Instructions:

1. There are a total of 4 problems (including the Extra Credit problem) on 6 pages. Check that your copy of the exam has all of the problems.

2. *All work must be shown* to receive credit for a correct answer. (A brief description of your logic is also acceptable.)

3. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

4. *No calculators!* If you believe you need to use a calculator you are doing something wrong!!

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Good Luck!
1. (24 points) [6 points each] Evaluate each limit. Remember to indicate every application of l'Hôpital's Rule.

(a) \( \lim_{x \to 0} \frac{\cos x}{x^2} = +\infty \) because \( \lim_{x \to 0} \cos x = 1 \)

and \( \lim_{x \to 0} x^2 = 0 \) (from the right).

(b) \( \lim_{t \to 0^+} \sqrt{t} \ln t = \lim_{t \to 0^+} \frac{\ln t}{t^{1/2}} = \lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{2}t^{-3/2}} = \lim_{t \to 0^+} -2t^{1/2} = 0. \)

(c) \( \lim_{t \to 0} \frac{\ln ((1 + t)^2)}{t} = \lim_{t \to 0} \frac{2 \ln (1 + t)}{t} = \lim_{t \to 0} \frac{2(1 + t)}{1} = \lim_{t \to 0} 2(1 + t) = 2. \)

(d) \( \lim_{\theta \to 0} \frac{\sin \theta - \tan \theta}{\theta^2} = \lim_{\theta \to 0} \frac{\cos \theta - \sec^2 \theta}{2\theta} = \lim_{\theta \to 0} \frac{-\sin \theta - 2\sec \theta (\sec \theta \tan \theta)}{2} \)

\( = \lim_{\theta \to 0} \frac{-\sin \theta - 2 \frac{\sin \theta}{\cos \theta}}{2} = \frac{0}{2} = 0. \)

Note: There are many other ways to work this problem.
2. (24 points) [6 points each] Evaluate each definite integral.

(a) \[ \int_{-\infty}^{0} e^{2x} \, dx = \lim_{A \to -\infty} \int_{A}^{0} e^{2x} \, dx = \lim_{A \to -\infty} \frac{1}{2} e^{2x} \bigg|_{A}^{0} = \lim_{A \to -\infty} \left( \frac{1}{2} e^{0} - \frac{1}{2} e^{1} \right) = \frac{1}{2} - 0 = \frac{1}{2} \]

(b) \[ \int_{2}^{\infty} \frac{1}{x(\ln x)^{2}} \, dx = \lim_{A \to \infty} \int_{2}^{A} \frac{1}{x(\ln x)^{2}} \, dx = \lim_{A \to \infty} \int_{\ln 2}^{A} \frac{1}{u^{2}} \, du \]
\[ u = \ln x \quad z = \ln x \quad du = \frac{1}{x} \, dx \quad x = A \rightarrow u = \ln A \]
\[ = \lim_{A \to \infty} \left( -u^{-1} \right)_{\ln 2}^{A} = \lim_{A \to \infty} -\ln A + \frac{1}{\ln 2} \]
\[ = \frac{1}{\ln 2} \]

(c) \[ \int_{-1}^{1} \frac{1}{1 - x} \, dx = \lim_{t \to 1^{-}} \int_{-t}^{1} \frac{1}{1 - x} \, dx = \lim_{t \to 1^{-}} -\ln (1 - t) + \ln (2) = +\infty \]

(d) \[ \int_{0}^{\infty} \frac{1}{x^{5/3}} \, dx = \lim_{A \to \infty} \int_{0}^{A} x^{-5/3} \, dx = \lim_{A \to \infty} \left( \frac{3}{2} x^{2/3} \right)_{0}^{A} \]
\[ = \lim_{A \to \infty} \left( \frac{3}{2} A^{2/3} \right) \]
3. (32 points) [8 points each] Determine if each series is absolutely convergent, conditionally convergent, or divergent.

(a) \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k}} \)

This series diverges, so the original series is not absolutely convergent.

(b) \( \sum_{k=1}^{\infty} (1 + \frac{1}{k})^k \)

This series diverges by the nth Term Test.

(c) \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{e^{k^2}} \)

This series converges absolutely by the Ratio Test.

(d) \( \sum_{k=1}^{\infty} k^2 \left( \frac{2}{3} \right)^k \)

This series converges absolutely by the Ratio Test.
Extra Credit (10 points) The Cauchy distribution (with parameter $\theta = 0$) has probability density function

$$f(x) = \frac{1}{\pi (1 + x^2)} \quad \text{for } -\infty < x < \infty.$$  

(a) Verify that the Cauchy distribution (with parameter $\theta = 0$) is a probability density function.

$$\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{\infty} f(x) \, dx = \lim_{B \to -\infty} \int_{-B}^{0} \frac{1}{1+x^2} \, dx + \lim_{A \to \infty} \int_{0}^{A} \frac{1}{1+x^2} \, dx$$

$$= \lim_{B \to -\infty} \left( \frac{1}{\pi} \arctan 0 - \frac{1}{\pi} \arctan B \right) + \lim_{A \to \infty} \left( \frac{1}{\pi} \arctan A - \frac{1}{\pi} \arctan 0 \right)$$

(b) Explain why $\int_{-\infty}^{\infty} x f(x) \, dx = \infty$.

$$\int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{0} x f(x) \, dx + \int_{0}^{\infty} x f(x) \, dx = \lim_{B \to -\infty} \int_{-B}^{0} \frac{x}{1+x^2} \, dx + \lim_{A \to \infty} \int_{0}^{A} \frac{x}{1+x^2} \, dx$$

$$= \lim_{B \to -\infty} \left( \frac{1}{\pi} \arctan 0 - \frac{1}{\pi} \arctan B \right) + \lim_{A \to \infty} \left( \frac{1}{\pi} \arctan A - \frac{1}{\pi} \arctan 0 \right) = \frac{1}{\pi} (-\pi) + \frac{1}{\pi} (\pi) = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) It does not make any sense to say the mean of this distribution is infinity. The remainder of this problem introduces the concepts needed to create a meaningful definition of the mean for the Cauchy distribution.

Let $A > 0$. Show that $\int_{-A}^{0} xf(x) \, dx = -\int_{0}^{A} xf(x) \, dx$.

$$\int_{-A}^{0} xf(x) \, dx = \int_{-A}^{0} (-u) f(-u) (-\, du) = \int_{A}^{0} u f(u) \, du = \int_{A}^{0} u f(u) \, du$$

$$= -\int_{0}^{A} u f(u) \, du = -\int_{0}^{A} f(x) \, dx.$$

(d) The Cauchy principal value of the integral $\int_{-\infty}^{\infty} F(x) \, dx = \infty$ is defined to be

$$\lim_{A \to \infty} \int_{-A}^{A} F(x) \, dx$$

whenever this limit exists. Find the Cauchy principle value of $\int_{-\infty}^{\infty} xf(x) \, dx$.  