Notice that the inside front cover of the text book has a list of common distributions. It gives formulas for the means and variances of each and the m.g.f. for most.

2. In this exercise one should try to recognize the distribution type by the given m.g.f. and then use the formulas to solve the rest of the problem. Also, remember that when specifying a distribution, it is equally important to specify the parameters as well.

a. \( M(t) = (0.3 + 0.7e^t)^5 \)
   
i. binomial; \( n=5, \ p = 0.7 \)
   \[ \mu = np = (5)(0.7) = 3.5 \]
   \[ \sigma^2 = np(1-p) = 1.05 \]
   
   iii. \( P(1 \leq X \leq 2) = P(X = 1) + P(X = 2) = 5C_1 (0.7)^1(0.3)^4 + 5C_2 (0.7)^2(0.3)^3 \)
   \[ = 0.0284 + 0.1323 = 0.1607 \]

   b. \( M(t) = 0.3e^t \), \( t < -\ln(0.7) \)
   
i. geometric; \( p = 0.3 \)
   \[ \mu = 1/p = 10/3 \]
   \[ \sigma^2 = (1 – p) / p^2 = (0.7)/(0.09) = 7.7778 \]
   
   iii. \( (1 \leq X \leq 2) = P(X = 1) + P(X = 2) = (0.7)^0(0.3)^1 + (0.7)^1(0.3)^1 \)
   \[ = 0.3 + 0.21 = 0.51 \]

   d. \( M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t} \)
   
i. \( f(x) = \{ 0.3, \ x = 1 \)
   \[ \] 0.4, \ x = 2
   \[ \] 0.2, \ x = 3
   \[ \] 0.1, \ x = 4 \}
   , this is not a common distribution.
   
   ii. \( \mu = E(X) = M'(0) = 0.3e^{(0)} + 0.8e^{2(0)} + 0.6e^{3(0)} + 0.4e^{4(0)} = 2.1 \)
   \[ \sigma^2 = E(X^2) - \mu^2 = M''(0) - \mu^2 = 0.3e^{(0)} + 1.6e^{2(0)} + 1.8e^{3(0)} + 1.6e^{4(0)} - \mu^2 = \]
   \[ = 5.3 - 4.41 = 0.89 \]
   
   iii. \( P(1 \leq X \leq 2) = P(X = 1) + P(X = 2) = 0.3 + 0.4 = 0.7 \)

   f. \( M(t) = \sum_{x=1}^{10} (0.1) e^{tx} \)
   
i. uniform; \( m = 10 \)
   
   ii. \( \mu = (m + 1)/2 = 11/2 = 5.5 \)
   \[ \sigma^2 = (m^2 – 1) / 12 = 99/12 = 8.25 \]
   
   iii. \( P(1 \leq X \leq 2) = P(X = 1) + P(X = 2) = 0.1 + 0.1 = 0.2 \)

7. Let \( X = \) the number of bags under three pounds
   \( X = \) geometric; \( p = 0.04 \)

   a. \( P(X \geq 20) = P(X > 19) = (1 – p)^{19} = (0.96)^{19} = 0.4604 \)
   b. \( P(X \leq 20) = 1 - (1 – p)^{20} = 1 - (0.96)^{20} = 0.5580 \)
   c. \( P(X = 20) = (0.96)^{19}(0.04) = 0.0184 \)
13. Let $X$ = number of flips of a fair coin needed to observe heads-tails in consecutive flips.

This problem has a negative binomial distribution since this is a sequence of Bernoulli trials and we wish to find the probability that in $x$ number of trials, we will observe exactly two “successes”. The fact that one the first success is a heads and the other a tails does not complicate things since they both have the same probability.

a. $X =$ negative binomial: $r = 2, p = 0.5$

$$f(x) = \binom{x-1}{2-1} (0.5)^2 (0.5)^{x-2} = (x-1)(0.5)^x, \ x = 2, 3, 4, \ldots$$

b. $\mu = r/p = (2)/(1/2) = 4$

$$\sigma^2 = r(1-p)/p^2 = (2)(0.5)/(0.25) = 4$$

$$\sigma = \sqrt{\sigma^2} = 2$$

c.

i. $P(X \leq 3) = P(X = 2) + P(X = 3) = [(2 - 1)/4)] + [(3 - 1)/ 8] = 0.5$

ii. $P(X \geq 5) = 1 - P(X \leq 4) = 1 - [P(X = 2) + P(X = 3) + P(X = 4)]$

$$= 1 - [(1/4) + (1/4) + (3/16)] = 1 - (11/16) = 5/16$$

iii. $P(X = 3) = 1/4$

21. $M(t) = (44/120) + (45/120)e^t + (20/120)e^{2t} + (10/120)e^{3t} + (1/120)e^{5t}$

a. $\mu = E(X) = M'(0) = [0 + 45 + 40 + 30 + 5]/120 = 120/120 = 1$

$$\sigma^2 = E(X^2) - \mu^2 = M''(0) - \mu^2 = [0 + 45 + 80 + 90 + 25]/120 - 1$$

$$= (240/120) - 1 = 1$$

b. $P(X \geq 1) = 1 - P(X=0) = 1 - (44/120) = 76/120 = 0.63$

Notice that from the m.g.f. one can rebuild the p.m.f. of $x$.

$$f(x) = \begin{cases} 
(44/120), & x = 0 \\
(45/120), & x = 1 \\
(20/120), & x = 2 \\
(10/120), & x = 3 \\
(0/120), & x = 4 \\
(1/120), & x = 5 
\end{cases}$$

c.
**Extra Credit**

19. In this problem, it suffices to show that $M(0) = 1$, since in that case

$R'(0) = M'(0) = E(X) = \mu$

$R''(0) = M''(0) - [M'(0)]^2 = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2 = \sigma^2$

$M(t) = \sum_{x \in S} f(x)e^{tx}$

$M(0) = \sum_{x \in S} f(x)e^{(0)x} = \sum_{x \in S} f(x) = 1 \square$

This makes sense because $M'(0)$ and $M''(0)$ give the first and second moments respectively. So $M(0)$ would give the zeroth moment or the sum of the p.m.f. over the entire sample space. And of course, all p.m.f. 's by definition sum to one over their sample space.

20b. Use the result from 19 to find the mean and variance of the binomial distribution.

$M(t) = (1 - p + pe^{tx})^n$, $R(t) = \ln[(1 - p + pe^{tx})^n] = n \ln(1 - p + pe^{tx})$

$R'(t) = \frac{npe^{tx}}{1 - p + pe^{tx}}$ $R''(t) = \frac{(1 - p + pe^{tx})(npe^{tx}) - (npe^{tx})(pe^{tx})(1 - p + pe^{tx})}{(1 - p + pe^{tx})^2}$

$R'(0) = \frac{np}{1 - p + p} \square$ $R''(0) = \frac{(1 - p + p) np - (np)(p)}{(1 - p + p)^2}$

$R'(0) = np \square$ $R''(0) = np - (np)(p) = np(1 - p) \square$