	2003
California							
Rest of US							

Use the following Maple commands to plot the population in California, and the population in the rest of the US, versus years, on the same graph. Include a printed copy of your graph with this project.

```
with(plots):                                       # load plots package
yr := vector( [ $ 1990 ..  2003)] );               # vector [1990 1991 ... 2003]
ptCA := zip( (x, y) -> [x, y], yr, row(P, 1) );    # pts for California pop
ptUS := zip( (x, y) -> [x, y], yr, row(P, 2) );    # pts for rest-of-US pop
plotCA := plot(ptCA, color=red):                   # plot of California pop
plotUS := plot(ptUS, color=blue):                  # plot of rest-of-US pop
display( [ plotCA, plotUS ],                       # display 2 plots in 1 graph
  title="CA and US pop (in millions)" );
```

3. Instead of assuming the total population is constant, suppose that the population in California and the rest of the US is actually increasing each year because of immigration, say 0.1 million people into California and 2 million into the rest of the US each year. Then if data is expressed in millions and $\mathbf{x}$ is the population vector this year, $M\mathbf{x} + \begin{bmatrix} 0.1 \\ 2.0 \end{bmatrix}$ will yield be the population vector for the next year.

   Use the next set of Maple commands to calculate the new population predictions for 1990–2003.

```
d := vector( 2, [ 0.1, 2.0 ] ):   # annual immigration (in millions)
x := x0/10.^6 :                   # rescale population data to millions
P := evalm( x ):                  # put 1990 data in first column of P
for i from 1991 to 2003 do        # for each year ...
  x := evalm( M &* x + d);        # – update populations
  P := augment( P, x );           # – add new column to P
end do:
evalm( P );                       # let's see the final result
```

   Record the data from P in the table below. Round each number to 5 digits.

Population (in millions) assuming external migration

Year	1990	1991	1992	1993	1994	1995	1996
California							
Rest of US							

Year	1997	1998	1999	2000	2001	2002	2003
California							
Rest of US							

   Create a plot showing the annual populations of California and the rest of the US for 1990 – 2000. Include a printed copy of this graph with this project. (Since the plot is likely to be only a small part of your Maple worksheet, you may find it more convenient to select and copy this graphics region to a new worksheet and then print the new, smaller, worksheet.)

No content or questions on this page. Use this space for additional work if needed.

Purpose	To study the owl population with several survival rates for juveniles.
Prerequisites	Section 1.10 (and the Opening Example for Chapter 5)
Maple commands used	`add`, `for .. do .. end do`, `plot`, and `zip`; `augment`, `col`, and `row` from the `linalg` package; `owldat` from the `laylinalg` package; `display` from the `plots` package.

The background information for this project can be found in the Opening Example to Chapter 5. The fundamental idea is that spotted owls have three distinct life stages: first year (juvenile), second year (subadult), and third year and beyond (adult) where j_k, s_k and a_k denote the number of owls in each stage in year k. Let $\mathbf{x}_k = \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$ and define the transition matrix $\mathrm{A} = \begin{bmatrix} 0 & 0 & 0.33 \\ t & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$. where the parameter t is the juvenile survival rate. Then $\mathbf{x}_{k+1} = \mathrm{A}\mathbf{x}_k$. The text reports that the population will eventually die out if $= 0.18$ but not if $t = 0.30$. The purpose of this project is to investigate this claim.

1. Let $t = 0.18$ and suppose there are 100 owls in each life stage in 1997. Load the `laylinalg` package, then use the command `owldat();` to load the transition matrix, A, (with $t = 0.18$) and the initial population vector, `x0`, in 1997.

   (a) The population in each stage and the total population in 1998 can be found with the following Maple commands:

```
x := x0;                             # initial population in 1997
x := evalm( A &* x );                # next year's populations
total := add( x[i], i=1..3 );        # total population
```

   Record the 1998 populations, rounded to the nearest integer, in the table at the top of the next page. Repeat the last two lines two more times to obtain the 1999 and 2000 populations. Record all results in the table below.

Spotted owl population when the juvenile survival rate is $t = 0.18$

Year	1997	1998	1999	2000	2010	2020
Juvenile						
Subadult						
Adult						
Total						

(b) Continuing to execute these lines manually becomes tiresome (and it can be difficult to keep track of the years as well). Instead, use the following repetition loop to calculate the annual population vectors through 2020 and store the results in the matrix P.

```
x := x0;                        # initial population in 1997
P := evalm( x ):                # put 1997 data in first column of P
for i from 1998 to 2020 do      # for each year ...
  x := evalm( A &* x );         # – update populations
  P := augment( P, x );         # – add new column to P
end do:
evalm( P );                     # let's see the final result
```

A simple way to select the data needed to fill in the table is illustrated below:

```
T := augment( col(P,1..4), col(P,14), col(P,24) );   # select cols of P
vector( 6, i -> add( T(j,i), j=1..3 ) );             # total population
```

(c) The next set of Maple commands prepares and plots these results in a single graph. Include this graph with your project.

```
with(plots):                                              # load plots package
yr := vector( [ $ 1997 ..  2020)] ):
ptJ := zip( (x, y) -> [x, y], yr, row(P, 1) ):            # points for juveniles
ptS := zip( (x, y) -> [x, y], yr, row(P, 2) ):            # points for subadults
ptA := zip( (x, y) -> [x, y], yr, row(P, 3) ):            # points for adults

plotJ := plot(ptJ, color=red):                            # plot of juveniles
plotS := plot(ptS, color=blue):                           # plot of subadults
plotA := plot(ptA, color=green):                          # plot of adults
display( [ plotJ, plotS, plotA ],
 title="Spotted owl populations with t=0.18" );
```

2. Repeat Question 1 with a juvenile survival rate of $t = 0.30$. To change the value of t in the transition matrix A, use the command: `A[2,1] := 0.30;`.

Spotted owl population when the juvenile survival rate is $t = 0.30$

Year	1997	1998	1999	2000	2010	2020
Juvenile						
Subadult						
Adult						
Total						

3. Create separate plots for the three spotted owl subpopulations when the juvenile survival rate is $t = 0.20$, $t = 0.24$, $t = 0.26$, and $t = 0.28$.

   Summarize how the three populations (and the total population) change between 1997 and 2020 for the different values of the juvenile survival rate. For what values of t do you think the owl will survive (forever)?

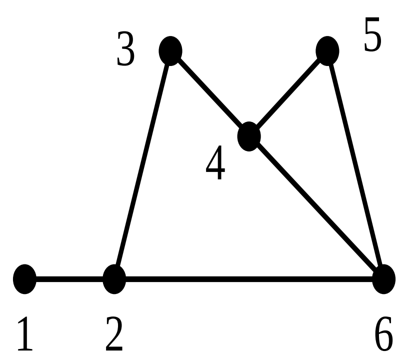
No content or questions on this page. Use this space for additional work if needed.

Purpose	To learn about graphs and adjacency matrices; to see one connection between graph theory and linear algebra.
Prerequisites	Section 2.1
Maple commands used	`adjdat` from the `laylinalg` package.

Definitions: A *graph* is a finite set of objects called nodes, together with some paths between some of the nodes, as illustrated below. A *path of length one* is a path that directly connects one node to another. A *path of length k* is a path made up of k consecutive paths of length one. The same length one path can appear more than once in a longer path; for example, 1–2–1 is a path of length two from node 1 to itself in the example below.

When the nodes are numbered from 1 to n, the *adjacency matrix* A of the graph is defined by letting $a_{ij} = 1$ if there is a path of length one, i.e., a direct path, between vertices i and j and $a_{ij} = 0$ otherwise.

Example: Verify that matrix A is the adjacency matrix for the graph shown below.



$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Theorem: (Interpretation of the powers of an adjacency matrix) If A is the adjacency matrix of a graph, then the (i, j) entry of A^k is a nonnegative integer which is the number of paths of length k from node i to node j.

1. To understand why the theorem is true, we will examine – by hand – the (6,3) entry of A^2. Using the "Row-Column Rule", the (6,3) entry of A^2 is $a_{61}a_{13}+a_{62}a_{23}+a_{63}a_{33}+a_{64}a_{43}+a_{65}a_{53}+a_{66}a_{63}$. Use the following table to complete this computation.

Term	$a_{61}a_{13}$	$a_{62}a_{23}$	$a_{63}a_{33}$	$a_{64}a_{43}$	$a_{65}a_{53}$	$a_{66}a_{63}$	(6,3) entry of A^2
Explicit Product		(1)(1)					
Simplified Product		1					

2. Observe that the product $a_{62}a_{23} = (1)(1) = 1$ says that there is one length two path connecting nodes 6 and 3 (the intermediate node is node 2). Explain what each of the remaining five terms in the sum for the (6,3) entry of A^2 tells about paths of length 2 from node 6 to node 3.

3. The adjacency matrix A for the graph shown above can be obtained with the commands:

```
with( laylinalg );
adjdat();
```

   (a) Find, and record, A^2 and A^3.
   Remember to use `evalm` to force Maple to display the resulting matrix.

A^2	A^2

(b) Notice that the (1,2) entry of A^2 is zero, so there are no paths of length two from node 1 to node 2. Verify this by studying the graph. Similarly, notice that the (6,6) entry of A^3 is 2, so there are two paths of length three from node 6 to itself; study the graph to see that they are 6–4–5–6 and 6–5–4–6.

   i. How many paths of length two go from node 4 to itself? What are they?

   ii. How many paths of length three go from node 4 to node 5? What are they?

   iii. Which pair(s) of nodes are connected with the most paths of length 2? How many?

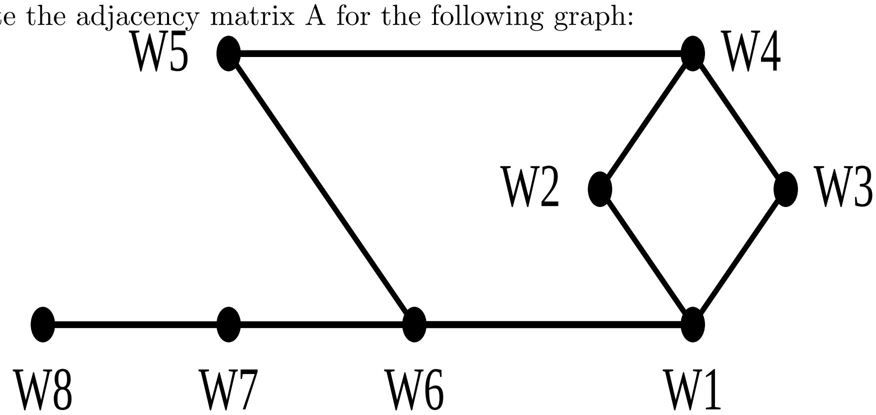
   iv. Which pairs of nodes are not connected by any path of length 2 or 3? What are they?

Definition: A graph is said to have *contact level* k between node i and node j if there is a path of length less than or equal to k from node i to node j.

(c) Suppose A is the adjacency matrix of a graph. Explain why you must calculate the sum $A + A^2 + \ldots + A^k$ in order to decide which pairs of nodes have contact level k ?

4. Eight workers, denoted W1, .., W8, handle a potentially dangerous substance. Safety precautions are taken, but accidents do happen occasionally. It is known that if a worker becomes contaminated, s/he could spread this through contact with another worker. The graph below shows which workers have direct contact with which others.

   (a) Write the adjacency matrix A for the following graph:



   (b) Enter this adjacency matrix A in your Maple session and answer the following questions:

      i. Which workers have contact level 3 with W3?

      ii. Which workers have contact level 3 with W7?

   (c) What is the smallest k such that every worker has contact level k with every other worker? Explain how you know your answer is correct. (Hint: Use Maple to examine A, $A + A^2$, $A + A^2 + A^3$, etc.)

   (d) Define what is meant when a worker is "dangerous". Be very specific so anyone could decide whether a worker was "dangerous" according to your definition.

   (e) Using your definition, answer the following questions. Be sure to explain your answers and verify that they are consistent with your definition of "dangerous".

      i. Which workers are the most dangerous if contaminated?

      ii. Which workers are the least dangerous if contaminated?

Purpose	To use linear algebra to analyze a production model for an open sector economy
Prerequisites	Sections 2.1 and 2.6 (Exercise 13)
Maple commands used	`add` and `seq`; `diag` and `inverse` from the `linalg` package; `c2s6` from the `laylinalg` package.

Read Exercise 13 in Section 2.6. In this problem, the US economy is divided into 7 sectors. Each sector produces goods and each sector uses some of the output of the other sectors. There is also an open sector (i.e., a sector which only consumes). The consumption matrix is called C and **d** denotes the demand vector for the open sector. The hope is that it is possible to find production levels that meet each sector's needs. That is, that there will be a nonnegative solution, **x**, to the equation $\mathbf{x} = C\mathbf{x} + \mathbf{d}$. Notice this equation can be rewritten as $(I - C)\mathbf{x} = \mathbf{d}$.

The matrix C in this exercise has nonnegative entries and each column sum is less than one. Therefore, by Theorem 11, $I - C$ will be invertible and, when **d** is a non-negative vector, the unique production vector **x** which satisfies $(I - C)\mathbf{x} = \mathbf{d}$ will also be nonnegative.

Notes:

- Maple's `inverse` command can be used to calculate the solution to any linear system whose coefficient matrix is invertible. You are encouraged to use the `inverse` command in this project.

1. Enter the consumption matrix, C, and final demand vector, **d**, in a Maple session. Assuming the `laylinalg` package has been loaded, the command `c2s6( 13 );` can be used to define C and **d**.

   (a) Verify that each column sum is less than one. The column sums of the 7×7 matrix C can be found with the command:

   ```
   seq( add( C[i, j], i=1..7 ), j = 1..7 );   # column sums for C
   ```

   (b) Use Maple to find the inverse of $(I - C)$ and then solve the linear system $(I - C)\mathbf{x} = \mathbf{d}$.

   ```
   Id := diag( 1 $ 7 );          # 7x7 identity matrix
   M := evalm( Id - C );         # M = I - C
   Minv := inverse( M );
   x := evalm( Minv &* d );      # production vector
   ```

   Report the production levels, rounded to three significant digits in the table below.

   (c) Repeat b) for two different nonnegative demand vectors, **d1** and **d2**, which seem reasonable to you. Record the demand vectors you selected and the corresponding production vectors (rounded to three significant digits) in the table.

d	**x**	**d1**	**x1**	**d2**	**x2**
$\begin{bmatrix} 74,000 \\ 56,000 \\ 10,500 \\ 25,000 \\ 17,500 \\ 196,000 \\ 5,000 \end{bmatrix}$					

2. Explain why it is important for the entries of the production vector to be nonnegative.

3. Use trial and error (and educated guessing) to find one entry in the first column of C which, when changed to a different positive number, has the property that solving $(\mathrm{I} - \mathrm{C})\mathbf{x} = \mathbf{d}$ with the new consumption matrix gives a solution that has at least one negative entry. Record your results, rounded to three significant digits, in the table.

modified C	$\mathbf{x}$

4. What do the numbers in your new matrix C say about the economy? Explain why your new matrix produces a nonfeasible demand vector? (Be sure your explanation utilizes appropriate linear algebra theorems.)

Purpose	To find the polynomial of order $(n-1)$ that passes through n data points.
Prerequisites	Section 2.2
Maple commands used	`interp` and `plot`; `inverse` from the `linalg` package.

When measuring a quantity that changes with time, for example, temperature at different times of the day, one often ends up with data that can be interpreted as points in a plane: (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), where y_k is a real number representing the measurement taken at time x_k. There are a variety of ways this data can be used. In Section 6.5 you will learn how to find "least-squares" approximations to the data. These approximations may not pass through, or interpolate, any of the data points. One way to find a function that interpolates the data is to piece together cubic polynomials that pass through consecutive pairs of data points with extra conditions that ensure their first and second derivatives are continuous.[1]

In this project we seek a single polynomial $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_kx^k$ that interpolates all n data points. From the n conditions $p(x_1) = y_1$, $p(x_2) = y_2$, $\ldots$, $p(x_n) = y_n$ it is clear that the polynomial must have degree $n-1$. All that remains is to determine appropriate values for the polynomial's coefficients: a_0, a_1, $\ldots$, a_{n-1}. Observe that the interpolating conditions can be written as $\mathrm{M}\mathbf{a} = \mathbf{y}$ where $\mathbf{a} = [a_0\ a_1\ \ldots\ a_{n-1}]^T$, $\mathbf{y} = [y_1\ y_2\ \ldots\ y_n]^T$, and M is an $n \times n$ matrix. (Recall that ^{T} denotes the transpose).

1. Find the $n \times n$ matrix M.

2. Read about Vandermonde matrices in Exercise 11 in the Supplementary Exercises for Chapter 2. Under what conditions is the matrix M in Question 1 invertible?

3. Assuming M is nonsingular, the interpolation problem is solved by $\mathbf{a} = \mathrm{M}^{-1}\mathbf{y}$. Is the polynomial through the points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) unique? Why or why not?

[1]For additional information about interpolation the supplemental project, Splines, for Section 1.2, is available on the Lay website (`http://www.laylinalgebra.com/`).

4. Select a character string consisting of at least 8 letters. One possible source for the string is to use your name – first, last, or a combination of the two.

   Use the coding scheme given below to generate data points from your string.

a	=	-12	b	=	-11	c	=	-10	d	=	-9	e	=	-8	f	=	-7
g	=	-6	h	=	-5	i	=	-4	j	=	-3	k	=	-2	l	=	-1
m	=	0	n	=	1	o	=	2	p	=	3	q	=	4	r	=	5
s	=	6	t	=	7	u	=	8	v	=	9	w	=	10	x	=	11
y	=	12	z	=	13												

   That is, let y_1 denote the numerical value of the first letter of your string, y_2 the numerical value of the second letter of your string, etc. For example, the string "marybeth" would be encoded (1,0), (2,-12), (3,5), (4,12), (5,-11), (6,-8), (7,7), (8,-5). Find the degree 7 polynomial through the points $(1, y_1)$, $(2, y_2)$, $\ldots$, $(8, y_8)$ for the string you selected.

   (a) List the 8 data points.

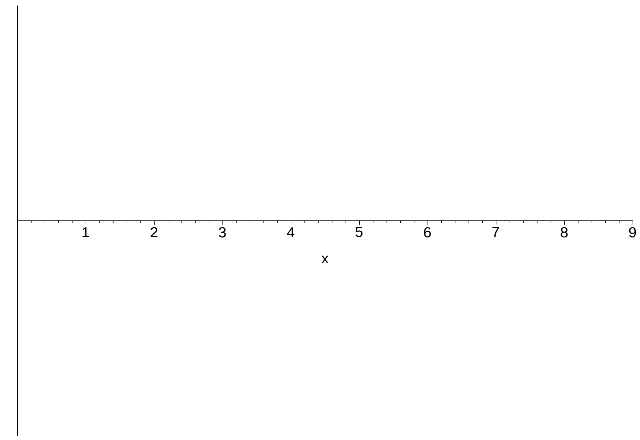
i	1	2	3	4	5	6	7	8
x_i								
y_i								

   (b) Give the Maple commands you use to enter the vector **y** and the matrix M.

   (c) What Maple command(s) did you use to find the coefficient vector **a**?

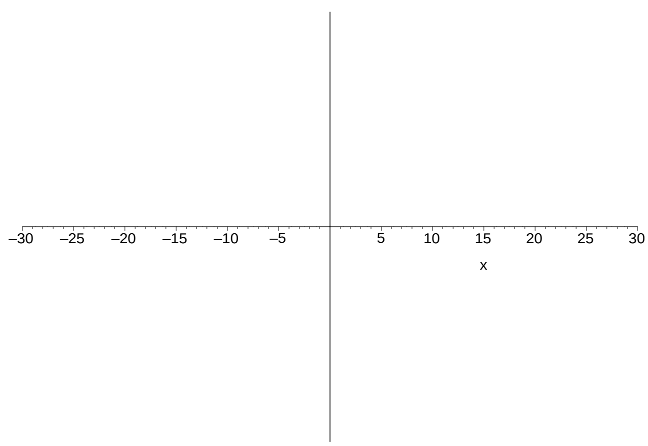
   (d) What is the interpolating polynomial for your data?

(e) Graph the interpolating polynomial for $1 \leq x \leq 8$ on the axes provided. *(Be sure to label the axes.)*



(f) Does the interpolating polynomial have any relative maxima or minima outside the interval $[1, 8]$? Explain.

(g) Graph your polynomial for $-30 \leq x \leq 30$ on the axes provided.



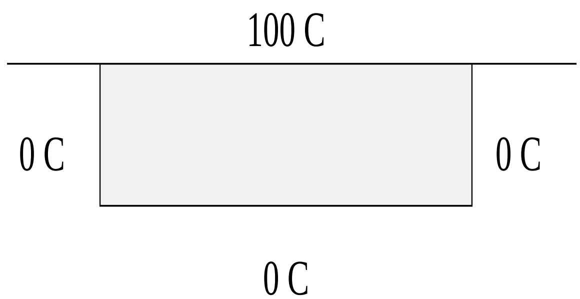
Extra Credit Look at the online help for the `interp` command. Use this information to give a single Maple command that returns the interpolating polynomial through the data points obtained from your name. How does this compare with the interpolating polynomial you found?
Submit your answer to this question on the back of this page or on separate paper.

No content or questions on this page. Use this space for additional work if needed.

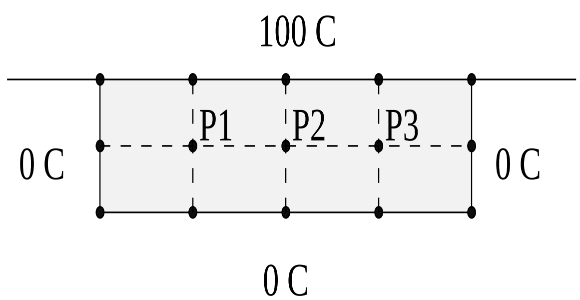
Purpose	Given temperature on a region's boundary, determine the steady-state temperature inside.
Prerequisites	Section 2.5
Maple commands used	`diag` and `inverse` from the `linalg` package.

Consider a situation in which the temperature is held constant on the surfaces of a beam with uniform cross-section. The temperature may be different at different points on the boundary, but does not change over time. To determine the steady-state temperature distribution inside the beam, look at a rectangular grid of points in a cross-section of the beam. At steady-state it is assumed that the temperature at each grid point is the average of the temperatures of the grid points to the north, south, east, and west of the original point.

1. Suppose a rectangular beam is exposed to boiling water on its top surface and ice water on the other three surfaces, as shown below.



If the cross-section is subdivided into a 2×4 grid, as shown below, there are three interior points (P_1, P_2, and P_3) at which the temperature is unknown.



Let t_i denote the steady-state temperature at node P_i for i=1, 2, 3. According to the assumption stated at the outset, the three steady-state temperatures must satisfy the linear equations:

$$
\begin{aligned}
t_1 &= \frac{1}{4}(100 + t_2 + 0 + 0) \\
t_2 &= \frac{1}{4}(100 + t_3 + 0 + t_1) \\
t_3 &= \frac{1}{4}(100 + 0 + 0 + t_2)
\end{aligned}
$$

(a) Show that these equations can be written in matrix notation as $4\mathbf{t} = \mathbf{b} + C\mathbf{t}$ where the vector of unknown temperatures is $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$, $\mathrm{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$.

(b) If the matrix $\mathrm{A} = 4\mathrm{I} - \mathrm{C}$ is invertible the steady-state temperatures are given by $\mathbf{t} = \mathrm{A}^{-1}\mathbf{b}$. Without computing the inverse of A, show that A is invertible.

(c) Assuming the matrix C and vector **b** have previously been defined in your Maple session, the following Maple commands can be used to compute the matrices A and A^{-1} and the solution vector **t**. The following Maple commands can be used to determine the temperature at the three interior nodes for the configuration in this question.

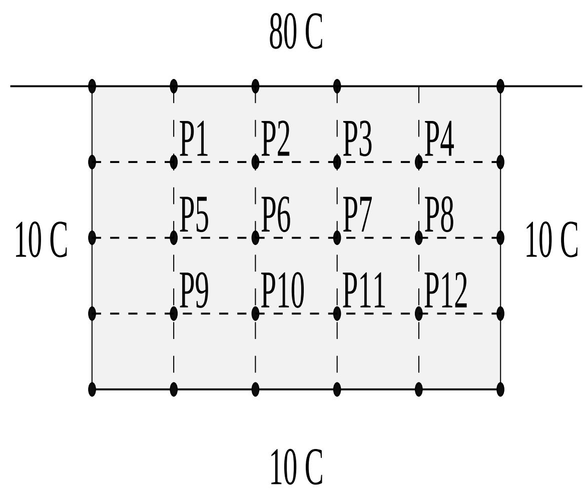
```
A := evalm( 4*diag( 1$3 ) -- C );   # create the matrix A
Ainv := inverse( A );               # compute the inverse of A
t := evalm( Ainv &* b );            # compute steady-state temperatures
```

Record your results in the table below.

A	A^{-1}	**t**

(d) Do these temperatures seem reasonable? Why? or Why not?

2. Consider the cross-section and grid shown below. Use the method described in Question 1 to determine the steady-state temperatures at the grid points.

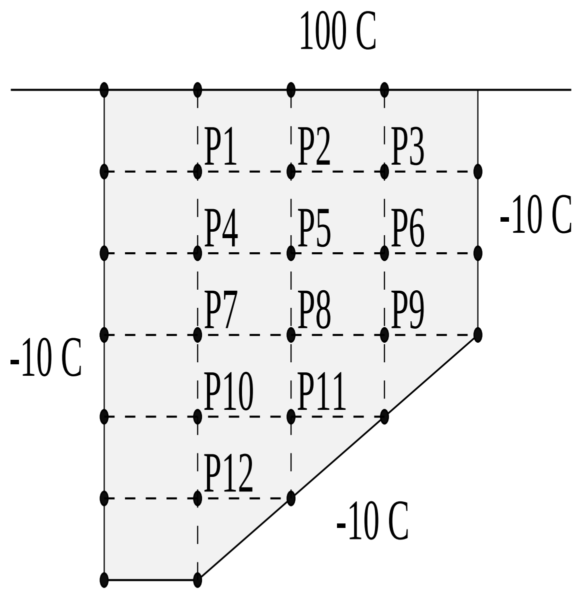


(a) Determine the temperature at the 12 interior points in this problem. Record your results in the table below.

A	A^{-1}	**t**

(b) Do these temperatures seem reasonable? Why? or Why not?

3. Consider the cross-section and grid shown below. Use the method described in Question 1 to determine the steady-state temperatures at the grid points.



(a) Record your results in the table below.

A	A^{-1}	**t**

(b) Do these temperatures seem reasonable? Why? or Why not?

Name ______________________

Purpose	To gain experience making and justifying conjectures about a family of structured matrices.
Prerequisites	Sections 2.2 and 3.2
Maple commands used	`for .. do .. end do` and `max`; `det` and `inverse` from the `linalg` package.

In this project you will attempt to determine the general formula for the determinant and inverse of a special collection of matrices. You will test your conjecture for as many matrices as possible but will not be asked to give complete proofs.

For each positive integer n, let M_n be the $n \times n$ matrix whose (i, j) entry is $m_{ij} = \max(i, j)$ for all i, $j = 1, 2, 3, \ldots, n$. For example, $\mathrm{M}_1 = \begin{bmatrix} 1 \end{bmatrix}$, $\mathrm{M}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$, $\mathrm{M}_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$, $\ldots$.

1. Find an efficient and painless way to define these matrices — even for large values of n. Record your way of entering the matrix M_n in the box below.

   <u>Hints</u>:

   - There are numerous ways to approach this problem.
   - One of the most straightforward is to use assignments of the form `M[i,j] := max(i,j);` inside a `do` loop (see the online help for `do` to see the appropriate syntax).
   - A more sophisticated solution involves uses only a single `matrix` command with a procedure as its third argument (see the online help for `matrix` for more details on this approach).

2. Find a formula for the determinant of M_n for all integers $n \geq 1$. Justify your conjecture with at least 5 examples.

3. Find the inverse matrices M_1^{-1}, M_2^{-1}, M_3^{-1}, and M_4^{-1}.

4. Find a formula for M_n^{-1} that is valid for all $n \geq 1$. Support your conjecture with several (additional) examples.

Purpose	To use linear algebra to analyze Markov chains and investigate their steady state vectors.
Prerequisites	Sections 2.1 and 4.9
Maple commands used	`add` and `for .. do .. end do`; `diag` and `rref` from the `linalg` package; `markdat` from the `laylinalg` package.

Read Exercises 2 and 12 in Section 4.9. These problems consider the Markov chain with the system matrix $P = \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$. Here there are three foods and the i, j entry of P is the probability that if an animal chooses food j on the first trial, it will choose food i on the second trial. Similarly, the (i, j) entry of P^2 is the probability that if an animal chooses food j on the first trial, it will choose food i on the third trial.

1. Load the system matrix with the command `markdat( );` (this assumes the `laylinalg` package is loaded).

   (a) Compute the matrix P^2. Record the result here.

   (b) Suppose an animal chooses food 1 on the first trial. Use P and P^2 to determine the probability the animal will:

   choose food 2 on the second trial ______________________

   choose food 2 on the third trial ______________________

   choose food 3 on the second trial ______________________

   (c) Use Maple to find the reduced echelon form of $P - I$. Recall that the 3×3 identity matrix can be defined using `Id := diag( 1 $ 3 );`. Then, `rref( P - Id );` returns the reduced echelon form of $P - I$. Use the space provided to record $P - I$, the reduced echelon form of $P - I$, the general solution, $\mathbf{x}$, to the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$, and a particular solution, $\mathbf{w}$, corresponding to a nonzero value of the free variable.

$P - I$	$P - I$ (ref)	$\mathbf{x}$	$\mathbf{w}$

(d) The steady-state vector, **q**, for P can be found by dividing the particular solution, **w**, by the sum of its components. The Maple commands to compute this vector are:

```
w := vector( 3, [ your data from (c) ] );   # a particular solution
s := add( w[i], i=1..3 );                   # sum of the components of w
q := evalm( w / s );                        # rescaled particular solution
```

Is **q** a (particular) solution to $(P - I)\mathbf{x} = \mathbf{0}$? Is **q** a probability vector? Explain.

(e) If P is a regular stochastic matrix, Theorem 18 says there is only one steady-state. This means regardless of the particular solution you found in (c) you will compute the same vector **q** in (d). Confirm this by repeating the computation in (d) using the general solution, **x** instead of the particular solution **w**. Use linear algebra to explain why this is true.

2. Read Exercises 4 and 14 in Section 4.9 and solve it as follows.

(a) Use the table below to record the matrix W and the initial vector $\mathbf{v}_0$ describing today's weather forecast. Explain the steps you took to ensure W is stochastic and $\mathbf{v}_0$ is a probability vector.

(b) Compute and record (in the table below) the probability vector for tomorrow's weather: $W\mathbf{v}_0$. What is the chance of bad weather tomorrow?

(c) Using the predicted weather for Monday as a new initial vector, $\mathbf{v}_1$. Compute and record (in the table below) the probability vector for Wednesday's weather: $W^2\mathbf{v}_1$. What is the chance of good weather on Wednesday?

(d) Use the method in Question 1(c) to calculate the steady-state vector, $\mathbf{q}$. In the long run, what is the probability the weather will be good on a given day?

W	$\mathbf{v}_0$	$W\mathbf{v}_0 = \mathbf{v}_1$	$W^2\mathbf{v}_1$	$\mathbf{q}$

3. According to Theorem 18, when P is stochastic and regular, and $\mathbf{v}$ is any probability vector, the sequence of vectors $\mathbf{v}$, $P\mathbf{v}$, $P^2\mathbf{v}$, ... converge, and the limit vector will be the steady-state vector of P. In other words, when the power k is big enough, $P^k\mathbf{v}$ will look like the unique steady-state vector. This is not an efficient way to calculate the steady-state vector, but it is interesting to see the sequence $\mathbf{v}$, $P\mathbf{v}$, $P^2\mathbf{v}$, ... converge for a few examples. The following Maple commands can be used to compute the first 10 terms in the sequence. If more terms are needed, change 10 to a larger number and re-execute.

```
v := vector( 3, [ 1, 0, 0 ] );   # initial vector
for i from 1 to 10 do            # increase 10 for a longer sequence
  v := evalm( P &* v );          # compute next vector in sequence
end do;
```

Estimate k for both Exercise 2, using matrix P, and Exercise 4, using matrix W. Use each of the initial vectors shown below and at least one more probability vector $\mathbf{v}$ of your own. For each $\mathbf{v}$, calculate $P^k\mathbf{v}$ until you find a big enough k so that $P^k\mathbf{v}$ looks like the steady-state vector for P (compare to the steady-state vectors you found in Questions 1(d) and 2(d)). Repeat this for each $\mathbf{v}$ and W, and record the smallest value of k which is big enough in each case. Record your results, including the initial vector that you chose, in the table below.

	Animal Experiment (using matrix P)				Weather Forecast (using matrix W)			
$\mathbf{v}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0.2\\0.6\\0.2\end{bmatrix}$	$\begin{bmatrix}0.35\\0.35\\0.30\end{bmatrix}$		$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0.2\\0.6\\0.2\end{bmatrix}$	$\begin{bmatrix}0.35\\0.35\\0.30\end{bmatrix}$	
k								

Explain, in the box below, the criteria used to determine when k was large enough to stop the iterations.

4. The matrices $P_1 = \begin{bmatrix} 0.7 & 0.2 & 0.6 \\ 0 & 0.2 & 0.4 \\ 0.3 & 0.6 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.7 & 0.2 & 0.6 \\ 0 & 0.2 & 0 \\ 0.3 & 0.6 & 0.4 \end{bmatrix}$, and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ can be loaded into a Maple session by executing the command `markdat( );`. (This might have been done in Question 1; it is not a problem to execute this command again here.)

   (a) Which of P_1, P_2, and P_3 are regular? Explain your answers.

P_1	
P_2	
P_3	

   (b) Use the method in Question 1 to calculate steady state vectors for P_1, P_2, and P_3. Record the steady state vectors in the table below. Use Theorem 18 or some calculations to decide whether the steady-state vector is unique.

	P_1	P_2	P_3
What steady-state vector, $\mathbf{q}$ is produced by the method in Question 1?			
Is $\mathbf{q}$ unique? (Briefly, why?)			
If $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, does $P^k\mathbf{v}$ converge as k increases? If not, what does happen?			

Purpose	To learn how to use Maple's `eigenvectors` command to construct the matrix of eigenvalues and the matrix of eigenvectors.
Prerequisites	Sections 5.1 and 5.3
Maple commands used	`op` and `seq`; `augment`, `diag` and `eigenvectors` from the `linalg` package; `cxeigdat` from the `laylinalg` package.

The following five matrices will be used in this project:

$$\mathrm{A} = \begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix} \mathrm{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathrm{C} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} \mathrm{D} = \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix} \mathrm{E} = \begin{bmatrix} 0 & 0 & 5 \\ 1 & 0 & 0 \\ 0 & 5 & 4 \end{bmatrix}$$

1. Calculate the eigenvalues of each of the four 2×2 matrices by hand. Record your eigenvalues in the table at the end of Question 2.

2. Repeat Question 1 on four new 2×2 matrices created as follows: if the original matrix has real eigenvalues, change the sign of *one* entry so that the resulting matrix has complex eigenvalues; if the original matrix has complex eigenvalues, change *one* entry in such a way that the resulting matrix has real eigenvalues.

Original Matrix	A	B	C	D
Original Eigenvalues				
Modified Matrix				
Modified Eigenvalues				

3. The five matrices (A, B, C, D, and E) can be loaded into a Maple session with the command: `cxeigdat( );`. Recall that the Maple name for matrix D is `DD`. The `eigenvectors` command can be used to obtain all eigenvalues and the corresponding eigenvectors for a matrix. Use the following sequence of commands to construct the diagonal matrix L and invertible matrix P with the property that AP = PL (the text calls the diagonal matrix D, but we already have a matrix D in this project). Repeat these computations for the other four matrices. Report your results in the table.

```
EV := [ eigenvectors( A ) ];                    # eigenvalues and eigenvectors
L := diag( seq( op(1,e)$op(2,e), e=EV ) );  # diagonal matrix of eigenvalues
P := augment( seq( op(op(3,e)), e=EV ) );    # invertible matrix of eigenvectors
```

Matrix	A	B	C	D (i.e., `DD`)	E
L					
P					
AP					
PL					

The results found in Questions 1–3 illustrate several important facts about eigenvectors and eigenvalues. These are summarized here and are discussed in detail in Section 5.5 of the text.

Notes:

- Real-value matrices can have complex-valued eigenvalues and eigenvectors.
- Complex eigenvalues and eigenvectors both appear in complex conjugate pairs. That is, if $\lambda = a + ib$ is an eigenvalue with eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$, then $\overline{\lambda} = a - ib$ is an eigenvalue with eigenvector $\overline{\mathbf{v}} = \mathbf{u} - i\mathbf{w}$.
- If A is an $n \times n$ matrix with n odd, then A has at least one real eigenvalue.

Purpose	To complete the analysis of the spotted owl population, including a determination of the critical juvenile survival rate that ensures survival of the species.
Prerequisites	Sections 5.1 and 5.6 and the Maple Project: *Initial Analysis of the Spotted Owl*.
Maple commands used	`abs`, `for .. do .. end do`, `map`, `plot`, and `zip`; `augment`, `eigenvalues`, and `eigenvectors` from the `linalg` package; `owldat` from the `laylinalg` package; `display` from the `plots` package.

Recall, from the Opening Example to Chapter 5 and the Maple Project "Initial Analysis of the Spotted Owl", that the spotted owl population is divided into three distinct life stages: first year (juvenile), second year (subadult), and third year and beyond (adult). Let $\mathbf{x}_k = \begin{bmatrix} j_k \\ s_k \\ a_k \end{bmatrix}$ be the state vector for year k, then $\mathbf{x}_{k+1} = \mathrm{A}\mathbf{x}_k$ with transition matrix $\mathrm{A} = \begin{bmatrix} 0 & 0 & 0.33 \\ t & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$. The parameter t is the *juvenile survival rate*. In the first part of this project the claim that the population will eventually die out if $t = 0.18$ but not if $t = 0.30$ was verified. The critical value of t that ensures long time survival of the spotted owl will be found in this project.

Definition: Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ sorted so that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. The eigenvalue λ_1 is called a *dominant eigenvalue* of A.

The special type of 3×3 matrix discussed in this project has one eigenvalue λ_1 such that $|\lambda_1| > |\lambda_i|$ for $i = 2$ and 3, so we will speak of "the" dominant eigenvalue λ_1. In fact, for the matrices here, λ_1 is even real and positive.

1. Recall that the command: `owldat( );` loads the transition matrix (and initial vector) for the spotted owl into your current Maple session.

   (a) Use the following Maple commands to find the dominant eigenvalue of the transition matrix for t=0.18, 0.19, ..., 0.30. Record these results, rounded to five significant digits, in the table below. Note that the "absolute value" of a complex number is its modulus.

```
for t from 0.18 to 0.30 by 0.01 do
  A[2,1] := t;                          # new value of t in transition matrix
  map( abs, [eigenvalues( A )] );       # absolute values of eigenvalues
end do;
```

t	0.18	0.19	0.20	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.30
λ_1													

   (b) The "critical value" of the juvenile survival rate is the value of t that corresponds to a transition matrix with dominant eigenvalue $\lambda_1 = 1$. For our purposes we will approximate the critical value of t. Define t_0 to be the smallest value of t with $\lambda_1 > 1$. Identify t_0 in the table in (a).

2. Let A be the transition matrix with $t = t_0$. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ denote the eigenvectors of A, let $\mathbf{x}_0$ be an initial vector, and suppose c_1, c_2, and c_3 are scalars such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.

   (a) Explain why the population of owls will not die out if the coefficient of the eigenvector corresponding to the dominant eigenvalue is not zero.
   <u>Hints</u>:
   - Do not find the eigenvectors or coefficients.
   - The explanation depends on the *properties* of the dominant eigenvalue, etc., not their specific values.

   (b) Record the three eigenvalues, and their corresponding eigenvectors, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, of A in the table at the end of Question 3. Notice that the dominant eigenvalue, λ_1, is real and positive and the other two eigenvalues, λ_2 and λ_3, are complex conjugates.

   (c) Solve the equation $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for c_1, c_2, and c_3 when the initial vector is $\mathbf{x}_0 = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$. Record the solution, rounded to two significant digits, in the following table. Verify that c_1 , the coefficient of the dominant eigenvalue, is not zero, hence the owl population does not die out for this initial vector. One way to do this computation is to create the matrix V whose columns are the eigenvectors of A and then solve the linear system $V\mathbf{c} = \mathbf{x}_0$ where $\mathbf{c}$ is the vector of coefficients. (Be sure the columns appear in the correct order!)

c_1	c_2	c_3

3. (a) Choose two new values of the juvenile survival rate t_1 and t_2 so that

$$t_0 - 0.01 < t_1 < t_0 < t_2 < t_0 + 0.01.$$

   Report the eigenvalues and eigenvectors for the associated transition matrices in the following table. (Round all numbers to two significant digits.)

t	λ_1	λ_2	λ_3	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$t_0 =$___						
$t_1 =$___						
$t_2 =$___						

(b) For each of these survival rates, t_1 and t_2, create one graph showing the size of each subpopulation for the years 1997 through 2020. Be sure each plot contains a caption that indicates the value of t. Attach both plots to this project.

Hints:

- The first step is to generate the yearly populations for 1997 – 2020 when $t = 0.30$.

```
with(plots):                    # load plots package

A[2,1] := 0.30;                 # change juvenile survival rate
x := x0;                        # initial population in 1997
P := evalm( x ):                # put 1997 data in first column of P
for i from 1998 to 2020 do      # for each year ...
  x := evalm( A &* x );         # – update populations
  P := augment( P, x );         # – add new column to P
end do:
```

- The next set of Maple commands displays the size of each subpopulation for $t = 0.30$. only minor modifications are needed for t_1 and t_2.

```
yr := vector( [ $ 1997 ..  2020)] ):
ptJ := zip( (x, y) -> [x, y], yr, row(P, 1) ):     # juveniles
ptS := zip( (x, y) -> [x, y], yr, row(P, 2) ):     # subadults
ptA := zip( (x, y) -> [x, y], yr, row(P, 3) ):     # adults

plotJ := plot(ptJ, color=red):                     # plot of juveniles
plotS := plot(ptS, color=blue):                    # plot of subadults
plotA := plot(ptA, color=green):                   # plot of adults
display( [ plotJ, plotS, plotA ],                  # combined plot
  title="Spotted owl populations (t=0.30)" );
```

(c) What trends in the three age groups are apparent in the graphs? How do the plots with $t = t_1$ and $t = t_2$ differ? How are they similar? Are these results consistent with what you know about the dominant eigenvalue of the transition matrix? (It might help to look at the eigenvalues of the transition matrix for these values of t.)

Extra Credit .

(a) Let A = $\begin{bmatrix} 0 & 0 & a \\ t & 0 & 0 \\ 0 & b & c \end{bmatrix}$, and assume a, b, c, t are positive. Show that $f(\lambda) = -\lambda^3 + c\lambda^2 + abt$ is the characteristic polynomial of A.

(b) Prove that A has one positive (real) eigenvalue and that the other two eigenvalues of A must be complex conjugates. Let λ_1 denote the positive eigenvalue and let λ_2 and λ_3 denote the other two eigenvalues.

Hints:
- Since $f(\lambda)$ has only real coefficients, you can sketch its graph in R^2. It will be helpful to calculate the y-intercept and to use the derivative to find the turning points. Use this graph to explain why there is only one real root of $f(\lambda)$ and that this root is positive. Then, use properties about zeros of polynomials to explain why the other two zeros must be conjugate complex numbers.

(c) Prove that $\lambda_1 > |\lambda_2| = |\lambda_3|$, hence the real positive eigenvalue of A will always be the dominant eigenvalue for this type of matrix.

Hints:
- Show that $f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$.
- Use this to show $\lambda_1\lambda_2\lambda_3 = abt$.
- Next, show that $\lambda_2\lambda_3 = |\lambda_2|^2$ and so $\lambda_1|\lambda_2|^2 = abt$.
- Finally, explain why $-\lambda_1^3 + c\lambda_1^2 + abt = 0$.
- Put these findings together to conclude that $\lambda_1 > |\lambda_2|$.

(d) Assume $\lambda_1 = 1$ and use this to obtain a formula for the exact critical value of t. Evaluate your formula when $a = 0.33$, $b = 0.71$ and $c = 0.94$, and compare this with the critical value you found experimentally in Question 1. Are they essentially the same? Discuss what $\lambda_1 = 1$ means in the owl model. (For example, does it mean no births or deaths? If not, then what?)

Notes:
- Prepare your explanations on separate pages; attach these pages to this project.
- Be sure your explanations are well-organized and neatly presented.

Purpose	To learn about the Cayley-Hamilton Theorem
Prerequisites	Section 5.2
Maple commands used	`eval`;
	`charpoly` from the `linalg` package;
	`randomint` from the `laylinalg` package.

Cayley–Hamilton Theorem: square matrix A satisfies its characteristic equation.

That is, if $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0$ is the characteristic polynomial for A, then $p(\mathrm{A}) = 0$. Note that 0 is the $n \times n$ zero matrix. To evaluate the characteristic polynomial "at a matrix" it is essential to interpret the constant term, c_0, as the corresponding multiple of the $n \times n$ identity matrix, $c_0\mathrm{I}$. For example, the characteristic polynomial for $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $p(\lambda) = \lambda^2 - 5\lambda - 2$. Then, $p(\mathrm{A}) = \mathrm{A}^2 - 5\mathrm{A} - 2\mathrm{I}$.

1. Empirical evidence of the validity of the Cayley–Hamilton Theorem can be obtained by looking at randomly-selected matrices of various sizes. Use Maple to fill in the following table. (For $n = 8$, $p(\mathrm{A})$ can be evaluated with the command `evalm(eval(p,lambda=A));`.)

```
with( laylinalg ):
A := randomint( 2, 2 );                   # randomly-selected 2x2 example
p = det( A - lambda * diag( 1$2 ) );   # char polynomial for 2x2 matrix
```

n	A	$p(\lambda)$	$p(\mathrm{A})$ – written as the sum of $n+1$ matrices
2			
3			
4			
8			

No content or questions on this page. Use this space for additional work if needed.

Purpose	To investigate the pseudo-inverse of a matrix and its use in solving overdetermined systems.
Prerequisites	Section 6.5
Maple commands used	`inverse` and `transpose` from the `linalg` package; `implicitplot` from the `plots` package.

The transpose A^T of a matrix A is obtained by interchanging the rows and columns of A: the first row of A becomes the first column of A^T, the second row of A becomes the second column of A^T, and so forth. If A is an $n \times m$ matrix, then A^T is an $m \times n$ matrix. Therefore the matrix products AA^T and A^TA are always defined and yield $n \times n$ and $m \times m$ matrices, respectively. Since A^TA is a square matrix, it makes sense to ask whether it has an inverse. If it does, then the matrix $A^+ = (A^TA)^{-1}A^T$ is called the pseudo-inverse of A (sometimes the term generalized inverse is used).

Notes:

- This definition of the pseudo-inverse applies only when (A^TA) is invertible. This occurs if and only if A has linearly independent columns. (Why?)
- The singular value decomposition can be used to define the pseudo-inverse of any matrix. (See Example 7 in Section 7.4 of the text.)
- The pseudo-inverse is sometimes called the Moore-Penrose inverse.

1. Let A be an $m \times n$ matrix and assume A^TA is nonsingular. What size matrix is A^+, the pseudo-inverse of A?

2. Let A be a randomly-selected 4×4 matrix with integer entries. Use Maple to compute A^TA and check if it is invertible. If A^TA is not invertible, obtain a new random 4×4 matrix of integers. Find A^+, then compute A^+A and AA^+; report all of your findings in the table at the bottom of this page.
3. Repeat Question 2 with a randomly-selected 3×2 matrix with integer entries.
4. Repeat Question 2 with a randomly-selected 5×2 matrix with integer entries.
5. Repeat Question 2 with a randomly-selected 5×1 matrix with integer entries.

Size	A	A^+	A^+A	AA^+
4×4				
3×2				
5×2				
5×1				

6. Examples 1 and 2 in Section 6.5 demonstrate that the "least-squares" solution to $\mathrm{A}\mathbf{x} = \mathbf{b}$ is $\mathbf{w} = \mathrm{A}^{+}\mathbf{b}$. You will now verify that this statement is true in general.

(a) Show that, if A is invertible, then $\mathbf{w}$ is the exact solution, i.e., $\mathrm{A}\mathbf{w} = \mathbf{b}$.

(b) Show that, if $\mathbf{b}$ is in the range of A, then $\mathbf{w}$ is the exact solution.

(c) Consider the following system of linear equations:

$$\begin{array}{rcrcl} 5x_1 & + & 6x_2 & = & 7 \\ 3x_1 & - & 4x_2 & = & 8 \\ 2x_1 & + & 9x_2 & = & 5 \end{array}$$

Write these equations in the form $\mathrm{A}\mathbf{x} = \mathbf{b}$ and find the least-squares solution, $\mathbf{w}$. Compute $\mathrm{A}\mathbf{w}$. Is this result surprising? Record the (transposes of the) vectors $\mathbf{w}$ and $\mathrm{A}\mathbf{w}$ in the following table.

$\mathbf{w}$	
$\mathrm{A}\mathbf{w}$	

(d) Use the following template of Maple commands to graph the three lines. Then, identify on a hardcopy of the graph, the point that corresponds to the vector $\mathbf{w}$. You will need to supply an appropriate viewing window to create the final graph. (Attach the final graph to this project.)

```
with( plots ):
eq1 := 5*x1 + 6*x2 = 7;
eq2 := 3*x1 - 4*x2 = 8;
eq3 := 2*x1 + 9*x2 = 5;
implicitplot( {eq1, eq2, eq3}, x1= ___ .. ___, x2= ___ .. ___ );
```