#1. (a) \( A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \) \( \lambda_1 = -2 \) \( \lambda_2 = 1 \) so \( K = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \).

\( \lambda = -2 : \quad A - (-2)I = \begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix} \quad \text{so } u = [1, 0]^T \)

\( \lambda = 1 : \quad A - I = \begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{so } \text{null space basis } u = [1]^T \).

This gives us that \( A \) is diagonalized by \( P = [1, 1]^T \).

(b) \( A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \) has eigenvalues 3 to 0 so \( K = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \).

\( \lambda = 3 : \quad A - 3I = \begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix} \quad \text{so } u = [1, 0]^T \).

\( \lambda = 0 : \quad A - 0I = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \) has null space basis \( u = [1]^T \).

This matrix is also diagonalized by \( P = [1, 1]^T \).

(c) \( A = \begin{bmatrix} -2 & 0 \\ -6 & 1 \end{bmatrix} \) has eigenvalues -2 1 so \( K = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \).

\( \lambda = -2 : \quad A - (-2)I = \begin{bmatrix} 0 & 0 \\ -6 & 3 \end{bmatrix} \quad \begin{bmatrix} -2 & 0 \\ -6 & 3 \end{bmatrix} \quad \text{so } u = [1, 2]^T \).

\( \lambda = 1 : \quad A - I = \begin{bmatrix} 0 & 0 \\ -6 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ -6 & 0 \end{bmatrix} \quad \text{so } u = [1]^T \).

This matrix is diagonalized by \( P = [1, 0]^T \).
#2. \( M = \begin{bmatrix} 1 & -35 \\ 6 & -13 \end{bmatrix} \) has eigenvectors \( u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \).

(a) \( M \) is diagonalized by \( P = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \).

To find the eigenvalues of \( P \) we can compute (as in \( \text{87.1} \))

\[
M_u = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 2u \quad \text{so \ these \ eigenvector \ has \ \lambda = 2}.
\]

\[
M_v = \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 1u \quad \lambda = 1.
\]

Thus, \( K = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \), and we can write \( M = PKP^{-1} \).

(b) \( M^3 = (PKP^{-1})^3 = PK^3P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix} \]

\[
= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 24 & -54 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 100 & -245 \\ 42 & -97 \end{bmatrix}
\]

(c) \( M^{-1} = (PKP^{-1})^{-1} = PK^{-1}P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix} \]

\[
= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -7/2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -13/2 & 37/2 \\ -3 & -8 \end{bmatrix}
\]

#5. Claim. If \( A \) is similar to \( B \) and \( A \) is singular, then \( B \) must also be singular.

Proof: If \( A \) is singular, then at least one of \( A \)'s eigenvalues must be zero.

Because \( A \) is \( B \) are similar, they have the same eigenvalues.

Thus, at least one of \( B \)'s eigenvalues will also be zero which means \( B \) is singular. \( \square \)
#6. (a) A diagonalizable means A has a basis of eigenvectors.  
A invertible means A has non-zero eigenvalues.  
If A has a zero eigenvalue and a basis of eigenvectors 
then A is diagonalizable and not invertible.  
False  
(b) Similar to (a) except now if A has all eigenvalues non-zero 
and is deficient, then A is invertible & not diagonalizable.  
False.  
(c) If λ is an eigenvalue of A with algebraic multiplicity n  
it is diagonalizable only when λ also has geometric  
multiplicity n. Since this is not guaranteed, this  
statement is False.  
(d) If A is diagonalized by P, then 
\[ A = P \Lambda P^{-1} \] and \[ A^2 = P \Lambda^2 P^{-1} \]  
so that \[ A^2 \] is also diagonalized by P. False  
(e) If A has n distinct eigenvalues, then each of their  
eigenspaces has dimension 1 and there is a basis of eigenvectors. 
This means A can be diagonalized. True  

#8. \[ H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \] has eigenvalue \( \lambda = 1 \) with algebraic multiplicity 2.  
\[ H - 1I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \] has \( 1 \) free variable so the geometric multiplicity is 1.  
H cannot be diagonalized because H turns all vectors not on the  
Geometric x-axis to the right (HV is not parallel to v when v is not on the x-axis).  
* Use Mode: H: = Matrix ([[1,1], [0,1]]);  
Head1t (H)