Instructions:

1. There are a total of 4 problems on 5 pages. Check that your copy of the exam has all of the problems.

2. Calculators may not be used for any portion of this exam.

3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.

4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

5. Check your work. If I see clear evidence that you checked your answer (when possible) and you clearly indicate that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

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I hope you have a great Fall Break!
1. (24 points) [12 points each]
   (a) Find the general solution to $y'' - 2y' + 3y = 0$.

   \[ y = e^{rt} : \quad P(r) = r^2 - 2r + 3 = 0. \]

   \[ r = \frac{1}{2} \left( 2 \pm \sqrt{4 - 12} \right) = \frac{1}{2} \left( 2 \pm \sqrt{-8} \right) = \frac{1}{2} \left( 2 \pm 2\sqrt{2}i \right) = 1 \pm \sqrt{2}i. \]

   \[ y_1 = e^t \cos(\sqrt{2}t) \]

   \[ y_2 = e^t \sin(\sqrt{2}t) \]

   \[ y(t) = c_1 y_1 + c_2 y_2 = c_1 e^t \cos(\sqrt{2}t) + c_2 e^t \sin(\sqrt{2}t) \]

   (b) Find the general solution to $y''' - 3y'' + 3y' - y = 0$.

   \[ y = e^{rt} : \quad P(r) = r^3 - 3r^2 + 3r - 1 = 0 \]

   \[ (r-1)^3 = 0 \]

   \[ r = 1 \text{ (mult. 3)}. \]

   \[ y_1 = e^t \]

   \[ y_2 = te^t \]

   \[ y_3 = t^2 e^t \]

   \[ y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3 \]

   \[ = c_1 e^t + c_2 te^t + c_3 t^2 e^t. \]
2. (20 points) [5 points each — yes, I know $5 	imes 5 = 25$; there are 5 extra points in this problem]

(a) An object with mass 10 kg oscillates on the end of a spring with period 2 s. Explain why the spring constant is $k = 10\pi^2$.

The differential equation is $my'' + ky = 0$.

Because the period is $2$, we know $y = c_1 \cos(\pi t) + c_2 \sin(\pi t)$; i.e., $r = \pm \pi i$ are the roots of $\text{Re}(r) = 0$. Thus: $m(\pi i)^2 + k = 0$

\[ k = -m(-\pi^2) = 10\pi^2. \]

**NOTE**: Assume there is no damping.

(b) Continuing (a), for what value of the damping coefficient, $b$, is the motion critically damped?

\[ m' + by' + ky = 0. \]

\[ 10y'' + by' + 10\pi^2 = 0. \]

\[ y = e^{rt}; \quad \text{Re}(r) = 10r^2 + 6r + 10\pi^2 = 0. \]

\[ r = \frac{1}{2} \left( \frac{-6 \pm \sqrt{36 - 4 \times 10\pi^2}}{20} \right) \]

Critically damped when $b^2 - 400\pi^2 = 0$.\[ b = 20\pi \]

(c) Find the Wronskian of $y_1 = \ln(x)$ and $y_2 = x \ln(x)$.

\[ W[y_1, y_2] = \det \begin{bmatrix} \ln(x) & x \ln(x) \\ \frac{1}{x} & 1 + x \ln(x) \end{bmatrix} = \det \begin{bmatrix} \ln(x) & x \ln(x) \\ \frac{1}{x} & \ln(x) + 1 \end{bmatrix} \]

\[ = (\ln(x))^2 - \frac{1}{x} \times x \ln(x) = (\ln(x))^2 + \ln(x) - \ln(x) \]

\[ = (\ln(x))^2. \]

(d) Explain why the two functions in (c) could not be solutions to a homogeneous second-order linear differential equation on the interval $(0, \infty)$.

Notice that $W[y_1, y_2](1) = (\ln(1))^2 = 0$, but that $W[y_1, y_2](x) \neq 0$ for all $x > 0$. By Abel’s Theorem, the Wronskian of $2$ lin indep solns must be either always zero or never zero.

(e) Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions to the $n$th-order linear differential equation

\[ y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = 0. \]

Use Abel’s Theorem to show that the Wronskian, $W[y_1, y_2, \ldots, y_n]$, is a constant.

**NOTE**: Note that there is no $y^{(n-1)}$ term.

\[ W' = -p \cdot W = 0 \implies W = \text{constant} \]

(by Abel’s Theorem)
3. (28 points) Consider the differential equation

\[ \sin(x) y'' + (1 + \sin(x)) y' + y = 0. \]

(a) [10 points] Show that any solution of the form \( y = e^{rx} \) must satisfy

\[ (r^2 + r) \sin(x) + (r + 1) = 0. \]

\[ y = e^{rx}, \quad y' = r e^{rx}, \quad y'' = r^2 e^{rx} \]

\[ 0 = \sin(x) y'' + (1 + \sin(x)) y' + y = \]

\[ = \sin(x) \left( r^2 e^{rx} (1 + \sin(x)) e^{rx} + e^{rx} \right) \]

\[ = e^{rx} \left( (r^2 + r) \sin(x) + (r + 1) \right) \quad \text{so} \quad (r^2 + r) \sin(x) + (r + 1) = 0. \]

(b) [10 points] Because the above condition must be satisfied for all \( x \), the only solution is \( r = -1 \). That is, \( y_1 = e^{-x} \). A second solution will be sought in the form \( y_2(x) = v(x) y_1(x) \). Find the first-order differential satisfied by \( u = v' \).

\[ y_2 = v e^{-x}; \quad y_2' = v' e^{-x} - v e^{-x} \]

\[ y_2'' = v'' e^{-x} - 2v' e^{-x} + v e^{-x} \]

\[ 0 = \sin(x) \left( v'' e^{-x} - 2v' e^{-x} + v e^{-x} \right) + \sin(x) v' e^{-x} + \sin(x) v e^{-x} \]

\[ = \sin(x) e^{-x} v'' + (-2 \sin(x) + 1 + \sin(x)) v' + (1 + \sin(x) + 1) e^{-x} \]

\[ 0 = \left( \sin(x) v'' + (-2 \sin(x) + 1 + \sin(x)) v' + 0 v \right) e^{-x} \]

Because \( e^{-x} \neq 0 \) for all \( x \) real:

\[ \sin(x) v'' + (1 - 2 \sin(x)) v' = 0. \]

Let \( u = v' \):

\[ \sin(x) u'' + (1 + \sin(x)) u = 0 \]

(c) [4 points] With initial conditions \( y(\pi/4) = 1, y'(\pi/4) = -1 \), what is the largest interval on which a unique solution is guaranteed to exist by the Existence and Uniqueness Theorem (Theorem 2.4.2)?

\[ y' + \frac{1 + \sin(x)}{\sin(x)} y = 0. \]

Because \( \sin(x) \neq 0 \) for \( x \in (0, \pi/2) \), the E.U. Theorem tells us the solution must exist for at least \( 0 < x < \pi/2 \).

(d) [4 points] With initial conditions \( y(0) = 1, y'(0) = -1 \), what is the largest interval on which a unique solution is guaranteed to exist by the Existence and Uniqueness Theorem (Theorem 2.4.2)?

Because \( \sin(0) = 0 \), the work of \( y' \) is not continuous at \( x = 0 \), the E.U. Theorem has nothing to say about this problem.
4. (28 points) Consider the Cauchy-Euler equation

\[ x^2y'' - 2y = 0. \]

(a) [12 points] Find the general solution.

\[ x = e^t: \quad y'' - y' - 2y = 0. \]

\[ y = e^{rt} \quad r^2 - r - 2 = (r - 2)(r + 1) \]

\[ r = 2, -1 \]

\[ y_1 = e^{2t} \]

\[ y_2 = e^{-t} \]

\[ y = c_1 e^{2t} + c_2 e^{-t} \]

\[ = c_1 x^2 + c_2 x^{-1} \]

(b) [8 points] Find the solution that satisfies the initial conditions

\[ y(1) = \alpha \quad y'(1) = 2. \]

\[ y(1) = c_1 + c_2 = \alpha \]

\[ y'(1) = 2c_1 - c_2 = 2 \]

\[ 3c_1 = 2 + \alpha \]

\[ c_1 = \frac{2 + \alpha}{3} \]

\[ c_2 = \alpha - c_1 = \alpha - \frac{2 + \alpha}{3} = \frac{\alpha - 2 - \alpha}{3} = \frac{2 - \alpha}{3} \]

\[ y = \frac{2 + \alpha}{3} x^2 + \frac{2 - \alpha}{3} x^{-1} \]

(c) [4 points] For what value(s) of \( \alpha \) does the solution approach 0 as \( x \to \infty \)?

\[ \lim_{x \to \infty} y(x) = 0 \quad \text{only when} \quad \frac{2 + \alpha}{3} = 0, \quad \text{i.e.} \quad \alpha = -2 \]

(d) [4 points] For what value(s) of \( \alpha \) is the solution bounded as \( x \to 0^+ \)?

\[ \lim_{x \to 0^+} y(x) \text{ exists only when} \quad \frac{2\alpha - 2}{3} = 0, \quad \text{i.e.} \quad \alpha = 1 \]