

# Differential Equations in the New Millennium: the Parachute Problem\*

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*Introductory courses in differential equations have traditionally consisted of a long list of solution techniques for special equations. This characterization is becoming increasingly inaccurate as more textbooks and courses are being designed around qualitative methods. One component of many revised courses is the discussion of real-life applications and modeling. The parachute problem will be used to illustrate several essential features of the improved courses. In particular, it will be seen that the traditional version of the parachute problem is not very realistic, but is easily improved without making the problem significantly more complicated.*

## INTRODUCTION

MATHEMATICAL MODELING is an increasingly essential skill for many engineers. The “parachute problem” is an appealing application that can be found in most differential equations textbooks [1, (p. 141, #19 and 20); 4, (p. 95, #10, 11, 20, and 21); 5, (p. 109, #20); 11, (pp. 112–114, Example 3 and #8)]. The typical formulation of the problem is:

A skydiver begins a jump at a specific height,  $x_0$ , above the ground and falls towards Earth under the influence of gravity. Assume the force due to air resistance is proportional to the velocity of the parachutist, with different constants of proportionality when the parachute is closed (free-fall) and open (final descent). Answer the following questions:

1. Given the conditions under which the parachute is deployed, how long does the jump last?
2. What is the velocity when the parachute is deployed and at landing? What are the terminal velocities of the different stages of the jump?
3. What is the latest time the parachute can be released and have the landing velocity below a specified safety threshold?

This problem, like most in traditional introductory courses, is intended to stimulate and exercise the student’s ability to find and manipulate explicit analytical solution formulas. However, this is not how an engineer typically encounters differential equations in subsequent courses—or the real world.

The assumptions stated in the problem description have several fundamental problems. For example, basic fluid mechanics shows that the

relationship between the drag force and velocity can be nonlinear [6, 12]. In the case of a parachute jump, the drag force is proportional to the *square* of the velocity. Moreover, the descriptions of the deployment and inflation of a parachute found in sport and military training guides (see [3] and [13], respectively) go into great detail about the release and inflation of the parachute. In particular, the transition from free-fall to final descent is not instantaneous. Several recent journal articles have begun to address these problems individually [2, 6, 7, 8, 9], but not in a systematic way based on fundamental principles.

The primary purpose of this paper is to illustrate the coordinated use of qualitative and theoretical results and real-world considerations that is the cornerstone of new pedagogical approaches for differential equations. The traditional parachute problem analysis is presented below. An improved model, based on the traditional analysis and additional physical information, is then developed and analyzed. Graphical and numerical solutions are used to verify that the motion stays within the design specifications of the parachute.

## THE TRADITIONAL PROBLEM

The “parachute problem” is a simple application of Newtonian mechanics,  $F_g + F_d = ma$ , to a skydiver of mass  $m$  with acceleration  $a$  that is subject to a gravitational force  $F_g$  and a drag force  $F_d$  due to air resistance. In the natural coordinate system in which  $x$  is the distance above the earth’s surface,  $a = dv/dt$  where  $v = dx/dt$  is velocity and  $F_g = -mg$  with  $g \approx 9.81 \text{ m/s}^2$  in MKS units. In many popular differential equations textbooks [1, (p. 141, #19

\* Accepted 2 July 1999.

and 20); 4, (p. 95, #10, 11, 20, and 21); 5, (p. 109, #20); 11, (pp. 112–114, Example 3 and #8)] the drag force is assumed to be proportional to velocity,  $F_d = -kv$ . The coefficient of drag,  $k$ , has one value, say  $k_1$ , when the skydiver is in free-fall and a second value,  $k_2$ , when the parachute is fully deployed. If the deployment occurs at time  $t_0$ ,

$$k = \begin{cases} k_1, & 0 \leq t < t_0 \\ k_2, & t \geq t_0. \end{cases} \quad (1)$$

At this point the problem can be posed as either a second-order ordinary differential equation (ODE) for position or as a first-order system of ODEs for the velocity and position. In many traditional discussions, only the second-order ODE would be considered. In this situation, however, the first-order system is much simpler to solve. During free-fall, the velocity satisfies the initial value problem:

$$m \frac{dv}{dt} = -mg - kv, \quad v(0) = 0 \quad (2)$$

with  $k = k_1$ . This equation can be solved either as a first-order linear ODE or as a separable ODE. After the first week of the course, most students can correctly find that the solution is:

$$v(t) = \frac{mg}{k_1} (e^{-(k_1/m)t} - 1). \quad (3)$$

The position is obtained by integrating the velocity with initial condition  $x(0) = x_0$ :

$$x(t) = x_0 - \frac{mg}{k} t - \frac{m^2 g}{k^2} (e^{-(k_1/m)t} - 1). \quad (4)$$

After the parachute is deployed the velocity and position can be found exactly as above, except that the initial conditions are  $v(t_0) = v(t_0^-)$  and  $x(t_0) = x(t_0^-)$  where, e.g.,  $v(t_0^-) = \lim_{t \rightarrow t_0^-} v(t) = (mg/k_1)(e^{-(k_1/m)t_0} - 1)$ . The formulae for the velocity and position are somewhat complicated, but are obtained as above. For example, the velocity is:

$$v(t) \begin{cases} \frac{mg}{k_1} (e^{-(k_1/m)t} - 1), & 0 \leq t < t_0 \\ \frac{mg}{k_1} (e^{-(k_1/m)t_0} - 1) e^{-(k_2/m)(t-t_0)} \\ \quad + \frac{mg}{k_2} (e^{-(k_2/m)(t-t_0)} - 1), & t \geq t_0. \end{cases} \quad (5)$$

The different terminal velocities of the two stages of the jump are easy to compute from the velocity. However, the terminal velocity,  $v_T$ , is even easier to find by setting  $dv/dt = 0$  in the equation of motion, (2), and solving for  $v$ :  $v_T = -mg/k$ . The direct use of the differential equation is an important feature of the new approach to teaching differential equations.

Another instance in which the differential equation is more useful than an explicit formula for the solution is the analysis of the acceleration of the skydiver. Contrast the computation of the acceleration using  $a = dv/dt$  (or, even worse,  $a = d^2x/dt^2$ ) with the direct substitution of the velocity into the equation of motion:  $a = -g - (k/m)v = -mge^{-(kt/m)}$ . Note that since  $k$  is discontinuous at  $t = t_0$ , the acceleration is also discontinuous at the time the parachute is deployed. Physically, however, the acceleration must be continuous [13, 14].

## THE PHYSICS OF SKYDIVING

The development of a more realistic model for a parachute jump will be based on the basic principles of fluid dynamics [10, 12]. The Navier-Stokes equations describe the motion of a body through a viscous fluid. The *speed* of the motion is frequently described in terms of the dimensionless Reynolds number,  $Re$ . In general,  $Re = \rho dv/\mu$  where:  $\rho$  is fluid density,  $d$  is a characteristic length,  $v$  is a characteristic velocity, and  $\mu$  is the fluid viscosity. Realistic Reynolds numbers range from  $O(1)$  for a dust particle in air or a larger object in a less viscous fluid to more than  $10^8$  for a submarine in water.

The Navier-Stokes equations contain both inertial and viscous forces [10]. The Reynolds number  $Re$  describes the relative importance of these forces in a given flow. When  $Re \ll 1$  viscous forces dominate and the drag force on a solid sphere of radius  $r$  is approximately linear in the velocity:  $F_d = -6\pi\mu r v$ . This approximation, which is also known as the *creeping flow* approximation [12], was discussed above. When  $Re > 10^3$  the inertial forces dominate and the drag force is approximately quadratic in the velocity.

To determine which of these models is most appropriate for a human falling through the atmosphere, it suffices to estimate the Reynolds number. The density  $\rho$  and viscosity  $\mu$  are essentially constant at altitudes appropriate for parachuting [12]:  $\rho \approx 1 \text{ kg/m}^3$ ,  $\mu \approx 1.5 \times 10^{-5} \text{ kg/m}\cdot\text{s}$ . Terminal velocity is a reasonable choice for the characteristic velocity. The landing impact, which generally occurs at the terminal velocity for the last stage of the jump, is frequently said to be comparable to a jump from a five-foot wall [13]:  $v \approx 5.3 \text{ m/s} \approx 17.4 \text{ ft/s}$ . A realistic terminal velocity during free-fall is  $v \approx 45 \text{ m/s} \approx 100 \text{ mile/hr}$ . A typical estimate for the characteristic length  $d$  in a flow around an object is the diameter of a disk which presents the same cross-section to the flow: a fully deployed parachute presents a cross-section of  $A \approx 44 \text{ m}^2$  giving  $d \approx 7.5 \text{ m}$ ; a skydiver in spread-eagle formation presents a cross-section of  $A \approx 0.5 \text{ m}^2$  giving  $d \approx 0.8 \text{ m}$ . Thus,  $Re > 10^6$  before and after parachute deployment and the creeping flow approximation is not valid!

For Reynolds numbers  $Re > 10^3$  the drag on a

Table 1. Drag coefficients of common shapes: note the drag coefficient for a cylinder gives force per unit length [12, p. 379, Table 12.2]

Shape	Reynolds number	$C_d$
Hemispherical shell	$Re > 10^3$	1.33
Disc	$Re > 10^3$	1.10
Flat strip	$Re > 10^3$	1.95
Cylinder	$10^3 < Re < 2 \times 10^5$	1.95
Cylinder	$Re > 5 \times 10^5$	$\approx 0.35$
Sphere	$10^3 < Re < 2 \times 10^5$	0.45
Sphere	$Re > 3 \times 10^5$	$\approx 0.20$

body which presents cross-sectional area  $A$  to the flow can be modeled by [12, (pp. 378–9)]:

$$F_d = \frac{1}{2} C_d A \rho v^2. \tag{6}$$

The coefficient of drag  $C_d$  is determined by the shape of the body (see Table 1).

Drag forces are produced by the skydiver’s body, the suspension lines, and the canopy. Several different canopy deployment schemes are discussed in [3] and, as noted on p. 236, “different deployment schemes change the number and magnitude of the impulses felt in re-accelerating the mass components to the velocity of the body”. A “canopy-first” deployment uses the inflating parachute to pull the risers and suspension lines to full extension. In the “lines-first” release the parachute remains in a deployment bag until the risers and suspension lines are fully extended. Combinations of these two schemes are also possible. The deployment schemes differ in the order in which the parachute system separates from the skydiver’s body. A completely realistic model of the deployment is beyond the scope of this discussion. We consider only the lines-first deployment scheme which can be modeled in three distinct stages, starting at time  $t_0$  when the ripcord is pulled. First, the suspension lines are released and become fully extended. At this time,  $t = t_1$ , the *snatch force* pulls the skydiver from the spread-eagle position into an upright position and the canopy begins to inflate. At  $t = t_2$  the canopy is fully inflated, *i.e.*, the first time when the cross-sectional area of the canopy reaches its projected steady-state value. Between times  $t_2$  and  $t_3$  the momentum of the surrounding air mass over-inflates the canopy before returning to the steady-state area for final descent ( $t > t_3$ ).

Both the skydiver’s body and the skydiving equipment generate separate drag forces during the different stages of deployment. Thus, the total drag force is

$$F_d = F_d^b + F_d^e = \frac{1}{2} \rho (C_d^b A^b + C_d^e A^e) v^2$$

where the superscripts  $b$  and  $e$  are used to distinguish the drag coefficients and cross-sectional areas of the skydiver’s body and equipment. This model ignores the drag force produced by the

suspension lines and assumes that the body and equipment are rigidly connected. In reality the suspension lines do produce drag and the entire system is elastic. The appropriate inclusion of these effects would lead to a slightly improved model.

To complete the model the shape and cross-sectional area of the body and equipment are required for each stage of the jump. The standard military parachute is a modification of the T-10, a flat skirt with a 35 ft ( $d_0 = 10.7$  m) nominal diameter and 10% extensions [3]. When fully inflated, the projected diameter is approximately 24.5 ft ( $d_p = 7.47$  m); the cross-sectional area is approximately 471 ft<sup>2</sup> ( $a_1 = 43.8$  m<sup>2</sup>). The suspension lines are 84% of the nominal diameter, *i.e.*,  $l = 8.96$  m. A typical skydiver in the head- or feet-first position can be represented as a 5’10” ( $h = 1.78$  m) long cylinder with cross-sectional area  $b_1 = 0.1$  m<sup>2</sup>. During free-fall, this position is unstable and difficult to maintain for more than a few seconds. In the stable spread-eagle position the body can be modeled as a flat rectangular strip with area  $b_0 = 0.5$  m<sup>2</sup>. The parachute and suspension lines weigh 13.85 lbs, the harness is another 10 lbs and the skydiver weights 190 lbs; the total mass is  $m = (13.85 + 10 + 190)/2.2 = 97.2$  kg. The time  $t_0$  when deployment begins depends on whether the ripcord is pulled by a static line connected to the jump plane, by the skydiver after a specified time delay or at a predetermined altitude. Training jumps for the parachute team at the United States Air Force Academy begin 4000 ft ( $x_0 = 1219$  m) above ground level with a  $t_0 = 10$  s free-fall [13]. (It is interesting to note that a free-fall lasting more than 13 s is grounds for removal from the team. This rule is based on the time needed to deploy the reserve parachute and still be able to make a safe landing. An investigation of the reserve chute is a good student project.) Independent of the value of  $t_0$ , the snatch force occurs around  $t_1 - t_0 = 0.5$  s after the ripcord is pulled and the opening force occurs about  $t_2 - t_1 = 1.0$  s after the snatch force. The total time for this lines-first deployment is approximately 3.2 s, *i.e.*,  $t_3 - t_2 = 1.7$  s.

The extension of the suspension lines can be modeled with a separate initial value problem. However, for simplicity, it will be assumed that the length of the suspension lines increases linearly over the interval  $[t_0, t_1]$ . The cross-sectional area of the canopy could be modeled similarly by assuming a linear increase in the diameter. However, experimental data for the canopy area indicates this would be inappropriate [3, (p. 245)]. Let  $A_{1,2}^e$  denote an appropriate approximation to this data and  $A_{2,3}^e$  the cross-sectional area during over-inflation.

The definitions of the cross-sectional area and drag coefficient for the body and equipment at any time during the jump can be summarized as follows:

Table 2. Parameter estimates for a typical skydiver and equipment

$a_1$	$b_0$	$b_1$	$h$	$l$	$m$	$t_0$	$t_1$	$t_2$	$t_3$
43.8 m <sup>2</sup>	0.5 m <sup>2</sup>	0.1 m <sup>2</sup>	1.78 m	8.96 m	97.2 kg	10 s	10.5 s	11.5 s	13.2 s

$$A^b(t) = \begin{cases} b_0, & t \leq t_0 \\ b_0, & t_0 < t \leq t_1 \\ b_1, & t_1 < t \leq t_2 \\ b_1, & t_2 < t \leq t_3 \\ b_1, & t \geq t_3 \end{cases} \quad (7)$$

$$C_d^b(t) = \begin{cases} 1.95, & t \leq t_0 \\ 1.95, & t_0 < t \leq t_1 \\ 0.35h, & t_1 < t \leq t_2 \\ 0.35h, & t_2 < t \leq t_3 \\ 0.35h, & t \geq t_3 \end{cases} \quad (8)$$

$$A^e(t) = \begin{cases} 0.0, & t \leq t_0 \\ b_1, & t_0 < t \leq t_1 \\ A_{1,2}^e(t), & t_1 < t \leq t_2 \\ A_{2,3}^e(t), & t_2 < t \leq t_3 \\ a_1, & t \geq t_3 \end{cases} \quad (9)$$

$$C_d^e(t) = \begin{cases} 0.0, & t \leq t_0 \\ 0.35l \frac{t-t_0}{t_1-t_0}, & t_0 < t \leq t_1 \\ 1.33, & t_1 < t \leq t_2 \\ 1.33, & t_2 < t \leq t_3 \\ 1.33, & t \geq t_3. \end{cases} \quad (10)$$

Estimates of the parameters in equations (7)–(10) for a typical skydiver and equipment are collected in Table 2.

The improved model for the velocity of the skydiver is the nonlinear initial value problem:

$$m \frac{dv}{dt} = -mg + kv^2, \quad v(0) = 0 \quad (11)$$

where

$$k = \frac{1}{2} \rho (C_d^b A^b + C_d^e A^e)$$

$$= \frac{1}{2} \rho \begin{cases} 1.95b_0, & t \leq t_0 \\ 1.95b_0 + 0.35b_1 l \frac{t-t_0}{t_1-t_0}, & t_0 < t \leq t_1 \\ 0.35b_1 h + 1.33A_{1,2}^e(t), & t_1 < t \leq t_2 \\ 0.35b_1 h + 1.33A_{2,3}^e(t), & t_2 < t \leq t_3 \\ 0.35b_1 h + 1.33a_1, & t \geq t_3. \end{cases} \quad (12)$$

Observe that equation (11) with (12) differs from

the original model, equation (2) with (1), in two important ways. The linear initial value problem equation (2) is replaced by the nonlinear problem (11) and the coefficient of drag in (12) contains significantly more real-world modeling than the piecewise constant function in (1). The improved model will be complete when the functions  $A_{1,2}^e$  and  $A_{2,3}^e$  are defined. This is deferred until appropriate smoothness and transition conditions are developed in the next section.

### ANALYSIS OF THE MODEL

Many introductory courses omit the section on existence and uniqueness theory. When it is included, the typical “application” of the theory is to determine the intervals for which a solution is known to exist or the initial conditions for which a unique solution exists for all time. The analysis of the model derived above is based on a simple application of the standard existence and uniqueness theory for first-order initial value problems [5, 11]. This use of the theory provides a more realistic example of the utility of theoretical results. The theorems and their proofs are not difficult for students to understand.

**Theorem.** Assume  $A_{1,2}^e$  is continuous on  $(t_1, t_2)$  and  $A_{2,3}^e$  is continuous on  $(t_2, t_3)$ . There is exactly one continuous solution to equation (11) on  $t > 0$ .

*Proof.* The idea is to apply the classical existence and uniqueness theory on each sub-interval. The hypotheses guarantee that the coefficient is continuous on each of the five subintervals.

Consider the initial value problem (11) on  $(0, t_0)$ . The standard theory provides a unique solution in the space of differentiable functions on  $(0, t_0)$ . Use the value of this solution as the initial condition to create an IVP on  $(t_0, t_1)$ . This problem has a unique differentiable solution on  $(t_0, t_1)$ . In the same way, differentiable solutions are obtained on  $(t_1, t_2)$ ,  $(t_2, t_3)$ , and  $(t_3, \infty)$ . The piecewise-defined function obtained by assembling each solution on the appropriate interval is a solution to equation (11) for all  $t > 0$ .

Note that the solution guaranteed by the above theorem is continuous on  $(0, \infty)$  but may fail to be differentiable at any of  $t_0, t_1, t_2$ , and  $t_3$ .

To investigate the smoothness of the solutions at the endpoints of the different stages of the jump, recall that the acceleration can be obtained directly from the differential equation:  $a = dv/dt = -g + (1/m)kv^2$ . Since  $g$  and  $m$  are constants and  $v$  is continuous (and non-zero), the acceleration is

continuous when  $k$  is continuous. This proves the following corollary to the theorem.

**Corollary.** *The solution to equation (11) is differentiable on  $(0, \infty)$  if and only if  $k$  is continuous on  $(0, \infty)$ .*

The continuity of  $k$ , as given in equation (12), is fairly easy to determine. Continuity of  $A_{1,2}^e$  and  $A_{2,3}^e$  on their respective subintervals ensures continuity of  $k$  on the interior of each subinterval. The choice of the linear function to model the extension of the suspension lines guarantees continuity at  $t_0$ . Continuity at  $t_1$  requires that:

$$1.95b_0 + 0.35b_1l = 0.35b_1h + 1.33A_{1,2}^e(t_1).$$

Continuity at  $t_2$  follows when  $A_{1,2}^e(t_2) = A_{2,3}^e(t_2)$ . Note that the definition of  $t_2$  as the time when the opening shock is felt implies that the cross-sectional area is  $a_1$ . Lastly,  $A_{2,3}^e(t_3) = a_1$  implies that  $k$  is continuous at  $t_3$ . These findings are summarized in the following lemma.

**Lemma 1.** *If  $A_{1,2}^e$  is continuous on  $(t_1, t_2)$  with boundary conditions*

$$A_{1,2}^e(t_1) = \frac{1.95b_0 + 0.35b_1(l - h)}{1.33} \text{ and } A_{1,2}^e(t_2) = a_1$$

*and if  $A_{2,3}^e$  is continuous on  $(t_2, t_3)$  with boundary conditions*

$$A_{2,3}^e(t_2) = A_{2,3}^e(t_3) = a_1,$$

*then  $k$  is continuous on  $(0, \infty)$ .*

The time derivative of the acceleration is the jerk,  $j = da/dt$ . Differentiating equation (9) produces:

$$\begin{aligned} j(t) &= \frac{v(t)}{m}(k'(t)v(t) + 2k(t)a(t)) \\ &= \frac{v(t)}{m} \left( \frac{2}{m}k(t)^2v(t)^2 + k'(t)v(t) - 2gk(t) \right) \end{aligned} \quad (13)$$

which immediately gives conditions under which the acceleration is differentiable.

**Lemma 2.** *If, in addition to the conditions in Lemma 1 above,  $A_{1,2}^e$  and  $A_{2,3}^e$  are differentiable, then the jerk is piecewise continuous on  $(0, \infty)$ . Discontinuities can occur only at the endpoints of the subintervals.*

*Proof.* The additional hypotheses ensure that  $k$  is piecewise continuous on  $(0, \infty)$ . The piecewise continuity of the jerk is now apparent.

Note that equation (13) can be used to show that if  $k$  is differentiable on  $(0, \infty)$  the jerk is continuous and, hence, the acceleration is smooth. The additional constraints on  $A_{1,2}^e$  and  $A_{2,3}^e$  necessary to make  $k$  differentiable are not difficult to obtain, but will not be pursued further in this paper.

Experimental data for the canopy area during deployment of the T-10 is presented in [3, (p. 246, Figure 6.10B)]. The area appears to be essentially exponential. (This is further confirmed by the information for a 28-ft solid flat circular parachute [3, (p. 245, Figure 6.10A)].) Let:

$$A_{1,2}^e(t) = \alpha_0 e^{\beta_0(t-t_1)/(t_2-t_1)}.$$

The conditions in Lemma 1 are satisfied when:

$$\alpha_0 = \frac{1.95b_0 + 0.35b_1(1 - h)}{1.33} \text{ and } \beta_0 = \log\left(\frac{a_1}{\alpha_0}\right).$$

Specific information about overinflation for the T-10 is more difficult to obtain. One plausible, simple functional form satisfying the hypotheses of Lemma 1 is:

$$A_{2,3}^e(t) = \alpha_1 \left( 1 + \beta_1 \sin\left(\pi \frac{t - t_2}{t_3 - t_2}\right) \right)$$

where the parameter  $\beta_1$  represents the relative increase in cross-sectional area above the nominal projected area ( $d_p$ ). Experimental data suggests that the maximum cross-sectional area is approximately 115% of  $d_p$ . Thus, a reasonable choice is  $\beta_1 = 0.15$ .

The model and values for all of its parameters are now completely determined. A numerical solution of the problem can be created and graphed using a software package such as *Maple*, *Mathematica*, or *MATLAB*. Special care must be exercised when plotting a discontinuous function, particularly one defined as the solution of an initial value problem. The most common numerical methods for initial value problems assume the solution has a certain smoothness. One way to avoid this problem is to use a numerical method to compute the solution to equation (11) and compute the acceleration and jerk in terms of the velocity as discussed above.

### VERIFICATION OF THE MODEL

Prior to looking at a numerical approximation to the solution, recall that the terminal velocity can be determined directly from equation (11):

$$v_T = -\sqrt{\frac{mg}{k}} = -\sqrt{\frac{2mg}{\rho(C_d^b A^b + C_d^e A^e)}}. \quad (14)$$

With  $k$  as defined in equation (12) and the numerical parameters given above, the free-fall terminal velocity is  $v_T \approx -44.2$  m/s  $\approx -98.9$  miles/hr while the impact velocity should be approximately  $v_T \approx -5.72$  m/s  $\approx -12.8$  miles/hr. The free-fall terminal velocity is exceptionally close to the 100 miles/hr estimate given above. The impact velocity is about 10% higher than the landing

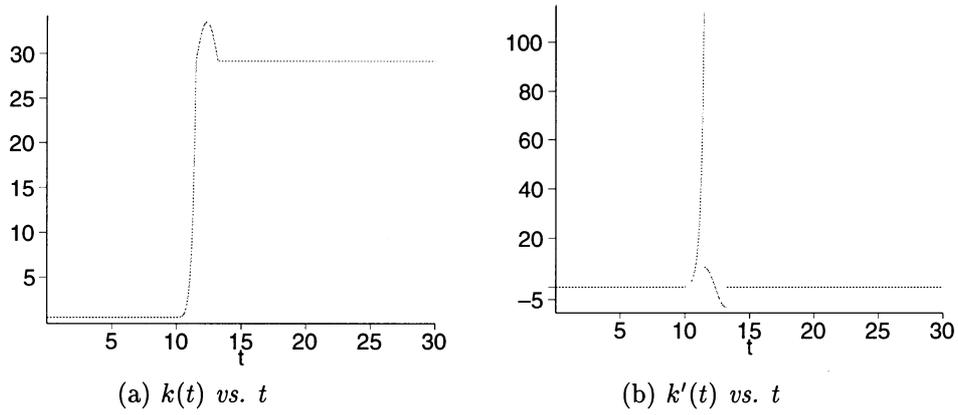


Fig. 1. Plots used to estimate time of impact: (a) the drag coefficient  $k$  and (b) its derivative  $k'$  during the first 30 seconds of the jump.

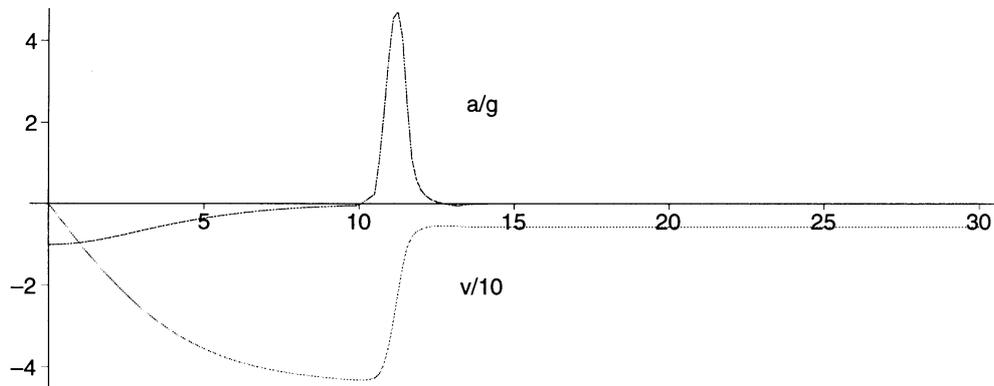


Fig. 2. Velocity and acceleration for the first 30 seconds of the jump.

velocity for a free-fall from a 5 ft wall. A quick calculation shows that to decrease the impact velocity to  $v_T \approx -5.5$  m/s the drag coefficient for  $t > t_3$  by would have to increase about 10% from its current value of 29.16 kg/m to approximately 32.21 kg/m. One possible source for this extra drag is the suspension lines.

According to the analysis above, the velocity and acceleration should be continuous and the jerk

should be piecewise continuous. Figure 1(a) shows that the drag coefficient is continuous. It is difficult to see that  $k'$  has a jump discontinuity at  $t = t_0$ , all other jump discontinuities in  $k'$  are clearly visible in Figure 1(b). A good reinforcement of the discontinuity of  $k'$  is to have the students explicitly compute the derivative of equation (12) and check for continuity at the transitions from one stage to the next.

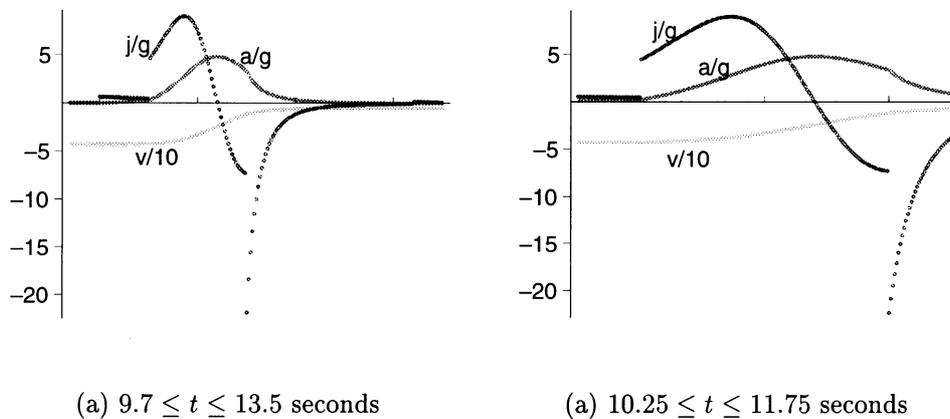


Fig. 3. Velocity, acceleration, and jerk (a) during canopy deployment and (b) showing the snatch ( $t = 10.5$ ) and opening forces ( $t = 11.5$ ).

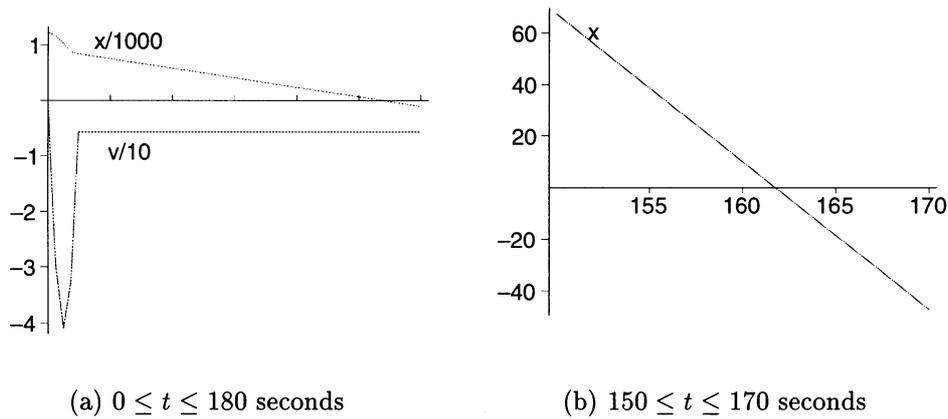


Fig. 4. Plots used to estimate time of impact: (a) position and velocity during first three minutes of jump and (b) position near time of landing ( $150 \leq t \leq 170$ ).

Figure 2 provides additional verification of the terminal velocity and smoothness results. The spike in the acceleration contains both the snatch force and opening force. Figure 3 provides a closer look at the velocity, acceleration, and jerk during each stage of canopy deployment. This model does not do a good job of capturing the snatch force. To obtain a larger snatch force it would be necessary to have a larger jump in  $k'$  at  $t_1$ . A more careful modeling of the line extension and inclusion of the canopy's porosity are two ways to improve this part of the model. The opening force has a magnitude of approximately 4.9g; this is in strong agreement with the data presented in [3, Figure 6.7(b)].

The analysis of the solution concludes with an estimate of the time when the skydiver returns to solid ground. Notice that once the motion approaches terminal velocity, the position is essentially linear. Simplifying even further, the motion appears to be piecewise linear with slope given by the free-fall and final descent terminal velocities, respectively. Under this assumption, the skydiver falls approximately  $t_0|v_T| \approx 442$  m during free-fall and spends a little less than 140 s in final descent. As this analysis overestimates the velocity during free-fall, this actual landing time should be slightly longer than 150 s. The landing time predicted by Figure 4(b) is 162 s.

*Maple* reports that at  $t = 162$  s,  $x \approx -1.34763$  m and  $v \approx -5.71886$  m/s. Since the motion is essentially linear at this time, linear interpolation yields an improved landing time of  $t = 161.674$  s. At this time, *Maple* reports the height is about 2 cm—less than one inch!

## CONCLUSION

Some of the new pedagogical methods being used to teach an introductory course in differential equations have been illustrated. While knowledge of solution techniques is still essential, modeling, qualitative analysis, and the mathematical theory of ordinary differential equations are also quite important. Each of these topics played an important role in the development and analysis of an improved model for a parachute jump.

While the new model is an improvement over the model found in traditional textbooks and recent journal articles, it does not include all of the physics. The derivation of models that eliminate some of these simplifications make excellent student projects. Specific suggestions for improving the model are to consider the elasticity and drag forces of the suspension lines and to include the porosity of the canopy. A more challenging exercise is to derive a model for all three components of motion and some of the handling characteristics of the parachute. Note, in particular, that a tangential velocity component allows for faster final descents without sacrificing safety (at least in terms of vertical landing forces).

The authors used *Maple* and *Mathematica* to assist with some of the symbolic manipulations, numerical computations and visualization. Copies of a supplemental *Maple* worksheet and *Mathematica* notebook can be found on the authors' homepages. The URLs are:

<http://www.math.sc.edu/~meade/publ.html>  
<http://www.math.mtu.edu/~struther/publ.html>

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