#5. This is similar to §5.2 #15. We know \( \{v_1, ..., v_n\} \) is a linearly independent set in \( \mathbb{R}^k \) and want to conclude that \( \{Av_1, ..., Av_n\} \) is also linearly independent (assuming \( A \) is invertible). Consider
\[
d_1(Av_1) + d_2(Av_2) + ... + d_n(Av_n) = A(d_1v_1 + d_2v_2 + ... + dv_n) = 0.
\]
Because \( A \) is invertible the only solution to \( Ax = 0 \) is \( x = 0 \). So, \( d_1v_1 + d_2v_2 + ... + dv_n = 0 \) and this is true only when \( d_1 = d_2 = ... = d_n = 0 \). So, \( \{Av_1, Av_2, ..., Av_n\} \) must be linearly independent. Because there are not these vectors they must also span \( \mathbb{R}^n \), hence they are a basis for \( \mathbb{R}^n \).
7. (a) $W_1 = \text{Span}\{u_1, \ldots, u_k\}$ and $W_2 = \text{Span}\{v_1, \ldots, v_k\}$

with $W_1 \cap W_2 = \{0\}$.

To show that $\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$ is lin. independent, look at
\[c_1 u_1 + \ldots + c_j u_j + d_1 v_1 + \ldots + d_k v_k = 0\]

$\iff c_1 u_1 + \ldots + c_j u_j = -(d_1 v_1 + \ldots + d_k v_k)$

Now, the only way a linear combination of the $u_i$ can be equal to a linear combination of the $v_i$ is when both linear combinations are zero:
\[c_1 u_1 + \ldots + c_j u_j = 0 \implies c_1 = c_2 = \ldots = c_j = 0\]
\[d_1 v_1 + \ldots + d_k v_k = 0 \implies d_1 = d_2 = \ldots = d_k = 0\]

so the set of $j+k$ vectors is linearly independent.

(b) $W_1 = \text{Span}\{u_1\}$, $W_2 = \text{Span}\{v_1, v_2\}$, with $W_1 \cap W_2 = \{0\}$.

Then $\{u_1, v_1, v_2\}$ is a linearly independent set of vectors in $\mathbb{R}^3$. Thus $\text{Span}\{u_1, v_1, v_2\}$ has dimension 3, i.e. $\text{Span}\{u_1, v_1, v_2\} = \mathbb{R}^3$.

$\dim W_1 + \dim W_2 = \dim (\mathbb{R}^3)$. 