Chapter 12

Applications of Series

The preceding chapter developed several tests for determining the convergence or divergence of infinite series. This chapter applies infinite series to approximate functions such as e^x and $\sin(\sqrt{x})$, evaluate integrals, and calculate limits of the indeterminate form "zero-over-zero". After a section devoted to complex numbers, we will use them to show that there is a close link between trigonometric and exponential functions.

12.1 Taylor Series

Section 5.4 introduced the n^{th}-order Taylor polynomial of a function f centered at a as the polynomial P_n that agrees with f and its first n derivatives at $x = a$:

$$\begin{aligned} P_n(x;a) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

The sequence of Taylor polynomials $P_0(x;a)$, $P_1(x;a)$, $\ldots$, $P_n(x;a)$, $\ldots$ can now be viewed as the sequence of partial sums of the infinite series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This series is called the **Taylor series at a associated with the function f**. When $a = 0$, the series is also called the **Maclaurin series associated with f**.

A partial sum of a Taylor series is a Taylor polynomial; a partial sum of a Maclaurin series is a Maclaurin polynomial.

EXAMPLE 1 Show that the limit of the Maclaurin series associated with e^x is e^x,

SOLUTION By Section 5.4 the series is $\sum_{k=0}^{\infty} \frac{x^n}{n!}$. We want to show taht the series converges to e^x. The absolute ratio test shows that the series converges, but it does not tell us that its limit is e^x.

Also by Section 5.4, the difference between $f(x)$ and its Maclaurin polynomial up through the power x^n has the form

$$\frac{f^{(n+1)}(c_n)}{(n+1)!}x^{n+1} \tag{12.1.1}$$

for some number c_n between 0 and x. In the case $f(x) = e^x$, we have $f^{(n+1)}(x) = e^x$, Hence $f^{(n+1)}(c_n) = e^{c_n}$. Thus the "error" (12.1.1) equals

$$\frac{e^{c_n}x^{n+1}}{(n+1)!}.$$

For $x > 0$, we know $c_n < x$ so $c^{c_n} < e^x$; for $x < 0$, $c_n < 0$, so $e^{c_n} < 1$. In either case e^{c_n} is less than a fixed number, which we call M. Thusl n, $e^{c_n} < M$ for all n. Keeping in mind that x is fixed, we see that

$$\lim_{n\to\infty} \frac{|e^{c_n}x^{n+1}|}{(n+1)!} \leq M\frac{|x|^{n+1}}{(n+1)!}. \tag{12.1.2}$$

It was shown in Section 11.2 that $\lim_{n\to\infty} k^n/n!$ is 0 for any fixed number k. Thus (12.1.2) approaches 0 as $n \to \infty$, which means that the sum of the series is e^x. We have, for every number x,

For all x
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This provides a way to estimate e^x using only addition, multiplication, and division. In particular, when $x = 1$, it gives a series representation of e:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots .$$

Euler used this formula to evaluate e to 23 decimal places (without the aid of a calculator). ◇

EXAMPLE 2 Use the Maclaurin series in Example 1 to estimate $\sqrt{e} = e^{1/2}$ with an error of at most 0.001.

SOLUTION The error in using the front end $\sum_{k=0}^{n}(1/2)^k/k!$ has the form

$$e^{c_n}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)!}$$

where c_n is between 0 and 1/2. Then $e^{c_n} < e^{1/2}$, which is less than 2, because $2^2 > 3$. So we want to find n large enough so that

$$\frac{2\left(\frac{1}{2}\right)^{n+1}}{(n+1)!} < 0.001.$$

To find such a number n, we experiment, making a little table, with 4-decimal place accuracy We stop at $n = 4$ with an error less than 0.001. Rounded to

n	1	2	3	4
$2\left(\frac{1}{2}\right)^{n+1}/(n+1)!$	0.2500	0.0417	0.0026	0.0005

Table 12.1.1:

five decimal places, the estimate for $\sqrt{e}$ is

$$1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} \approx 1.64843,$$

which is close to what a calculator shows: 1.6487. ◇

In Section 5.4 the Maclaurin polynomial associated with $\sin(x)$ was computed. Using that result, we conclude that the Maclaurin series associated with $\sin(x)$ is

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots. \quad (12.1.3)$$

The next Example shows that its sum is $\sin(x)$.

EXAMPLE 3 Show that $\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} = \sin(x)$.
SOLUTION To show that the series converges to $\sin(x)$ we must show that the difference between $\sin(x)$ and $\sum_{k=0}^{n}(-1)^k x^{2k+1}/(2k+1)!$ approaches 0 as $n \to \infty$.

To do this we again make use of Lagrange's formula, which involves the higher derivatves of $\sin(x)$, which are $\pm\sin(x)$ and $\pm\cos(x)$. In any case, if $f(x) = \sin(x)$, $\left|f^{(n)}(x)\right| \le 1$. Thus we have

$$\left|\frac{f^{(n+1)}(c_n)x^{n+1}}{(n+1)!}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Because the expression $|x|/(n+1)!$ approaches 0 as $n \to \infty$, no matter how large x is, the difference between the Maclaurin polynomials and $\sin(x)$ approaches 0 as their degree is chosen larger. We conclude that teh Taylor series (12.1.3) converges to $\sin(x)$ for all x. ◇

Therefore, we may write

For all x

$$sin(x) = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

In a similar manner, we have

For all x

$$cos(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}.$$

Taylor Series in Powers of $x - a$

Just as there are Taylor polynomials "around 0," there are such polynomials around any number, a. The Taylor series around a aassociated with $f(x)$ involves powers of $x - a$ instead of powers of x $(= x - 0)$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{n!}(x-a)^k.$$

This series may or may not converge; if it converges, it may not converge to $f(x)$.

EXAMPLE 4 Find the Taylor series associated with $1/x$ in powers of $x-1$.
SOLUTION Here $f(x) = 1/x$. This table shows a few of the higher derivatives evaluated at 1. In general,

n	1	2	3	4	5
$f^{(n)}(x)$	$-1/x^2$	$2/x^3$	$\frac{-3\cdot 2}{x^4}$	$\frac{4\cdot 3\cdot 2}{x^5}$	$\frac{-5\cdot 4\cdot 3\cdot 2}{x^6}$
$f^{(n)}(1)$	-1	2	$-3!$	$4!$	$-5!$

Table 12.1.2:

$$f^{(n)}(1) = (-1)^n n!.$$

Thus the typical term in the Taylor series around 1 is

$$\frac{(-1)^n n!(x-1)^n}{n!} = (-1)^n (x-1)^n.$$

The series begins

$$1 - (1-x) + (1-x)^2 - (1-x)^3 + \cdots.$$

By the n^{th} term test, the series does not converge if $|x-1| > 1$, that is, $x > 2$ or $x < 0$.

If $x = 0$, the series becomes $\sum_{k=0}^{\infty}(-1)^n(-1)^n = \sum_{k=0}^{\infty} 1$, which, by the n^{th} term test,does not converge. Similarly, it does not converge when $x = 2$. To deal with x in $(0, 2)$ we use the absolute-ratio test,examining

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-1)^{n+1}}{(-1)^n (x-1)^n} \right| = \lim_{n\to\infty} |x-1| = |x-1|.$$

So, if $|x-1| < 1$, the series converges. But, does it converge to $1/x$?

The Lagrange formula for the remainder is

$$\frac{f^{(n+1)}(c_n)(x-1)^{n+1}}{(n+1)!} = \frac{(n+1)!}{(c_n)^{n+1}} \frac{(x-1)^{n+1}}{(n+1)!} = \frac{(x-1)^{n+1}}{(c_n)^{n+1}} \quad (12.1.4)$$

where c_n is betwteen 1 and x. So we need to show that $|(x-1)/c_n|^{n+1} \to 0$ as $n \to \infty$. There is trouble.

For instance, if x is near 0, $|x-1|$ is near 1 and c_n may be near 0 for we know only that c_n is between x and 1. Perhaps the ratio $|(x-1)/c_n|$ is a large number.

However, if x is in $(1,2)$ then we have $c_n > 1$ while $|x-1| < 1$, so

$$0 < \frac{|x-1|}{c_n} < |x-1|.$$

Thus the remainder approaches 0 as $n \to \infty$. So we see that for x in $(1,2)$, $1-(x-1)+(x-1)^2-(1-x)^3-\cdots = 1/x$. The Lagrange formula justifies the same conclusion for x in $(-1/2,1)$, but doesn't help for x in $(0,1/2]$, as Exercise 31 shows.

However, $1-(x-1)+(x-1)^2-(1-x)^3-\cdots$ is a geometric series with first term 1 and ratio $r = -(x-1)$. It converges to

$$\frac{1}{1-r} = \frac{1}{1-(1-(-(x-1)))} = \frac{1}{1-x-1} = \frac{1}{x}.$$

This argument covers all x in $(0,2)$ at once. ◇

The General Binomial Theorem

Appendix C reviews the binomial theorem.

$\binom{r}{k} = \frac{r!}{k!(r-k)!}$

If r is 0 or a positive integer, $(1+x)^r$, when multiplied out, is a polynomial of degree r. Its Maclaurin series has only a finite number of nonzero terms, the one of highest degree being x^r. The formula

$$(1+x)^r = \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} x^k$$

is known as the **binomial theorem**. It can also be written as

To remember it, recall that the coefficient of x^k has k factors in both the numerator and denominator. The factors in the numerator start from r and decrease by one. The factors in the denominator start from 1 and increase by 1.

$$(1+x)^r = \sum_{k=0}^{r} \frac{r(r-1)\cdots(r-(k-1))}{1\cdot 2\cdots k} x^k.$$

Example 5 generalizes the binomial theorem to arbitrary exponents r.

EXAMPLE 5 Find the Maclaurin series associated with $f(x) = (1+x)^r$, where r is not 0 or a positive integer and determine its radius of convergence.
SOLUTION The following table will help in computing $f^{(k)}(0)$:

k	$f^{(k)}(x)$	$f^{(k)}(0)$
0	$(1+x)^r$	1
1	$r(1+x)^{r-1}$	r
2	$r(r-1)(1+x)^{r-2}$	$r(r-1)$
3	$r(r-1)(r-2)(1+x)^{r-2}$	$r(r-1)(r-2)$
$\vdots$	$\vdots$	$\vdots$
k	$r(r-1)\cdots(r-k+1)(1+x)^{r-k}$	$r(r-1)(r-2)\cdots(r-k+1)$

Table 12.1.3:

Consequently, the Maclaurin series associated with $(1+x)^r$ is

$$1 + rx + \frac{r(r-1)}{1\cdot 2}x^2 + \frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}x^3 + \cdots . \qquad (12.1.5)$$

Note that the series has an infinite number of non -zero terms (it does not stop) if r is not a positive integer or 0.

For $x = 0$, the series clearly converges. So consider $x \neq 0$. The presence of x^k, which can be positive or negative, and of $k!$ in the denominator of the general term suggests using the absolute-ratio test. Let a_k be the term in the Maclaurin series for $(1+x)^r$ that contains the power x^k. Then

$$a_k = \frac{r(r-1)(r-2)\cdots(r-k+1)}{1\cdot 2\cdot 3\cdots k}x^k,$$

and

$$a_{k+1} = \frac{r(r-1)(r-2)\cdots(r-k+1)(r-k)}{1\cdot 2\cdot 3\cdots k(k+1)}x^{k+1}.$$

Thus

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{r(r-1)(r-2)\cdots(r-k+1)(r-k)}{1\cdot 2\cdot 3\cdots k(k+1)}x^k}{\frac{r(r-1)(r-2)\cdots(r-k+1)}{1\cdot 2\cdot 3\cdots k}x^{k+1}}\right| = \left|\frac{r-k}{k+1}x\right|.$$

Since r is fixed,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = |x|.$$

By the absolute-ratio test, series (12.1.5) converges when $|x| < 1$ and diverges when $|x| > 1$. ◇

To find the interval of convergence, consider the series with $x = \pm 1$.

In Example 5 it was shown that for $|x| < 1$ the Maclaurin series associated with $(1+x)^r$ converges to something, but does it converge to $(1+x)^r$?

Let us check the case $r = -1$. When $r = -1$, series (12.1.5) becomes

$$1 + (-1)x + \frac{(-1)(-2)}{1 \cdot 2}x^2 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3}x^3 + \cdots ,$$

or

$$1 - x + x^2 - x^3 + \cdots .$$

This series is a geometric series with first term 1 and ratio $-x$. It therefore converges for $|x| < 1$. Moreover, it does represent the function $(1 + x)^r = (1 + x)^{-1}$. (See Exercises 34 to 37 in Section 12.4.)

It is true that for $|x| < 1$ series (12.1.5) does converge to $(1 + x)^r$. The fact that $(1+x)^r$ is represented by the series (12.1.5) is known as the **general binomial theorem** or, simply, the **binomial theorem**. Series (12.1.5) is called the **binomial expansion** of $(1 + x)^r$.

Exercises 34 to 37 in Section 12.4 outline a proof that (12.1.5) represents $(1 + x)^r$ for $|x| < 1$.

Summary

The Taylor series at a associated with a function is the series whose partial sums are its n^{th}-order Taylor polynomials. This series represents the original function only for inputs such that the remainder of the n^{th}-order Taylor polynomial approaches zero as $n \to \infty$: $\lim_{n\to\infty} R_n(x, a) = 0$. The Lagrange form of the remainder, Theorem 5.4.1 from Section 5.4, helps to show that the remainder converges to zero, though, as Example 5 illustrates, in some cases it may not be strong enough to do that.

Function	Series	Interval of Convergence
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	all x: $(-\infty, \infty)$
$\sin(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	all x: $(-\infty, \infty)$
$\cos(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	all x: $(-\infty, \infty)$
$\frac{1}{x}$	$\sum_{k=0}^{\infty} (-1)^k (x-1)^k = \sum_{k=0}^{\infty} (1-x)^k$	$0 < x < 2 : (0, 2)$
$(1+x)^r$	$\sum_{k=0}^{\infty} \frac{r(r-1)(r-2)\cdots(r-k+1)}{1\cdot 2\cdot 3\cdots k} x^k$	$\|x\| < 1$

Table 12.1.4:

EXERCISES for Section 12.1 *Key:* R–routine, M–moderate, C–challenging

1.[R] State without using any mathematical symbols the formula for the terms of a Taylor series of a function around a number that may not be zero.

2.[R] State without using any mathematical symbols the formula for the terms of a Maclaurin series of a function.

In Exercises 3 to 9 compute the Maclaurin series associated with the given function

3.[R] $1/(1+x)$

4.[R] $1/(1-x)$

5.[R] $\ln(1+x)$

6.[R] $\ln(1-x)$

7.[R] $\sin(x)$

8.[R] e^{-x}

9.[R] $\sqrt{1+x}$

10.[R] Let $f(x) = e^x$. Show that $\lim_{n\to\infty} R_n(x;0) = 0$ for any negative number x. This completes the proof that the exponential function is represented by its Maclaurin series for all numbers x (see Example 2).

11.[R] Show that the Maclaurin series associated with $\sin(x)$ represents $\sin(x)$ for all x.

12.[R] Show that the Maclaurin series associated with e^{-x} represents e^{-x} for all x.

13.[R]

(a) Why will there be no terms of even degree in the Maclaurin series for $\arctan(x)$? (That is, all terms of the form x^{2k} have coefficient zero.)

(b) Obtain the first two non-zero terms of the Maclaurin series for $\arctan(x)$.

In Section 12.4 we use a shortcut to find the entire series.

14.[R]

(a) Use the Lagrange formula to show that the Maclaurin series associated with $1/(1+x)$ represents $1/(1+x)$ for all $-1/2 < x < 1$. HINT: Examine $R_n(x;0)$.

(b) Use the fact that it is a geometric series to show that the representation holds for $-1 < x < 1$.

15.[R] Show that the Taylor series in powers of $x-a$ for e^x represents e^x for all x.

16.[R] Show that the Taylor series in powers of $x - a$ for $\cos(x)$ represents $\cos(x)$ for all x.

17.[R]

(a) Write out the first four terms of the binomial expansion of $(1+x)^{-2} = 1/(1+x)^2$.

(b) What is the coefficient of the general term x^n?

18.[R] Write out the first four terms of the binomial expansion of $(1 + x)^{1/2} = \sqrt{1+x}$.

19.[R] What is the typical term in the Maclaurin series associated with $(1 - x)^r$? HINT: Exploit the binomial expansion of $(1 + x)^r$; don't start from scratch.

20.[C] Suppose one uses the Maclaurin series for e^x to find e^{100}.

(a) What are the first four terms?

(b) Does the series converge to e^{100}?

(c) If your answer to (b) is "yes" how many terms would you use to estimate e^{100} with an error less than 0.005?

(d) Which term in the series is largest?

21.[R]

(a) Use the Maclaurin series for e^x to estimate $\sqrt[3]{e}$ to three decimal places.

(b) Compare your answer in (a) to the value of $\sqrt[3]{e}$ returned by your calculator.

22.[R] Find the Maclaurin series associated with $\ln(1 + x)$.

23.[M] This problem examines three ways to estimate the error in using a front-end of $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ to estimate e^{-1}.

(a) Use the Lagrange formula to obtain an estimate of the error in using the front-end up through $(-1)^m/m!$ to estimate e^{-1}

(b) Estimate the error by noticing the series is alternating and the terms decrease in absolute value

(c) Estimate the error by comparing $\sum_{k=m+1}^{\infty} \left| \frac{(-1)^k}{k!} \right|$ to a geometric series, which is easy to sum.

(d) Which of the three methods provides the smallest (best) estimate of the error?

24.[M]

(a) Use the Taylor series around $\pi/4$ to estimate $\cos(50°)$ to two decimal places. (That is, with an error less than 0.005.) Approximate π by 3.14 and $\sqrt{2}$ by 1.4142.

(b) Check your calculation by calculating $\cos(50°)$ with your calculator.

25.[C] Do there exist any polynomials $p(x)$ such that $\sin(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

26.[C] Do there exist any polynomials $p(x)$ such that $\ln(x) = p(x)$ for all x in the interval $[1, 1.0001]$? Explain.

27.[C] Let f be a function that has derivatives of all orders for all x. Assume that $\left|f^{(n)}(x)\right| \leq n$ for all 100^n. Show why $f(x)$ is represented by its Maclaurin series for all x.

28.[M]

(a) From the Maclaurin series for $\cos(x)$ obtain the Maclaurin series for $\sin(x)^2$. HINT: Use a trigonometric identity.

(b) From (a), and another trigonometric identity, obtain the Maclaurin series for $\cos(x)^2$.

Exercises 29 and 30 present a non-zero function whose Maclaurin series has the value 0 for all x, and therefore does not represent the function. This function is so "flat" at the origin that all its derivatives are zero there.

29.[C] The following steps show that $\lim_{x\to 0} \frac{e^{1/x^2}}{x^n} = 0$ for all positive numbers n:

(a) Why does it suffice to consider only $x > 0$?

(b) Let $v = 1/x^2$ and translate the limit to

$$\lim_{v\to\infty} v^{n/2} e^{-v}.$$

(c) This limit is similar to a limit treated in Section 5.5. Show that it equals 0.

(d) Show that $\lim_{n\to\infty} \frac{p(x)e^{-1/x^2}}{x^n} = 0$ for any polynomial $p(x)$.

30.[C] Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$.

(a) Show f is continuous at 0.

(b) Show f is differentiable at 0.

(c) Show that $f'(0) = 0$.

(d) Show that $f''(0) = 0$.

(e) Explain why $f^{(n)}(0) = 0$ for all integers $n \geq 0$.

(f) What is the Maclaurin series associated with f?

(g) Why does the example use e^{-1/x^2} instead of the simpler $e^{-1/x}$.

31.[C] Explain why it is not possible to use the Lagrange formula to show that the Taylor series in powers of $(x - 1)$ associated with $1/x$ converges to $1/x$ for x in $(0, 1/2)$.

12.2 Two Applications of Taylor Series

If a Taylor series associated with a function $f(x)$ represents the function, then any front end (or Taylor polynomial) approximates $f(x)$. This can be used to evaluate some indeterminate limits and to estimate some definite integrals.

Using a Taylor Series to Find a Limit

The next example shows how series can be used to evaluate the limit of a quotient that is an indeterminate form.

EXAMPLE 1 Find $\lim_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1+2x}}{\sqrt{1+2x}-\sqrt{1+4x}}$.

SOLUTION This limit could be bound by l'Hôpital's method. However, it is faster to use Taylor series.

For a number r, and $|x|<1$, the binomial theorem asserts that

$$(1+x)^r = 1 + rx + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots.$$

Thus the limit is

$$\begin{aligned}
&\lim_{x\to 0}\frac{\left(1+\frac{1}{2}x+\frac{\frac{1}{2}\frac{-1}{2}}{2!}x^2+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}x^3+\cdots\right)-\left(1+\frac{1}{2}(2x)+\frac{\frac{1}{2}\frac{-1}{2}}{2!}(2x)^2+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}(2x)^3+\cdots\right)}{\left(1+\frac{1}{2}(2x)+\frac{\frac{1}{2}\frac{-1}{2}}{2!}(2x)^2+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}(2x)^3+\cdots\right)-\left(1+\frac{1}{2}(4x)+\frac{\frac{1}{2}\frac{-1}{2}}{2!}(4x)^2+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}(4x)^3+\cdots\right)}\\
=&\lim_{x\to 0}\frac{\left(x\left(\frac{1}{2}+\frac{\frac{1}{2}\frac{-1}{2}}{2!}x+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}x^2+\cdots\right)\right)-\left(x\left(\frac{1}{2}(2)+\frac{\frac{1}{2}\frac{-1}{2}}{2!}4x+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}8x^2+\cdots\right)\right)}{\left(x\left(\frac{1}{2}(2)+\frac{\frac{1}{2}\frac{-1}{2}}{2!}4x+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}8x^2+\cdots\right)\right)-\left(x\left(\frac{1}{2}(4)+\frac{\frac{1}{2}\frac{-1}{2}}{2!}16x+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}64x^2+\cdots\right)\right)}\\
=&\lim_{x\to 0}\frac{\left(\frac{1}{2}+\frac{\frac{1}{2}\frac{-1}{2}}{2!}x+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}x^2+\cdots\right)-\left(\frac{2}{2}+\frac{\frac{2}{2}\frac{-2}{2}}{2!}x+\frac{\frac{2}{2}\frac{-2}{2}\frac{-6}{2}}{3!}x^2+\cdots\right)}{\left(\frac{1}{2}+\frac{\frac{1}{2}\frac{-1}{2}}{2!}x+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}x^2+\cdots\right)-\left(\frac{2}{4}+\frac{\frac{4}{4}\frac{-2}{2}}{2!}x+\frac{\frac{4}{2}\frac{-4}{2}\frac{-12}{2}}{3!}x^2+\cdots\right)}\\
=&\frac{\frac{1}{2}-\frac{2}{2}}{\frac{2}{2}-\frac{4}{2}}=\frac{1}{2}
\end{aligned}$$

◇

In Example 1 we needed only enough terms of each series to know the smallest power of x that appears in the numerator and in the denominator. The next example illustrates this.

EXAMPLE 2 Find $\lim_{x\to 0} \dfrac{\sin(x^2)}{\sqrt{1+3x^2}-1}$.

SOLUTION Using just enough of the Maclaurin series for $\sin(x^2)$ and $\sqrt{1+3x^2}$, we have

$$\lim_{x\to 0} \frac{\sin(x^2)}{\sqrt{1+3x^2}-1} = \lim_{x\to 0} \frac{x^2-\cdots}{1+\frac{1}{2}(3x^2)-1} = \frac{2}{3}.$$

◇

Using a Taylor Series to Estimate an Integral

The integral describes the "bell curve."

In statistics, the integral $\int_{-\infty}^{b}(1/\sqrt{2\pi})e^{-x^2/2}\,dx$ is of major importance. Since $e^{-x^2/2}$ does not have an elementary antiderivative, the integral must be estimated by other means. Tables of values of this function can be found in almost any mathematical handbook.

The next example shows how to estimate $\int_a^b f(x)\,dx$ when $f(x)$ is represented by a Taylor series.

EXAMPLE 3 Use the Maclaurin series for e^x to estimate $\int_0^1 e^{-x^2}\,dx$.
SOLUTION The first step is to obtain the Maclaurin series for the integrand: e^{-x^2}. Because

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we can replace x with $-x^2$ to obtain

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots . \qquad (12.2.1)$$

For $0 \le |x| \le 1$, (12.2.1) is a convergent alternating series. Every partial sum that ends with a negative term is smaller than e^{-x^2}; every partial sum that ends with a positive term is larger than e^{-x^2}. For example,

$$1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} < e^{-x^2} < 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!}.$$

Hence

$$\int_0^1 \left(1 - x^2 + \tfrac{x^4}{2!} - \tfrac{x^6}{3!}\right)\,dx < \int_0^1 e^{-x^2}\,dx < \int_0^1 \left(1 - x^2 + \tfrac{x^4}{2!} - \tfrac{x^6}{3!} + \tfrac{x^8}{4!}\right)\,dx,$$

or

$$1 - \tfrac{1}{3} + \tfrac{1}{5\cdot 2!} - \tfrac{1}{7\cdot 3!} < \int_0^1 e^{-x^2}\,dx < 1 - \tfrac{1}{3} + \tfrac{1}{5\cdot 2!} - \tfrac{1}{7\cdot 3!} + \tfrac{1}{9\cdot 4!}.$$

From this it follows that

$$0.742 < \int_0^1 e^{-x^2}\,dx < 0.748.$$

◇

Summary

The Taylor series associated with a function can be used to evaluate somoe indeterminate limits and to estimate definite integrals.

EXERCISES for Section 12.2 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 4 use Taylor series to find the limits.

1.[R] $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt{1+3x}-1}$.

2.[R] $\lim_{x\to 0} \frac{\sin(4x)}{\sqrt{1+3x}-1}$.

3.[R] $\lim_{x\to 0} \frac{e^{x^2}-1}{\sin(x^2)}$.

4.[R] $\lim_{x\to 0} \frac{\cos(x)-1+\frac{x^2}{2}}{\sin(x^4)}$.

In Exercises 5 to 11 find the limits two ways. First use a Taylor series and then again using l'Hôpital's rule.

5.[R] $\lim_{x\to 0} \frac{\cos(x)e^{2x^2}-1}{x\sin(x)}$.

6.[R] $\lim_{x\to 0} \frac{\sqrt{1+3x}(e^x-1)x}{1-\cos(2x)}$.

7.[R] $\lim_{x\to 0} \frac{\cos(x)-\sqrt{1+x}}{\cos(2x)-\sqrt[3]{1+2x}}$.

8.[R] $\lim_{x\to 0} \frac{\ln(1+3x)}{\sin(2x)}$.

9.[R] $\lim_{x\to 0} \frac{e^{x^2}-1}{e^{3x62}-1}$.

10.[R] $\lim_{x\to 0} \frac{\left(\sin(x^2)+e^{x^3}-1\right)\sqrt[3]{5+x}}{\sqrt{1+5x^2}-1}$.

11.[R] $\lim_{x\to 4} \frac{(8-2x)e^{x^2}}{\sqrt[3]{4-x}}$. NOTE: See Exercise **??** for the binomial theorem for $(a+b)^r$.

12.[R]

(a) Write out the first four terms of the binomial series for $(1+x)^{-2}$

(b) What is the general form?

13.[R] Find the limit in Example 1 by l'Hôpital's rule.

14.[M]

(a) Show $\int_0^1 (e^x-1)/x\,dx$ is finite, even though the integrand is not defined at 0.

(b) Show that $1 + \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} + \frac{1}{4 \cdot 4!} + \frac{1}{5 \cdot 5!}$ is an estimate of the integral in (a).

(c) The error in using the sum in (b) is $\frac{1}{6 \cdot 6!} + \frac{1}{7 \cdot 7!} + \frac{1}{8 \cdot 8!} + \frac{1}{9 \cdot 9!} + \cdots$. Show that this is less than $\frac{1}{6 \cdot 6!}\left(1 + \frac{1}{7}\left(\frac{1}{7}\right) + \frac{1}{7}\left(\frac{1}{7}\right)^2 + \frac{1}{7}\left(\frac{1}{7}\right)^3 + \cdots\right)$.

(d) From (c) deduce that the error is less than 0.000237.

15.[M]

(a) Show that for x in $[0, 2]$

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \le e^x - 1 \le x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^2 x^{n+1}}{(n+1)!}.$$

(b) Use (a) to find $\int_0^2 \frac{e^x - 1}{x}\, dx$ to three decimal places.

16.[M] Find $\int_0^1 \frac{1-\cos(x)}{x}\, dx$ to three decimal places, using an approach like that in Exercise 15.

17.[M] Estimate $\int_0^\infty e^{-5x^2}\, dx$ following these steps:

(a) Find a number b such that

$$\int_b^\infty e^{-5x^2}\, dx < 0.0005.$$

(Use the fact that $e^{-5x^2} < e^{-5x}$ for $x > 1$.)

(b) Let b be the number you found in (a). Estimate $\int_0^b e^{-5x^2}\, dx$ with an error of less than 0.0005. (Use the Maclaurin series for e^{-5x^2}.)

(c) Combine (a) and (b) to get a two decimal place estimate of $\int_0^\infty e^{-5x^2}\, dx$.

18.[M] Estimate $\int_0^\infty \frac{\cos(x^6/100)-1}{x^6}\, dx$, following these steps:

(a) Find a number b such that

$$\left| \int_b^\infty \frac{\cos(x^6/100) - 1}{x^6}\, dx \right| < 0.001.$$

(Use the fact that $|\cos(x)| \le 1$.)

(b) Let b be the number you found in (a). Estimate

$$\int_0^b \frac{\cos(x^6/100) - 1}{x^6}\, dx,$$

with an error less that 0.001. (Use the Maclaurin series for $\cos(x)$.)

(c) Combine (a) and (b) to get a two decimal place estimate for

$$\int_0^\infty \frac{\cos(x^6/100) - 1}{x^6}\, dx.$$

19.[C] Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by

(a) the Fundamental Theorem of Calculus (approximate π to 3 decimal places),

(b) Simpson's method (six sections),

(c) trapezoid method (six sections),

(d) using the first six non-zero terms of the series $1 - x^2 + x^4 - \cdots$ for $1/(1+x^2)$.

20.[C] If $|a/b| < 1$, use the binomial theorem to expand $(a+b)^r$ as the sum of terms of the form ca^pb^q.

21.[C] If $b/a| < 1$, use the fundamental threorem to expand $(a+b)^4$ as the sum of terms of trhe form ca^pb^q. (As a check, the series starts with b^r.)

22.[C] Write out the first four (4) terms of the series for $(8+x)^{1/3}$ if (a) $x > 8$, and (b) $x < 8$.

23.[C]

Sam: I was playing with the binomial theorem.

Jane: Is that possible?

Sam: I looked at $(3+5)^{1/3}$, which I know is two. But I can write it as $5^{1/3}\left(1 + \frac{3}{5}\right)$ and get

$$5^{1/3}\left(1 + \frac{1}{3}\frac{3}{5} + \frac{1}{3}\frac{-2}{3}\left(\frac{3}{5}\right)^2 + \cdots\right)$$

so

$$2 = -5^{1/3} + \frac{1}{3}5^{-2/3}(3) - \frac{1}{9}5^{-5/3}3^2 + \cdots.$$

Jane: That's a fancy way to estimate 2.

Sam: But I can write $(3+5)^{1/3}$ as $3^{1/3}\left(1+\frac{5}{3}\right)^{1/3}$ and get

$$2 = 3^{1/3} + \frac{1}{3}3^{-2/5}5^{1/3} + \cdots$$

Jane: Another nutty way to estimate 2.

Sam: My point is that they can't both be right.

Can they both be right?

24.[C] Repeat Exercise 19 for $\int_0^1 \frac{dx}{1+x^3}$.

25.[C] In R. P. Feynman, *Lectures on Physics*, Addison-Wesley, Reading, MA 1963, this statement appears in Section 15.8 of Volume 1:

> An approximate formula to express the increase of mass, for the case when the velocity is small, can be found by expanding $m_0/\sqrt{1-v^2/c^2} = m_0(1-v^2/c^2)^{-1/2}$ in a power series, using the binomial theorem. We get
>
> $$m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots\right).$$
>
> We see clearly from the formula that the series converges rapidly when v is small and the terms after the first two or three are neglible.

Check the expansion and justify the equation.

26.[C] A fluid mechanics text has the following argument in a discussion of flow through a nozzle:

> The pressure p equals
>
> $$\left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/(1-\gamma)}.$$
>
> By the binomial theorem and the fact that $v^2 = M^2\gamma RT$:
>
> $$p = 1 - \frac{1}{2}\frac{v^2}{RT} + \frac{\gamma(2\gamma-1)}{8}M^4 + \cdots.$$

Fill in the steps. NOTE: γ is the specific heat, which is about 1.4, and M is a Mach number, which is in the range 1 to 2.

27.[C]

(a) The ellipse $x^2/a^2 + y^2/b^2 = 1$ for $a \leq b$ has the parameterization

$$x = a\cos(t), \qquad y = b\sin(t).$$

Show that the arc length of one quadrant of an ellipse is

$$\int_0^{\pi/2} b\sqrt{1 - \left(1 - \left(\frac{a}{b}\right)^2\right)\sin(t)^2}\ dt.$$

NOTE: The integrand does not have an elementary antiderivative.

(b) If $a < b$, the integral in (a) has the form $\int_0^{\pi/2} b\sqrt{1-k^2\sin(t)^2}\ dt$, where $0 < k < 1$.

The "elliptic integral"

$$E = \int_0^{\pi/2} b\sqrt{1-k^2\sin(t)^2}\ dt$$

is tabulated in mathematical handbooks for many values of k in $[0,1]$. Using the binomial theorem and the formula for $\int_0^{\pi/2}\sin^n(\theta)\ d\theta$ (Formula 73 in the table of integrals), obtain the first four non-zero terms of E as a series of powers of k^2.

28.[C] Assume that $f(x)$ has a continuous fourth derivative. Let M_4 be the maximum of $|f^{(4)}(x)|$ for x in $[-1,1]$. Show that

$$\left| \int_{01}^{1} f(x)\ dx - f\left(\frac{1}{\sqrt{3}}\right) - f\left(\frac{-1}{\sqrt{3}}\right)\right| \leq \int 7M_4 250.$$

HINT: Use the representation $f(x) = f(0) + f'(0)x + f''(0)x^2/2 + f^{(3)}(0)x^3/6 + f^{(4)}x^4/24$, where c depends upon x.

12.3 Power Series and Their Interval of Convergence

Our use of Taylor polynomials to approximate a function led us to consider series of the form

$$\sum_{k=0}^{\infty} b_k(x-a)^k = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_k(x-a)^k + \cdots .$$

Such a series is called a **power series** in $x-a$. If $a=0$, we obtain a series in powers of x:

$$\sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k + \cdots .$$

We will now look at some properties of power series and see that they behave very much like polynomials.

The Radius of Convergence of a Power Series

For each fixed choice of x, a power series becomes a series with constant terms.

The power series $b_0 + b_1 x + b_2 x^2 + \cdots$ certainly converges when $x=0$. It may or may not converge for other choices of x. However, as Theorem 12.1.3 will show, if the series converges at a certain value c, it converges at any number x whose absolute value is less than $|c|$, that is, throughout the interval $(-|c|,|c|)$. Since the proof of Theorem 12.3.1 uses the comparison test and the absolute-convergence test, it offers a nice review of important concepts from Chapter 11.

Theorem 12.3.1. *Let c be a nonzero number such that Assume that $\sum_{k=0}^{\infty} b_k c^k$ converges. Then, if $|x| < |c|$, $\sum_{k=0}^{\infty} b_k x^k$ converges. In fact, it converges absolutely.*

The x's for which the series converges form an interval with 0 at its midpoint.

The proof is at the end of this section.

By Theorem 12.3.1, the set of numbers x such that $\sum_{k=0}^{\infty} b_k x^k$ converges has no holes. In other words, it is one unbroken piece, which includes the number 0. Moreover, if r is in the set, so is the entire open interval $(-|r|,|r|)$.

There are two possibilities. In the first case, there are arbitrarily large r's such that the series converges for x in $(-r,r)$. This means that the series converges for all x. In the second case, there is an upper bound on the numbers r such that the series converges for x in $(-r,r)$. It is shown in advanced calculus that there is then a smallest upper bound on the r's; call it R.

See Figure 12.3.1.

Consequently, either

1. $b_0 + b_1x + b_2x^2 + \cdots$ converges for all x

or

2. there is a positive number R such that $b_0 + b_1x + b_2x^2 + \cdots$ converges for all x such that $|x| < R$ but diverges for $|x| > R$.

Note that convergence or divergence at R and $-R$ is not mentioned.

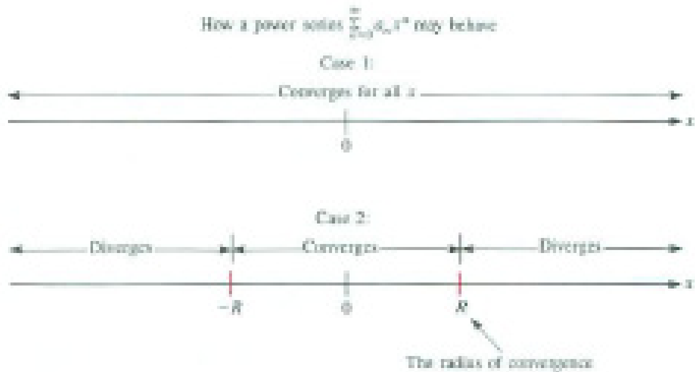


Figure 12.3.1:

In the second case, R is called the **radius of convergence** of the series. In the first case, the radius of convergence is said to be infinite, $R = \infty$. For the geometric series $1 + x + x^2 + \cdots + x^k + \cdots$, $R = 1$, since the series converges when $|x| < 1$ and diverges when $|x| > 1$. (It also diverges when $x = 1$ and $x = -1$.) A power series with radius of convergence R may or may not converge at R and at $-R$. These observations are summarized as Theorem 12.3.2.

Theorem 12.3.2. *Radius of Convergence Let R be the radius of convergence for the power series $\sum_{k=0}^{\infty} b_k x^k$. If $R = 0$, the series converges only for $x = 0$. If R is a positive number, the series converges for $|x| < R$ and diverges for $|x| > R$. If R is ∞, the series converges for all x.*

EXAMPLE 1 Find the radius of convergence, R, for $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{k+1}x^k}{k} + \cdots$.

SOLUTION Because of the presence of x^k and the fact that x may be negative, use the absolute-ratio test. The absolute value of the ratio of successive terms is

$$\left| \frac{\frac{(-1)^{k+2}x^{k+1}}{k+1}}{\frac{(-1)^{k+1}x^k}{k}} \right| = \frac{k}{k+1}|x|.$$

As $k \to \infty$, $k/(k+1) \to 1$. Thus,

$$\lim_{k\to\infty} \frac{k}{k+1}|x| \quad = \quad |x|.$$

Consequently, by the absolute-ratio test, if $|x| < 1$ the series converges. If $|x| > 1$, it diverges.

The absolute-ratio test takes care of $|x| < 1$ and $|x| > 1$.

The radius of convergence is $R = 1$. It remains to see what happens at the endpoints, 1 and -1.

Checking convergence at $x = 1$

For $x = 1$, we obtain the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

This series converges, by the alternating-series test.

Checking convergence at $x = -1$

What about $x = -1$? The series becomes

$$(-1) - \frac{(-1)^2}{2} + \frac{(-1)^3}{3} - \frac{(-1)^4}{4} + \cdots + \frac{(-1)^{k+1}(-1)^k}{k} + \cdots$$

or

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{k} + \cdots ,$$

which, being the negative of the harmonic series, diverges.

All told, this series converges only for $-1 < x \leq 1$. Figure 12.3.2 records what we found.

Figure 12.3.2:

◇

EXAMPLE 2 Find the radius of convergence of

$$\sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots ,$$

the Maclaurin series for e^x.

SOLUTION Because of the presence of the powers x^k, the factorial $k!$, and that x may be positive or negative, the Absolute-Ratio Test is the logical test to use. The absolute value of the ratio between successive terms is

$$\left|\frac{\frac{x^{k+1}}{(k+1)!}}{\frac{x^k}{k!}}\right| = \frac{k!}{(k+1)!}|x| = \frac{|x|}{k+1}.$$

Since

$$\lim_{k\to\infty}\frac{|x|}{k+1} = 0,$$

the limit of the ratio between successive terms is 0. Since 0 is less than 1, the series converges for all x. The radius of convergence R is infinite. ◇

A case where $R = \infty$

The next example represents the opposite extreme, $R = 0$.

EXAMPLE 3 Find the radius of convergence of the series

$$\sum_{k=1}^{\infty} k^k x^k = 1x + 2^2x^2 + 3^3x^3 + \cdots + k^kx^k + \cdots .$$

SOLUTION The series converges for $x = 0$.

Every power series converges for at least one value of x.

If $x \neq 0$, consider the k^{th} term k^kx^k, which can be written as $(kx)^k$. As $k \to \infty$, $|kx| \to \infty$. By the n^{th} term test, this series diverges. In short, the series converges only when $x = 0$. The radius of convergence is $R = 0$. ◇

A case where $R = 0$

The Radius of Convergence of $\sum_{k=0}^{\infty} b_k(x-a)^k$

Just as a power series in x has an associated radius of convergence, so does a power series in $x - a$. To see this, consider any such power series,

$$\sum_{k=0}^{\infty} b_k(x-a)^k = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots . \qquad (12.3.1)$$

Let $u = x - a$. Then series (12.3.1) becomes

$$\sum_{k=0}^{\infty} b_k u^k = b_0 + b_1u + b_2u^2 + \cdots . \qquad (12.3.2)$$

Series (12.3.2) has a certain radius of convergence R. That is, (12.3.2) converges for $|u| < R$ and diverges for $|u| > R$. Consequently (12.3.1) converges for $|x-a| < R$ and diverges for $|x-a| > R$. The number R is called the radius of convergence of the series (12.3.1).

R may be zero, positive, or infinite.

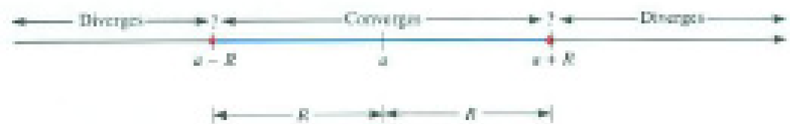


Figure 12.3.3:

As Figure 12.3.3 suggests, the series $\sum_{k=0}^{\infty} b_k(x-a)^k$ converges in an interval $(a-R, a+R)$, whose midpoint is a. The question marks in Figure 12.3.3 indicate that the series may converge or may diverge at the ends of the interval, $a-R$ and $a+R$. These cases must be decided separately.

These observations are summarized in the following theorem.

Theorem 12.3.3. *Let R be the radius of convergence for the power series $\sum_{k=0}^{\infty} b_k(x-a)^k$. If $R=0$, the series converges only for $x=a$. If R is a positive real number, the series converges for $|x-a|<R$ and diverges for $|x-a|>R$. If $R=\infty$, the series converges for all numbers x.*

EXAMPLE 4 Find all values of x for which

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(x-1)^k}{k} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \quad (12.3.3)$$

converges.

SOLUTION Note that this is Example 1 with x replaced by $x-1$. Thus $x-1$ plays the role that x played in Example 1. Consequently, series (12.3.3) converges for $-1 < x-1 \le 1$, that is, for $0 < x \le 2$, and diverges for all other values of x. Its radius of convergence is $R=1$. The set of values where the series converges is an interval $(0,2]$.

The convergence of (12.3.3) is recorded in Figure 12.3.4. ◇

Proof of Theorem 12.3.1 (in Section 12.1)

Proof (of Theorem 12.3.1)

Since $\sum_{k=0}^{\infty} b_k c^k$ converges, the k^{th} term $a_k c^k$ approaches 0 as $k \to \infty$. Thus

Figure 12.3.4:

there is an integer N such that for $k \geq N$, $|b_k c^k| \leq 1$, say. From here on, consider only $k \geq N$. Now,

$$b_k x^k = b_k c^k \left(\frac{x}{c}\right)^k.$$

Since

$$\left|b_k x^k\right| = \left|b_k c^k\right| \left|\frac{x}{c}\right|^k,$$

it follows that for $k \geq N$,

$$\left|b_k x^k\right| \leq \left|\frac{x}{c}\right|^k \qquad (\text{since } |b_k c^k| \leq 1 \text{ for } k \geq N).$$

The series $\sum_{k=0}^{\infty} \left|\frac{x}{c}\right|^k$ is a geometric series with the ratio $|x/c| < 1$. Hence it converges.

Since $|b_k x^k| \leq \left|\frac{x}{c}\right|^k$ for $k \geq N$, the series $\sum_{k=0}^{\infty} \left|b_k x^k\right|$ converges (by the comparison test). Thus $\sum_{k=N}^{\infty} b_k x^k$ converges (in fact, absolutely). Putting in the front end, $\sum_{k=0}^{N-1} b_k x^k$, we conclude that the series $\sum_{k=0}^{\infty} b_k x^k$ converges absolutely if $|x| < |c|$. •

You may wonder why it's called "radius of convergence," when no circles seem to be involved. Section 13.6, which uses complex numbers, explains why.

Summary

Motivated by Taylor series, we investigated series of the form $\sum_{k=0}^{\infty} b_k x^k$ and, more generally, $\sum_{k=0}^{\infty} b_k (x-a)^k$. Associated with each such series is a radius of convergence R. (If the series converges for all x, we take R to be infinite.) If $\sum_{k=0}^{\infty} b_k x^k$ has radius of convergence R, then it converges (absolutely) for all x

in $(-R, R)$, but diverges for all x such that $|x| > R$. Similarly, if $\sum_{k=0}^{\infty} b_k(x-a)^k$ has radius of convergence R, it converges for all x such that x is in $(a-R, a+R)$ but diverges for $|x-a| > R$. Convergence or divergence at the endpoints of the interval of convergence must be checked separately. You may wonder why R is called the "radius of convergence" when there is no circle present. This mystery is explained in Sections 13.5 and 13.6.

EXERCISES for Section 12.3 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 12 draw the appropriate diagrams (like Figure 12.3.4) showing where the series converge or diverge. Explain your work.

1.[R] $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

2.[R] $\sum_{k=1}^{\infty} \frac{x^k}{\sqrt{k}}$

3.[R] $\sum_{k=0}^{\infty} \frac{x^k}{3^k}$

4.[R] $\sum_{k=1}^{\infty} k^2 e^{-k} x^k$

5.[R] $\sum_{k=0}^{\infty} \frac{2k^2+1}{k^2-5} x^k$

6.[R] $\sum_{k=1}^{\infty} \frac{x^k}{k}$

7.[R] $\sum_{k=0}^{\infty} \frac{x^k}{(2k)!}$

8.[R] $\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$

9.[R] $\sum_{k=0}^{\infty} \frac{x^k}{(2k+1)!}$

10.[R] $\sum_{k=0}^{\infty} k! x^k$

11.[R] $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$

12.[R] $\sum_{k=1}^{\infty} \frac{2^k x^k}{n}$

13.[R] Assume that $\sum_{k=0}^{\infty} b_k x^k$ converges for $x = 9$ and diverges when $x = -12$. What, if anything, can be said about

(a) convergence when $x = 7$?

(b) absolute convergence when $x = -7$?

(c) absolute convergence when $x = 9$?

(d) convergence when $x = -9$?

(e) divergence when $x = 10$?

(f) divergence when $x = -15$?

(g) divergence when $x = 15$?

14.[R] Assume that $\sum_{k=0}^{\infty} b_k x^k$ converges for $x = -5$ and diverges when $x = 8$. What, if anything, can be said about

(a) convergence when $x = 4$?

(b) absolute convergence when $x = 4$?

(c) convergence when $x = 7$?

(d) absolute convergence when $x = -5$?

(e) convergence when $x = -9$?

(f) convergence when $x = -9$?

15.[R] If $\sum_{k=0}^{\infty} b_k x^k$ converges whenever x is positive, must it converge whenever x is negative?

16.[R] If $\sum_{k=0}^{\infty} b_k 6^k$ converges, what can be said abou the convergence of

(a) $\sum_{k=0}^{\infty} b_k (-6)^k$?

(b) $\sum_{k=0}^{\infty} b_k 5^k$?

(c) $\sum_{k=0}^{\infty} b_k (-5)^k$?

In Exercises 17 to 28 draw the appropriate diagrams showing where the series converge and diverge.

17.[R] $\sum_{k=0}^{\infty} \frac{(x-2)^k}{k!}$

18.[R] $\sum_{k=0}^{\infty} \frac{(x-1)^k}{k3^k}$

19.[R] $\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+3}$

20.[R] $\sum_{k=0}^{\infty} \frac{(x-4)^k}{2k+1}$

21.[R] $\sum_{k=0}^{\infty} \frac{k(x-2)^k}{2k+3}$

22.[R] $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k\ln(k)}$

23.[R] $\sum_{k=0}^{\infty} \frac{(x+3)^k}{5^k}$

24.[R] $\sum_{k=0}^{\infty} k(x+1)^k$

25.[R] $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k^2}$

26.[R] $\sum_{k=0}^{\infty} (-1)^k frac(x+4)^k k+2$

27.[R] $\sum_{k=0}^{\infty} k!(x-1)^k$

28.[R] $\sum_{k=0}^{\infty} \frac{k^2+1}{k^3+1}(x+2)^k$

In Exercises 29 to 34 write out the first five (non-zero) terms of the binomial expansion of the given functions.

29.[R] $(1+x)^{1/2}$

30.[R] $(1+x)^{1/3}$

31.[R] $(1+x)^{3/2}$

32.[R] $(1+x)^{-2}$
33.[R] $(1+x)^{-3}$
34.[R] $(1+x)^{-4}$

35.[R]

(a) If a power series $\sum_{k=0}^{\infty} b_k x^k$ diverges when $x=3$, at which x must it diverge?

(b) If a power series $\sum_{k=0}^{\infty} b_k (x+5)^k$ diverges when $x=-3$, at which x must it diverge?

36.[R] If $\sum_{k=0}^{\infty} b_k (x-3)^k$ converges for $x=7$, at what other values of x must the series necessarily converge?

37.[M] Find the radius of convergence of $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.

38.[M] If $\sum_{k=0}^{\infty} b_k x^k$ has a radius of convergence 3 and $\sum_{k=0}^{\infty} c_k x^k$ has a radius of convergence 5, what can be said about the radius of convergence of $\sum_{k=0}^{\infty}(b_k+c_k)x^k$?

39.[M]

(a) Use the first four nonzero terms of the Maclaurin series for $\sqrt{1+x^3}$ to estimate $\int_0^1 \sqrt{1+x^3}\, dx$. (This integral cannot be evaluated by the Fundamental Theorem of Calculus.)

(b) Evaluate the integral in (a) to three decimal places by Simpson's method.

40.[M]

(a) Write the first four terms of the Maclaurin series associated with $f(x) = (1+x)^{-3}$.

(b) Find a formula for the general term in the Maclaurin series associated with $f(x)$.

(c) Replace x by $-x$ in your answer to (b) to obtain the first four nonzero terms in the Maclaurin series for $(1-x)^{-3}$.

41.[M] What is the radius of convergence for the Maclaurin series associated with

(a) e^x

(b) $\sin(x)$

(c) $\cos(x)$

(d) $\ln(1+x)$

(e) $\arctan(x)$

(f) $(1+x)^{1/3}$

(g) $(1+2x)^{3/5}$

12.4 Manipulating Power Series

Where they converge, power series behave like polynomials. We can differentiate or integrate them term-by-term. We can add, subtract, multiply, and divide them. While most of the discussion will be on power series in x, the same ideas apply to power series in $(x-a)$. Proofs can be found in any advanced calculus text.

Differentiating a Power Series

In Section 3.3 we showed that you can differentiate the sum of a finite number of functions by adding their derivatives. Theorem 12.4.1 generalizes this to power series in x.

See the Sum and Difference Rules in Section 3.3

Theorem 12.4.1 (Differentiating a power series)**.** *Assume $R > 0$ and that $\sum_{k=0}^{\infty} b_k x^k$ converges to $f(x)$ for $|x| < R$. Then for $|x| < R$, f is differentiable, $\sum_{k=1}^{\infty} k b_k x^{k-1}$ converges to $f'(x)$, and*

$$f'(x) = b_1 + 2b_2x^2 + 3b_3x^3 + \cdots .$$

This theorem is *not* covered by the fact that the derivative of the sum of a *finite* number of functions is the sum of their derivatives.

Because f is differentiable it is continuous. Thus the limit as x approaches 0 of $\displaystyle\sum_{k=0}^{\infty} b_k x^k$ is b_0, the value of the series when $x = 0$. This property was used without justification in Example 1 in Section 12.2.

EXAMPLE 1 Obtain a power series for the function $1/(1-x)^2$ from the power series for $1/(1-x)$.

SOLUTION From the formula for the sum of a geometric series, we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{for } |x| < 1.$$

According to Theorem 12.4.1, differentiating both sides of this equation produces a valid equation, namely

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots \qquad \text{for } |x| < 1.$$

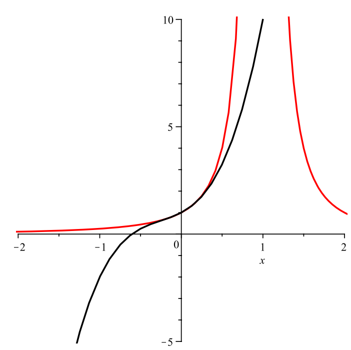


Figure 12.4.1:

This can be expressed in summation notation. The geometric series is $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. When we differentiate both sides of this equation, we obtain $\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}$. (See Figure 12.4.1.)

Note that the series can also be written as $\sum_{k=1}^{\infty}(k+1)x^k$ or $\sum_{k=1}^{\infty} kx^{k-1}$ or $\sum_{k=0}^{\infty}(k+1)x^k$.

Theorem 12.4.1 does not say anything about convergence at the endpoints of the interval of convergence. When $x = 1$ the series is $\sum_{k=1}^{\infty} k$ which diverges (because the terms of this series do not approach 0). This is not surprising, because the derivative (and, in fact, the original function) are not defined when $x = 1$. When $x = -1$, $\frac{1}{(1-x)^2} = \frac{1}{4}$, so the derivative of the function is well-defined. But, when the series for the derivative is evaluated at $x = -1$ we get the series $\sum_{k=0}^{\infty}(-1)^{k-1}k$. As when $x = 1$, the terms of this series do not converge to zero and the series diverges. ◇

Suppose that $f(x)$ has a power-series representation $b_0 + b_1x + b_2x^2 + \cdots$; Theorem 12.4.1 enables us to find the coefficients b_0, b_1, b_2, ...

Theorem 12.4.2 (Formula for b_k)**.** *Let R be a positive number and suppose that $f(x)$ is represented by the power series $\sum_{k=0}^{\infty} b_kx^k$ for $|x| < R$; that is,*

$$f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_kx^k + \cdots \qquad \text{for } |x| < R.$$

Then

$$b_k = \frac{f^{(k)}(0)}{k!}. \tag{12.4.1}$$

The proof is practically the same as the derivation of the formulas for the coefficients of Taylor polynomials in Section 5.4. It consists of repeated differentiation and evaluation of the higher derivatives at 0.

Theorem 12.4.2 also tells us that there can be at most one series of the form $\sum_{k=0}^{\infty} b_kx^k$ that represents $f(x)$, for the coefficients b_k are completely determined by $f(x)$ and its derivatives. That series must be the Maclaurin series we obtained in Section 12.1. For instance, the series $1+x+x^2+x^3+\cdots$, which sums to $1/(1-x)$ for $|x| < 1$ must be associated with $1/(1-x)$.

Integrating a Power Series

Just as we may differentiate a power series term by term, we can integrate it term by term.

Theorem 12.4.3. *(Integrating a power series) Assume that $R > 0$ and*

$$f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_kx^k + \cdots \qquad \text{for } |x| < R.$$

Then

$$b_0x + b_1\frac{x^2}{2} + b_2\frac{x^3}{3} + \cdots + b_k\frac{x^{k+1}}{k+1} + \cdots$$

converges for $|x| < R$, and

$$\int_0^x f(t)\ dt = b_0x + b_1\frac{x^2}{2} + b_2\frac{x^3}{3} + \cdots + b_k\frac{x^{k+1}}{k+1} + \cdots .$$

> **WARNING** *(Choosing Variables of Integration)* Note that t is used as the variable of integration. This is done to avoid writing $\int_0^x f(x)\ dx$, an expression in which x describes both the interval $[0, x]$ and the independent variable of the integrand.

The next example shows the power of Theorem 12.4.3.

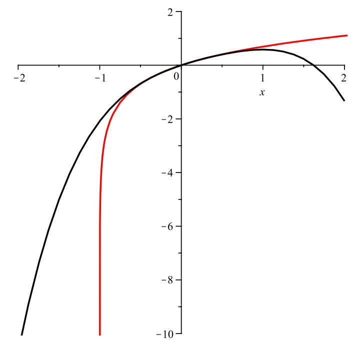


Figure 12.4.2:

EXAMPLE 2 Integrate the power series for $1/(1+x)$ to obtain the power series in x for $\ln(1+x)$.

SOLUTION Start with the geometric series $1/(1-x) = 1 + x + x^2 + \cdots$ for $|x| < 1$. Replace x by $-x$ and obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad \text{for } |x| < 1.$$

By Theorem 12.4.3, $\int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$ for $|x| < 1$.

Now,

$$\begin{aligned}\int_0^x \frac{dt}{1+t} &= \ln(1+t)|_0^x \\ &= \ln(1+x) - \ln(1+0) \\ &= \ln(1+x).\end{aligned}$$

Thus

Power series for $\ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \text{ for } |x| < 1.$$

◇

The power series for $\ln(1+x)$ can also be found using Theorem 12.4.2 on page 1028 but this requires calculating the derivatives of $\ln(1+x)$ and evaluating them at $x = 0$.

The derivation in Example 2 is more straightforward, and it gives the radius of convergence without additional work.

The Algebra of Power Series

In addition to differentiating and integrating power series, we may also add, subtract, multiply, and divide them just like polynomials, as Theorem 12.4.4 asserts.

Theorem 12.4.4. *The algebra of power series. Assume that*

$$f(x) = \sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + \cdots \quad \textit{for } |x| < R_1$$

and

$$g(x) = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots \quad \textit{for } |x| < R_2.$$

Let R be the smaller of R_1 and R_2. Then, for $|x| < R$,

$$\begin{aligned} f(x)+g(x) &= \sum_{k=0}^{\infty}(b_k+c_k)x^k = (b_0+c_0)+(b_1+c_1)x+(b_2+c_2)x^2+\cdots \\ f(x)-g(x) &= \sum_{k=0}^{\infty}(b_k-c_k)x^k = (b_0-c_0)+(b_1-c_1)x+(b_2-c_2)x^2+\cdots \\ f(x)g(x) &= (b_0c_0)+(b_0c_1+b_1c_0)x+(b_0c_2+b_1c_1+b_2c_0)x^2+\cdots \\ f(x)/g(x) &\ \textit{is}\ \ \textit{obtainable by long division, provided } g(x)\neq 0 \textit{ for all } |x|<R. \end{aligned}$$

EXAMPLE 3 Find the first four terms of the Maclaurin series for $e^x/(1-x)$.

SOLUTION There are at least three ways to approach this problem. The direct approach is to use Theorem 12.4.2; this requires finding the first three derivatives of $e^x/(1-x)$ evaluated at $x = 0$. A second idea is to divide the power series for e^x by $1-x$. The third idea is to multiply the power series for e^x and the power series for $1/(1-x)$.

See Exercise 6

As multiplication is generally easier to carry out than division, that is the option we choose. The power series for e^x is $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ (radius of convergence is ∞) and the power series for $1/(1-x)$ is $1+x+x^2+x^3+\cdots$ (radius of convergence is 1):

$$\begin{aligned} e^x\frac{1}{1-x} &= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3}+\cdots\right)\left(1+x+x^2+x^3+\cdots\right) \\ &= (1\cdot 1)+(1\cdot 1+1\cdot 1)x+\left(1\cdot 1+1\cdot 1+\frac{1}{2!}\cdots\right)x^2 \\ &\quad +\left(1\cdot 1+1\cdot 1+\frac{1}{2!}\cdots+\frac{1}{3!}\cdot 1\right)x^3+\cdots \\ &= 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^3+\cdots. \end{aligned}$$

According to Theorem 12.4.2, the power series for $e^x/(1-x)$, whose first four terms we just found, has radius of convergence $R = 1$, the smaller of 1 and ∞. ◇

EXAMPLE 4 Find the first four terms of the Maclaurin series associated with $e^x/\cos(x)$.

SOLUTION We attack this problem with Theorem 12.4.4. The Maclaurin series associated with $e^x/\cos(x)$ is the quotient of the Maclaurin series associated with e^x and $\cos(x)$. Long division shows us that

$$\frac{e^x}{\cos(x)} = 1+x+x^2+\frac{2x^3}{3}+\cdots$$

Even though the power series for e^x and $\cos(x)$ both have infinite radius of convergence, the fact that $\cos(\pi/2) = 0$ reduces the radius of convergence to $\pi/2$.

What happens when $|x| = \pi/2$?

We could have found the front-end of the Maclaurin series using Theorem 12.4.2, but this approach does not give any information about the radius of converges of this power series.

According to Theorem 12.4.2, the power series for $e^x/(1-x)$, whose first four terms we just found, has radius of convergence 1, the minimum of 1 and ∞. ◇

Power Series Around a

Power series in $x - a$

The various theorems and methods of this section were stated for power series in $x = x - 0$. Analogous theorems hold for power series in $x - a$. Such series may be differentiated and integrated term by term inside the interval in which they converge. For instance, Theorem 12.4.2 generalizes to the following result.

Theorem 12.4.5 (Formula for b_k)**.** *Let R be a positive number and suppose that $f(x)$ is represented by the power series $\sum_{k=0}^{\infty} b_k(x-a)^k$ for $|x-a| < R$; that is,*

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_k(x-a)^k + \cdots \qquad \textit{for } |x-a| < R.$$

Then

$$b_k = \frac{f^{(k)}(a)}{k!}.$$

The proof of Theorem 12.4.5 is similar to that of Theorem 12.4.2.

Endpoints

Each theorem in this section includes information on the radius of convergence of a power series obtained from another power series. Convergence at the endpoints is never mentioned; it must be checked separately in every case.

In Example 1 we found the power series in x for $1/(1-x)^2$ is

$$1 + 2x + 3x^2 + \cdots = \sum_{k=1}^{\infty} kx^{k-1} \qquad (12.4.2)$$

for $|x| < 1$. When $x = 1$ this series becomes $\sum_{k=1}^{\infty} k$, and, when $x = -1$ it is $\sum_{k=1}^{\infty} k(-1)^{k-1}$. Each of these series diverges because its terms do not approach 0 as $k \to \infty$. Thus, (12.4.2) converges only on the open interval $(-1, 1)$.

In Example 2 the power series for $\ln(1+x)$ was found to be

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \tag{12.4.3}$$

again for $|x| < 1$.

When $x = 1$ the series becomes $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. This is the alternating harmonic series, which converges to $\ln(2)$, as Exercise 29 shows. When $x = -1$ the series becomes $\sum_{k=1}^{\infty} \frac{-1}{k}$ which diverges because it is the negative of the harmonic series. This means the interval of convergence for (12.4.3) is $(-1, 1]$.

Some series converge at both endpoints. You can never tell what will happen until you check each endpoint.

How Some Calculators Find e^x

The power series in x for e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots.$$

For $x = 10$, this would give

$$e^{10} = 1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!} + \cdots + \frac{10^k}{k!} + \cdots.$$

Although the terms eventually become very small, the first few terms are quite large. (For instance, the fifth term, $10^4/4!$, is about 417.) So when x is large, the series for e^x provides a time-consuming procedure for calculating e^x.
Some calculators use the following method instead.
The values of e^x at certain inputs are built into the memory:

$$\begin{aligned} e^1 &\approx 2.718281828459 \\ e^{10} &\approx 22,026.46579 \\ e^{100} &\approx 2.6881171 \times 10^{43} \\ e^{0.1} &\approx 1.1051709181 \\ e^{0.01} &\approx 1.0100501671 \\ e^{0.001} &\approx 1.0010005002. \end{aligned}$$

To find $e^{315.425}$, say, the calculator makes use of the identities $e^{x+y} = e^x e^y$ and $(e^x)^y = e^{xy}$ and computes

$$\left(e^{100}\right)^3 \left(e^{10}\right)^1 \left(e^{1}\right)^5 \left(e^{0.1}\right)^4 \left(e^{0.01}\right)^2 \left(e^{0.001}\right)^5 \approx 9.71263198 \times 10^{136}.$$

This result is accurate to six decimal places.

Summary

We showed how to operate with power series to obtain new power series — by differentiation, integration, or an algebraic operation, such as multiplying or dividing two series. For instance, from the geometric series for $1/(1+x)$, you can obtain the series for $\ln(1+x)$ by integration, or the series for $-1/(1+x)^2$ by differentiation.

In many cases the radius of convergence for a derived power series can be determined directly from the radius of convergence of the original series and the operation performed. However, convergence at the endpoints must be checked for each series.

EXERCISES for Section 12.4 *Key:* R–routine, M–moderate, C–challenging

1.[R] Differentiate the Maclaurin series for $\sin(x)$ to obtain the Maclaurin series for $\cos(x)$.

2.[R] Differentiate the Maclaurin series for e^x to show that $D(e^x) = e^x$.

3.[R] Prove Theorem 12.4.2 by carrying out the necessary differentiations.

4.[R]

(a) Show that, for $|t| < 1$, $1/(1+t^2) = 1 - t^2 + t^4 - t^6 + \cdots$.

(b) Use Theorem 12.4.3 to show that, for $|x| < 1$, $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

(c) Give the formula for the k^{th} term of the series in (b).

(d) How many terms of the series in (b) are needed to approximate $\arctan(1/2)$ to three decimal places?

(e) Use the formula in (b) to estimate $\arctan(1/2)$ to three decimal places.

NOTE: Exercise 22 shows that the series in (b) converges to $\arctan(x)$ also when $x = -1$ and $x = 1$.

5.[R]

(a) Using Theorem 12.4.3, show that for $|x| < 1$,

$$\int_0^x \frac{dt}{1+t^3} = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots .$$

(b) Use (a) to express $\int_0^{0.7} dt/(1+t^3)$ as a series whose terms are numbers.

(c) How many terms of the series in (a) are needed to estimate $\int_0^{0.7} dt/(1+t^3)$ to three decimal places?

(d) Use (b) to evaluate $\int_0^{0.7} dt/(1+t^3)$ to three decimal places.

(e) Describe how you would evaluate $\int_0^{0.7} dt/(1+t^3)$ using the fundamental theorem of calculus. (*Do not carry out the details.*)

(f) Use a computer algebra system to find the exact value of $\int_0^{0.7} dt/(1+t^3)$.

6.[R]

(a) Find the first four nonzero terms of the Maclaurin series for $e^x/(1-x)$ by division of series. HINT: Keep the first five terms of e^x.

(b) Find the first four nonzero terms of the Maclaurin series for $e^x/(1-x^2)$ by using the formula for them in terms of derivatives.

7.[R]

(a) Find the first three nonzero terms of the Maclaurin series for $\tan(x)$ by dividing the series for $\sin(x)$ by the series for $\cos(x)$.

(b) Find the first two nonzero terms of the Maclaurin series for $\tan(x)$ by using the formula for the k^{th} term, $b_k = f^{(k)}(0)/k!$.

8.[R]

(a) Find the first four nonzero terms of the Maclaurin series for $(1-\cos(x))/(1-x^2)$ by division of series.

(b) Find the first four nonzero terms of the Maclaurin series for $(1-\cos(x))/(1-x^2)$ by multiplication of series.

In Exercises 9 and 10, obtain the first three nonzero terms in the Maclaruin series for the indicated functions by algebraic operations with known series. Also, state the radius of convergence.

9.[R] $e^x \sin(x)$

10.[R] $\frac{x}{\cos(x)}$

In Exercises 11 to 16 use power series to determine the limits.

11.[R] $\lim_{x\to 0} \frac{(1-\cos(x))^3}{x^6}$

12.[R] $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$

13.[R] $\lim_{x\to 0} \frac{\sin^2(x^3)e^x}{(1-\cos(x^2))^3}$

14.[R] $\lim_{x\to 0} \left(\frac{1}{\sin(x)} - \frac{1}{\ln(1+x)}\right)$

15.[R] $\lim_{x\to 0} \frac{(e^x-1)^2(\cos(3x))^2}{\sin(x^2)}$

16.[R] $\lim_{x\to 0} \frac{\sin(x)(1-\cos(x))}{e^{x^3}-1}$

17.[R] Estimate $\int_0^{1/2} \sqrt{x}e^{-x}\, dx$ to four decimal places.

18.[R] Let $f(x) = \sum_{k=0}^{\infty} k^2 x^k$.

(a) What is the domain of f?

(b) Find $f^{(100)}(0)$.

19.[R] Let $f(x) = \arctan(x)$. Making use of the Maclaurin series for $\arctan(x)$, find

(a) $f^{(100)}(0)$

(b) $f^{(101)}(0)$.

20.[M] Since $e^x e^y = e^{x+y}$, the product of the Maclaurin series for e^x and e^y should be the Maclaurin series for e^{x+y}. Check that for terms up to degree 3 in the series for e^{x+y}, this is the case.

21.[M]

(a) Give a numerical series whose sum is $\int_0^1 \sqrt{x}\sin(x)\, dx$.

(b) How many terms of the series in (a) are needed to approximate this integral to four decimal places?

(c) Use (a) to evaluate the integral to four decimal places.

22.[M] The Taylor series for $\arctan(x)$ is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. While the interval of convergence of this power series is easily found to be $[-1, 1]$, Theorem 12.4.3 tells us only that this series converges to $\arctan(x)$ on the open interval $(-1, 1)$.

(a) Show that, when $x = 1$, the given series is the Maclaurin series for $\arctan(1)$. Hint: Look at the Lagrange Form for the Remainder.

(b) Repeat (a), using $x = -1$.

(c) Because $\arctan(1) = \pi/4$, the Maclaurin series for $\arctan(1)$ provides one way to obtain approximations to π. Approximate π using the first 5 non-zero terms in the Maclaurin series for $\arctan(1)$.

(d) Estimate the error in the approximation to π found in (c).

(e) How many terms in the Maclaurin series are needed to obtain an approximate value of π accurate to 2 decimal places? 4 decimal places? 12 decimal places?

23.[M]

(a) From the Maclaurin series for $\cos(x)$, obtain the Maclaurin series for $\cos(2x)$.

(b) Exploiting the identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, obtain the Maclaurin series for $\sin^2(x)/x^2$.

(c) Estimate $\int_0^1 (\sin(x)/x)^2 \ dx$ using the first three nonzero terms of the series in (b).

(d) Find a bound on the error in the estimate in (c).

24.[M] Let $\sum_{k=0}^{\infty} b_k x^k$ and $\sum_{k=0}^{\infty} c_k x^k$ converge for $|x| < 1$. If, for all k, they converge to the same limit, must $b_k = c_k$?

25.[M] This exercise outlines a way to compute logarithms of numbers larger than 1.

(a) Show that every number $y > 1$ can be written in the form $(1 + x)/(1 - x)$ for some x in $(0, 1)$.

(b) When $y = 3$, find x.

(c) Show that if $y = (1 + x)/(1 - x)$, then $\ln(y) = 2(x + x^3/3 + \cdots + x^{2n+1}/(2n + 1) + \dots)$.

(d) Use (b) and (c) to estimate $\ln(3)$ to two decimal places. Hint: To control the error, compare a tail end of the series to an appropriate geometric series.

(e) Is the error in (d) less than the first omitted term?

26.[M] Sam has an idea: "I have a more direct way of estimating $\ln(y)$ for $y > 1$. I just use the identity $\ln(y) = -\ln(1/y)$. Because $1/y$ is in $(0, 1)$ I can write it as $1-x$, and x is still in (0,1). In short, $\ln(y) = -\ln(1/y) = -\ln(1-x) = x+x^2/2+x^3/3+\dots$. It's even an easier formula. And it's better because it doesn't have that coefficient 2 in front."

(a) Is Sam's formula correct?

(b) Use his method to estimate $\ln(3)$ to two decimal places.

(c) Which is better, Sam's method or the one in Exercise 25?

27.[M] Use the method of Exercise 25 to estimate ln(5) to two decimal places. Include a description of your procedure.

28.[C] Here are five ways to compute ln(2). Which seems to be the most efficient? least efficient? Explain.

(a) The series for $\ln(1+x)$ when $x=1$.

(b) The series for $\ln(1+x)$ when $x=\frac{-1}{2}$. NOTE: This gives $\ln\left(\frac{1}{2}\right)=-\ln(2)$.

(c) The series for $\ln((1+x)/(1-x))$ when $x=\frac{1}{3}$.

(d) Simpson's method applied to the integral $\int_1^2 \, dx/x$.

(e) The root of $e^x=2$. HINT: Use Newton's method.

29.[C] In the discussion of endpoints for the Maclaurin series for $\ln(1+x)$, we showed that the series converges for $x=1$, but we did not show that its sum is ln(2). To show that it does equal ln(2), integrate both sides of the following equation over $[0,1]$:

$$\frac{1+(-x)^{n+1}}{1+x}=1-x+x^2-x^3+\cdots+(-1)^n x^n.$$

HINT: Separate the left-hand side into two separate integrals. Then, take the limit as $n\to\infty$.

30.[C]

(a) Compute teh product of the Maclaurin series of degree 5 for e^x and e^y.

(b) How does the result compare with the first few terms of teh Maclaurin series for e^{x+y}?

31.[C]

(a) For which x does $\sum_{k=0}^{\infty} k^2x^k$ converge?

(b) Starting with the Maclaurin series for $x^2/(1-x)$, sum the series in (a).

(c) Does your formula seem to give the correct answer when $x=\frac{1}{3}$?

32.[C] This exercise uses power series to give a new perspective on l'Hôpital's rule. Assume that f and g can be represented by power series in some open interval containing 0:

$$f(x) = \sum_{k=0}^{\infty} b_k x^k \quad \text{and} \quad g(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Assume that $f(0) = 0$, $g(0) = 0$, and $g'(0) \neq 0$. Under these assumptions explain why

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}.$$

33.[C] If R. P. Feynman, *Lectures on Physics*, Addison-Wesley, Reading, MA, 1963, appears this remark:

> Thus the average velocity is
>
> $$\langle E \rangle = \frac{\hbar\omega(0 + x + 2x^2 + 3x^3 + \cdots)}{1 + x + x^2 + \cdots}.$$
>
> Now the two sums which appear here we shall leave for the reader to play with and have some fun with. When we are all finished summing and substituting for x in the sum, we should get — if we make no mistakes in the sum —
>
> $$\langle E \rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}.$$
>
> This, then, was the first quantum-mechanical formula ever known, or ever discussed, and it was the beautiful culmination of decades of puzzlement.

Have the aforementioned fun, given that $x = e^{-\hbar\omega/kT}$.

Exercises 34 to 37 outline a proof that the Maclaurin series associated with $(1+x)^r$ converges to $(1+x)^r$ for $|x| < 1$. This justifies the assertion that $(1+x)^r = \sum_{k=0}^{\infty} \begin{pmatrix} n \\ k \end{pmatrix} x^k$ for $|x| < 1$. The notation $\begin{pmatrix} n \\ k \end{pmatrix}$ stands for $\frac{n!}{k!(n-k)!}$.

34.[C] Show that

$$k \begin{pmatrix} r \\ k \end{pmatrix} + (k+1) \begin{pmatrix} r \\ k+1 \end{pmatrix} = r \begin{pmatrix} r \\ k \end{pmatrix}.$$

(This is needed in Exercise 35.) HINT: First, rewrite the equation as $(k+1) \begin{pmatrix} r \\ k+1 \end{pmatrix} = (r-k) \begin{pmatrix} r \\ k \end{pmatrix}$.

35.[C] Let $f(x) = \sum_{k=0}^{\infty} \binom{r}{k} x^k$.

(a) Find the interval of convergence for $f(x)$.

(b) Show that $(1+x)f'(x) = rf(x)$. HINT: First, write out the first four terms to see the pattern.

36.[C] Using the result from Exercise 35, show that the derivative of $f(x)/(1+x)^r$ is 0.

37.[C] Show that $f(x)/(1+x)^r = 1$, which implies that $\sum_{k=0}^{\infty} \binom{r}{k} x^k = (1+x)^r$. What is the interval of convergence

38.[C] Newton obtained the Maclaurin series for $\arcsin(x)$ with the aid of the binomial series for $\sqrt{1-x^2}$, as follows.
Consider the circle $x^2 + y^2 = 1$ and the point $Q = (x, y)$ on it, as shown in Figure 12.4.3. Then $\theta = \arcsin(x) = \angle QOR$.

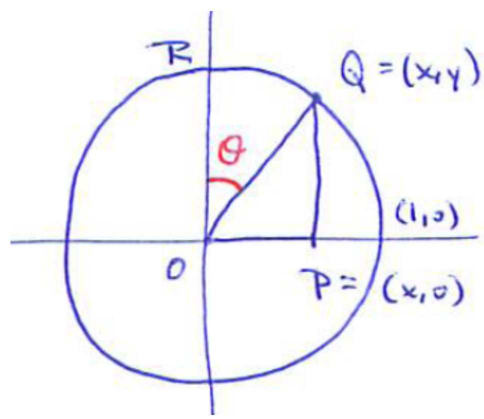


Figure 12.4.3:

(a) Then

$$\begin{aligned}\frac{\theta}{2} &= \text{area}OQR = \text{area}OPQR - \text{area}OPQ \\ &= \int_0^x \sqrt{1-t^2}\, dt - \frac{1}{2}x\sqrt{1-x^2}.\end{aligned}$$

Use this equation to obtain Newton's result:

$$\theta = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots \qquad (12.4.4)$$

(b) Use the fact that $\theta = \arcsin(x) = \int_0^x \frac{dt}{\sqrt{1-x^2}}$ to derive (12.4.4).

12.5 Complex Numbers

The number line of real numbers coincides with the x-axis of the xy coordinate system. With its addition, subtraction, multiplication, and division, it is a small part of a number system that occupies the plane, and which obeys the usual rules of arithmetic. This section describes that system, known as the **complex numbers**. One of the important properties of the complex numbers is that any nonconstant polynomial has a root; in particular, the equation $x^2 = -1$ has two solutions.

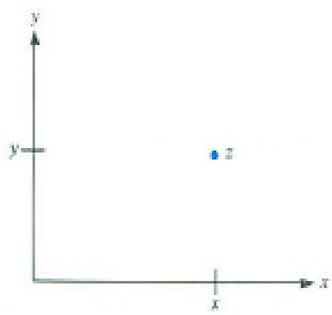


Figure 12.5.1:

The Complex Numbers

By a complex number z we mean an expression of the form $x + iy$ or $x + yi$, where x and y are real numbers and i is a symbol with the property that $i^2 = -1$. This expression will be identified with the point (x, y) in the xy plane, as in Figure 12.5.1. Every point in the xy plane may therefore be thought of as a complex number.

To add or multiply two complex numbers, follow the usual rules of arithmetic of real numbers, with one new proviso:

Whenever you see i^2, replace it by -1.

For instance, to add the complex numbers $3 + 2i$ and $-4 + 5i$, just collect like terms:

$$(3 + 2i) + (-4 + 5i) = (3 - 4) + (2i + 5i) = -1 + 7i.$$

(See Figure 12.5.2(a).) Addition does not make use of the fact that $i^2 = -1$. However, multiplication does, as Example 1 shows.

EXAMPLE 1 Compute the product $(2 + i)(3 + 2i)$.

SOLUTION We can multiply the complex numbers just as we would multiply binomials. (Recall the mnemonic FOIL for "first, outer, inner, last.") We have

$$(2+i)(3+2i) = 2\cdot3+2\cdot2i+i\cdot3+i\cdot2i = 6+4i+3i+2i^2 = 6+4i+3i-2 = 4+7i.$$

Figure 12.5.2(b) shows the complex numbers $2 + i$, $3 + 2i$, and their product $4 + 7i$. ◇

Note that $(-i)(-i) = i^2 = -1$. Both i and $-i$ are square roots of -1. The symbol $\sqrt{-1}$ traditionally denotes i rather than $-i$.

Real numbers are on the x-axis, imaginary on the y-axis.

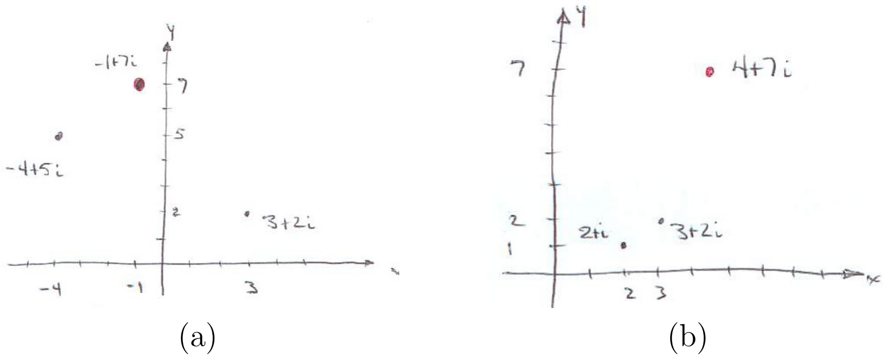


Figure 12.5.2:

A complex number that lies on the y-axis is called **imaginary**. Every complex number z is the sum of a real number and an imaginary number, $z = x + iy$. The number x is called the **real part of** z, and y is called the **imaginary part**. One often writes "$\operatorname{Re} z = x$" and "$\operatorname{Im} z = y$."

We have seen how to add and multiply complex numbers. Subtraction is straightforward. For instance,

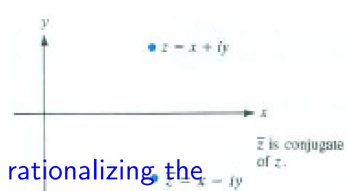
$$(3 + 2i) - (4 - i) = (3 - 4) + (2i - (-i)) = -1 + 3i.$$

conjugate of z

Division of complex numbers requires rationalizing the denominator. This involves the **conjugate** of a complex number. The conjugate of the complex number $z = x + yi$ is the complex number $x - yi$, which is denoted $\bar{z}$. Note that

$$\begin{aligned} z\bar{z} &= (x + yi)(x - yi) = x^2 + y^2 \\ z + \bar{z} &= (x + yi) + (x - yi) = 2x \\ \text{and} \qquad z - \bar{z} &= (x + yi) - (x - yi) = 2yi. \end{aligned}$$

Thus, $z\bar{z}$ and $z + \bar{z}$ are real, and $z - \bar{z}$ is imaginary. Figure 12.5.3 shows that $\bar{z}$ is the mirror image of z reflected across the x-axis. To "rationalize the denominator" means to find an equivalent fraction with a real-valued denominator. If the fraction is w/z, the denominator can be rationalized by multiplying by $\bar{z}/\bar{z}$.



rationalizing the denominator

Figure 12.5.3:

EXAMPLE 2 Compute the quotient $\frac{1+5i}{3+2i}$.
SOLUTION To rationalize the denominator, we multiply by $\frac{3-2i}{3-2i}$:

$$\frac{1+5i}{3+2i} = \frac{1+5i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{3 - 2i + 15i + 10}{9 - 6i + 6i + 4i^2} = \frac{13 + 13i}{13} = 1 + i.$$

◇

Every polynomial has a root in the complex numbers.

Now All Polynomials Have Roots

The complex numbers provide the equation $x^2 + 1 = 0$ with two solutions, i and $-i$. This illustrates an important property of complex numbers: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is any polynomial of degree $n \geq 1$, with real or complex coefficients, then there is a complex number z such that $f(z) = 0$. This fact, known as the **Fundamental Theorem of Algebra**, is illustrated in Example 3. Its proof requires advanced mathematics.

EXAMPLE 3 Solve the quadratic equation $z^2 - 4z + 5 = 0$.
SOLUTION By the quadratic formula, the solutions are

$$z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

The two solutions are $2 + i$ and $2 - i$.

These solutions can be checked by substitution in the original equation. For instance,

$$(2+i)^2 - 4(2+i) + 5 = (4 + 4i + i^2) - 8 - 4i + 5 = 4 + 4i - 1 - 8 - 4i + 5 = 0 + 0i = 0.$$

Yes, it checks. The solution $2 - i$ can be checked similarly. ◇

The sum of the complex numbers z_1 and z_2 is the fourth vertex (opposite O) in the parallelogram determined by the origin O and the points z_1 and z_2, as shown in Figure 12.5.4.

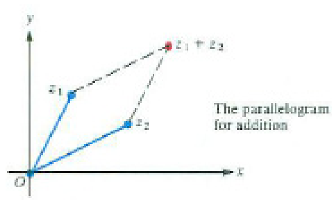


Figure 12.5.4:

A Concrete Approach to Complex Numbers

In our approach the number i came out of nowhere. We said only "whenever you see i^2 replace it by -1. There are other ways to introduce the complex numbers. We now present a second approach, one that is more concrete, though if you had not seen the approach in the text it might seem unmotivated and contrived.

In the approach we used, we calculated that $(a+bi)(c+di) = ac-bd+(ad+bc)i$. This suggests the second approach. Define the "product" of (a,b) and (c,d) to be $(c - bd, ad + bc)$. There is no mention of i. The definition of the "sum" is simpler: the "sum" of (a,b) and (c,d) is defined to be $(a+c, b+d)$. Finally, we define i to be the pair $(0,1)$.

This may suggest that we could introduce an addition and multiplication on points (a,b,c) in space. However, it has been shown that it is impossible to do this in a way that preserves the basic properties of arithmetic that we are used to. Interestingly, it is possible to do this in the space consisting of quadruples of the form (a,b,c,d). This is described on page 1050.

The Geometry of the Product

The geometric relation between z_1, z_2 and their product $z_1 z_2$ is easily described in terms of the magnitude and argument of a complex number. Each complex number z other than the origin is at a (positive) distance r from the origin and has a polar angle θ relative to the positive x-axis. The distance r is called the **magnitude of** z, and θ is called the **argument of** z. A complex number has an infinity of arguments differing from each other by an integer multiple of 2π. The complex number 0, which lies at the origin, has magnitude 0 and any polar angle as argument. In short, we may think of magnitude and argument as polar coordinates r and θ of z, with the restriction that r is nonnegative. The magnitude of z is denoted $|z|$. The symbol $\arg(z)$ denotes any of the arguments of z, it being understood that if θ is an argument of z, then so is $\theta + 2n\pi$ for any integer n.

Amplitude is a synonym for magnitude.

The symbols $|z|$ and $\arg(z)$

EXAMPLE 4

(a) Draw all complex numbers with magnitude 3.

(b) Draw the complex number z of magnitude 3 and argument $\pi/6$.

SOLUTION

(a) The complex numbers of magnitude 3 form a circle of radius 3 with center at 0. (See Figure 12.5.5.)

(b) The complex number of magnitude 3 and argument $\pi/6$ is shown (in red) in Figure 12.5.5.

◇

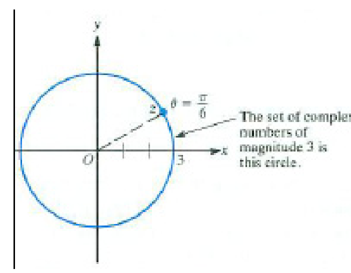


Figure 12.5.5: NOTE: Draw the point labeled z in red.

Note that $|x+iy| = \sqrt{x^2+y^2}$, by the Pythagorean theorem. Each complex number $z = x+iy$ other than 0 can be written as the product of a positive real number and a complex number of magnitude 1. To show this, let $z = x+iy$ have magnitude r and argument θ. Recalling the relation between polar and rectangular coordinates, we conclude that

$$z = r\cos(\theta) + r\sin(\theta)i = r(\cos(\theta) + i\sin(\theta)).$$

The number r is a positive real number. The magnitude of the number $\cos(\theta) + i\sin(\theta)$ is $\sqrt{\cos(\theta)^2 + \sin(\theta)^2} = 1$. Figure 12.5.6 shows the numbers r and $\cos(\theta) + i\sin(\theta)$, whose product is z. (The expression $\cos(\theta) + i\sin(\theta)$ appears so frequently when working with complex numbers that the shorthand notation $\operatorname{cis}(\theta)$ is used, that is, $\operatorname{cis}(\theta) = \cos(\theta) + i\sin(\theta)$. While this is convenient, you have to be careful not to confuse "cis" with "cos.")

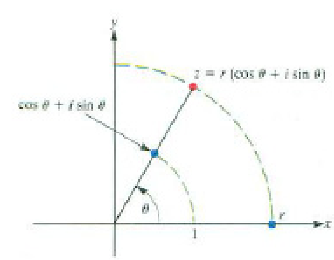


Figure 12.5.6: ARTIST: Draw the point for (b) in red.

The next theorem describes *how to multiply two complex numbers if they are given in polar form,* that is, in terms of their magnitudes and arguments.

Theorem. *Assume that z_1 has magnitude r_1 and argument θ_1 and that z_2 has magnitude r_2 and argument θ_2. Then the product $z_1 z_2$ has magnitude $r_1 r_2$ and argument $\theta_1 + \theta_2$.*

Proof

The last step uses the identities for $\cos(u+v)$ and $\sin(u+v)$.

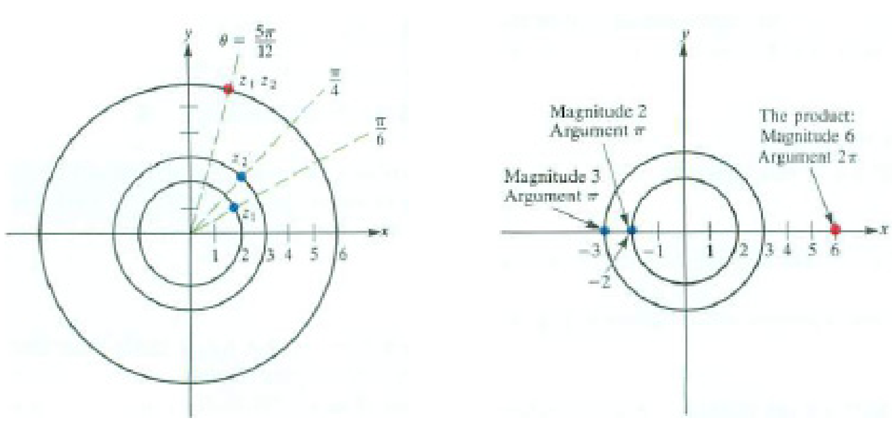
$$\begin{aligned} z_1 z_2 &= r_1(\cos(\theta_1) + i\sin(\theta_1))r_2(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \end{aligned}$$

Thus, the magnitude of $z_1 z_2$ is $r_1 r_2$ and the argument of $z_1 z_2$ is $\theta_1 + \theta_2$. This proves the theorem. •

In practical terms, this theorem says:

"To multiply two complex numbers, add their arguments and multiply their magnitudes."

EXAMPLE 5 Find $z_1 z_2$ for z_1 and z_2 in Figure 12.5.7(a).
SOLUTION z_1 has magnitude 2 and argument $\pi/6$; z_2 has magnitude 3 and argument $\pi/4$. Thus, $z_1 z_2$ has magnitude $2 \cdot 3 = 6$ and argument $\pi/6 + \pi/4 = 5\pi/12$. (See Figure 12.5.7 ◇



(a) (b)

Figure 12.5.7:

EXAMPLE 6 Using the geometric description of multiplication, find the product of the real numbers -2 and -3.
SOLUTION The number -2 has magnitude 2 and argument π. The number -3 has magnitude 3 and argument π. Therefore $(-2)\cdot(-3)$ has magnitude $2\cdot 3 = 6$ and argument $\pi+\pi = 2\pi$. The complex number with magnitude 6 and argument 2π is just our old friend, the real number 6. Thus $(-2)\cdot(-3) = 6$, in agreement with the statement "the product of two negative numbers is positive." (See Figure 12.5.7(b).) ◇

Division of Complex Numbers

See Exercise 29.

Division of complex numbers given in polar form is similar, except that the magnitudes are divided and the arguments are subtracted:

$$\frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right).$$

EXAMPLE 7 Let $z_1 = 6(\cos(\pi/2) + i\sin(\pi/2)$ and $z_2 = 3(\cos(\pi/6) + i\sin(\pi/6)$. Find *(a)* z_1z_2 and *(b)* z_1/z_2.
SOLUTION See Figure 12.5.8

(a)

$$\begin{aligned} z_1z_2 &= 6\cdot 3\left(\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)\right) = 18\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\ &= 18\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -9 + 9\sqrt{3}i. \end{aligned}$$

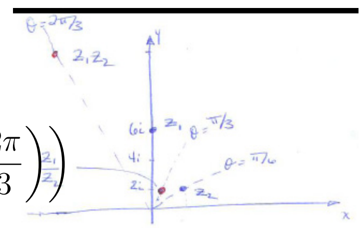


Figure 12.5.8:

(b)

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{6}{3}\left(\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\right) = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) \\ &= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i \end{aligned}$$

◇

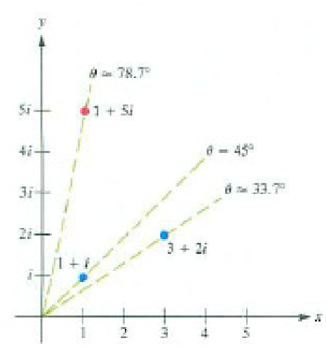


Figure 12.5.9:

EXAMPLE 8 Compute the product $(1+i)(3+2i)$ arithmetically and check the answer in terms of magnitudes and arguments.
SOLUTION

$$(1+i)(3+2i) = 3 + 2i + 3i + 2i^2 = 3 + 2i + 3i - 2 = 1 + 5i.$$

To check this calculation, we first verify that $|1+5i| = |1+i||3+2i|$. We have

$$\begin{aligned} |1+5i| &= \sqrt{1^2+5^2} = \sqrt{26}, \\ |1+i| &= \sqrt{1^2+1^2} = \sqrt{2}, \\ |3+2i| &= \sqrt{3^2+2^2} = \sqrt{13}. \end{aligned}$$

$\arg(x+iy) = \arctan(y/x)$ for $x+iy$ in the first or fourth quadrants.

Since $\sqrt{26} = \sqrt{2}\sqrt{13}$, the magnitude of $1+5i$ is the product of the magnitudes of $1+i$ and $3+2i$.

Next, consider the arguments. First, $\arg(1+5i) = \arctan(5) \approx 1.3734$. Similarly, $\arg(1+i) = \arctan(1) \approx 0.7854$ and $\arg(3+2i) = \arctan(2/3) \approx 0.5880$. Since $0.7854 + 0.5880 = 1.3734$, the argument of $1+5i$ is the sum of the arguments of $1+i$ and $3+2i$. (See also Figure 12.5.9.) ◇

Powers of z

How to compute z^n

When the polar coordinates of z are known, it is easy to compute the powers z^2, z^3, z^4, ... Let z have magnitude r and argument θ. Then $z^2 = z \cdot z$ has magnitude $r \cdot r = r^2$ and argument $\theta + \theta = 2\theta$. So, to square a complex number, just square it magnitude and double its argument (angle).

More generally, to compute z^n for any positive integer n, find $|z|^n$ and multiply the argument of z by n. In short, we have **DeMoivre's Law**:

$$(r(\cos(\theta) + i\sin(\theta)))^n . = r^n (\cos(n\theta) + i\sin(n\theta))$$

Example 9 illustrates the geometric view of computing powers.

EXAMPLE 9 Let z have magnitude 1 and argument $2\pi/5$. Compute and sketch z, z^2, z^3, z^4, z^5, and z^6.

SOLUTION Since $|z| = 1$, it follows that $|z^2| = |z|^2 = 1^2 = 1$. Similarly, for all positive integers n, $|z^n| = 1$; that is, z^n is a point on the unit circle with center O. All that remains is to examine the arguments of z^2, z^3, etc..

The argument of z^2 is twice the argument of z: $2(2\pi/5) = 4\pi/5$. Similarly, $\arg(z^3) = 6\pi/5$, $\arg(z^4) = 8\pi/5$, $\arg(z^5) = 10\pi/5 = 2\pi$, and $\arg(z^6) = 12\pi/5$. Observe that $z^5 = 1$, since it has magnitude 1 and argument 2π. Similarly, $z^6 = z$, since both z and z^6 have magnitude 1 and their arguments differ by an integer multiple of 2π. (Or, algebraically, $z^6 = z^{5+1} = z^5 \cdot z = 1 \cdot z = z$.) Figure 12.5.10 shows that the powers of z form the vertices of a regular pentagon. ◇

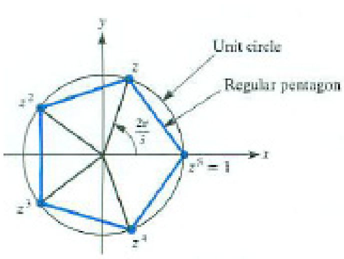


Figure 12.5.10:

The equation $x^5 = 1$ has only one real root, namely, $x = 1$. However, it has five complex roots. For instance, the number z shown in Figure 12.5.10

is a solution of $x^5 = 1$ since $z^5 = 1$. Another root is z^2, since $(z^2)^5 = z^{10} = (z^5)^2 = 1^2 = 1$. Similarly, z^3 and z^4 are roots of $x^5 = 1$. There is a total of five roots: 1, z, z^2, z^3, and z^4.

The powers of i.

The powers of i will be needed in the next section. They are $i^2 = -1$, $i^3 = i^2 \cdot i = (-1)i = -i$, $i^4 = i^3 \cdot i = (-i)i = -i^2 = 1$, $i^5 = i^4 \cdot i = i$, and so on. They repeat in blocks of four: for any integer n, $i^{n+4} = i^n$.

It is often useful to express a complex number $z = x + iy$ in polar form. Recall that $|z| = \sqrt{x^2 + y^2}$. To find θ, it is best to sketch z in order to see in which quadrant it lies. Although $\arctan(\theta) = y/x$ we cannot say that $\theta = arctan(y/x)$, since $\arctan(u)$ lies between $-\pi/2$ and $\pi/2$ for any real number u. However, the angle of z may lie in the second- or third-quadrant

For instance, to put $z = -2 - 2i$ in polar form, first sketch z, as in Figure 12.5.11. We have $|z| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$ and $\arg(z) = 5\pi/4$. Thus

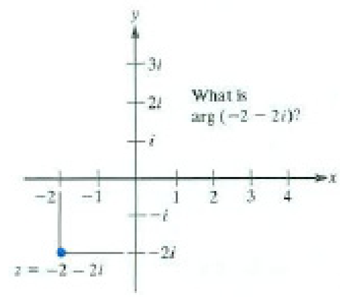


Figure 12.5.11:

$$z = \sqrt{8}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right).$$

Note that $\arctan(-2/(-2))$ is $\pi/4$ which is *not* an argument of z.

Roots of z

Each complex number z, other than 0, has exactly n n^{th} roots for each positive integer n. These can be found by expressing z in polar coordinates. If $z = r(\cos(\theta) + i\sin(\theta))$, that is, has magnitude r and argument θ, then one n^{th} root of z is

$$r^{1/n}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right).$$

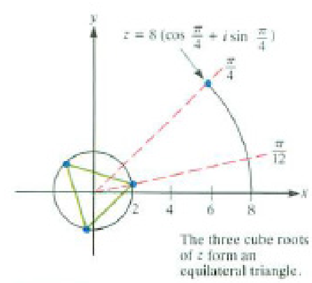


Figure 12.5.12:

To check that this is an n^{th} root of z, just raise it to the n^{th} power.

To find the other n^{th} roots of z, change the argument z from θ to $\theta + 2k\pi$, where $k = 1, 2, \ldots, n-1$. Then

$$r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$$

is also an n^{th} root of z. (Why?)

The n roots of the equation $z^n = a$ are the vertices of a regular polygon with n sides.

For instance, let $z = 8(\cos(\pi/4) + i\sin(\pi/4))$. Then the three cube roots of z all have magnitude $8^{1/3} = 2$. Their arguments are

$$\frac{\pi/4}{3} = \frac{\pi}{12}, \qquad \frac{\pi/4 + 2\pi}{3} = \frac{\pi}{12} + \frac{2\pi}{3}, \qquad \frac{\pi/4 + 4\pi}{3} = \frac{\pi}{12} + \frac{4\pi}{3}.$$

These three roots are shown in Figure 12.5.12, along with z.

From Point to Vector to Complex Number

Earlier in the text ordered pairs (x, y) stood for points in the plane. Then we enriched them by introducing an addition $(x, y) + (u, v) = (x + u, y + v)$. With this added structure we called (x, y) a vector and denoted it $< x, y >$.

Then we enriched the structure further by introducing a multiplication, (x, y) times $(u, v) = (xu - yv, xv + yu)$. We did that indirectly, by using a magical ”number” i whose square is -1 and saying that $(x + iy)(u + iv) = xu - yv + (xv + yu)i$. But there was no need to do that. Having defined the multiplication of ordered pairs, we then can define i as the particular ordered pair $(0, 1)$. While the vector structure easily generalizes to any dimensional space, the complex numbers do not. Only in dimensions one and two, the line and the plane, is it possible to impose a structure obeying the usual rules of arithmetic. Even in going to dimension two we lost an important property of the real numbers. What is that property? There is a structure in dimension four, called the ”quaternions,” but its multiplication is not commutative.

Summary

The real numbers, with which we all grew up, are just a small part of the complex numbers, which fill up the xy plane. We add complex numbers by a “parallelogram law.” To multiply them “we multiply their magnitudes and add their angles.” Using the complex numbers we can see that “negative real time negative real is positive,” since $180° + 180° = 360°$, which describes the positive x-axis. We also saw how to raise a complex number to a power and how to take its roots. We can now view points in the xy plane as “numbers.” However, mathematicians have shown that we cannot treat points in space as “numbers” that satisfy the usual rules of addition and multiplication.

EXERCISES for Section 12.5 *Key:* R–routine, M–moderate, C–challenging

In Execises 1 to 4 express the given quantities in teh form $x + iy$.
1.[R]

(a) $(2 + 3i) + (5 - 2i)$

(b) $(2 + 3i)(2 - 3i)$

(c) $\frac{1}{2-i}$

(d) $\frac{3+2i}{4-i}$

2.[R]

(a) $(2 + 3i)^2$

(b) $\frac{4}{3-i}$

(c) $(1 + i)(3 - i)$

(d) $\frac{1+5i}{2-3i}$

3.[R]

(a) $(1 + 3i)^2$

(b) $(1 + i)(1 - i)$

(c) i^{-3}

(d) $\frac{4+\sqrt{2}i}{2+i}$

4.[R]

(a) $(1 + i)^3$

(b) $\frac{i}{1-i}$

(c) $(3 + i)^{-1}$

(d) $(5 + 2i)(5 - 2i)$

In Exercises 5 to 8 express the number in polar form $r(\cos(\theta) + i\sin(\theta))$ with θ is $[0, 2\pi]$.

5.[R] $\sqrt{3} + i$

6.[R] $\sqrt{3} - i$

7.[R] $\sqrt{2} + \sqrt{2}i$

8.[R] $-4 + 4i$

In Exercises 9 to 12 express the number in both polar and rectangular forms.

9.[R] $(-1 + i)^{10}$

10.[R] $(\sqrt{3} + i)^4$

11.[R] $(2 + 2i)^8$

12.[R] $1 - \sqrt{3}i)^7$

13.[R] Rationalize the denominator in each fraction. That is, express the fraction as an equivalent fraction whose denominator is an integer.

(a) $\frac{1}{1+\sqrt{2}}$

(b) $\frac{1}{2-i}$

(c) $\frac{2-\sqrt{3}}{\sqrt{3}+2}$

(d) $\frac{3+2i}{i-3}$

14.[R] For each equation, (i) find all solutions, (ii) plot all solutions in the complex plane, and (iii) check that the solutions satisfies the equations.

(a) $x^2 + x + 1 = 0$

(b) $x^2 - 3x + 5 = 0$

(c) $2x^2 + x + 1 = 0$

(d) $3x^2 + 4x + 5 = 0$

15.[R]

(a) Use the quadratic formula to find all solutions of the equation $x^2 + x + 1 = 0$.

(b) Plot the solutions in (a).

(c) Check that the solutions in (a) satisfy $x^2 + x + 1 = 0$.

16.[R] Let z_1 have magnitude 2 and argument $\pi/6$, and let z_2 have magnitude 3 and argument $\pi/3$.

(a) Plot z_1 and z_2.

(b) Find $z_1 z_2$ using the polar form.

(c) Write z_1 and z_2 in the rectangular form $x + yi$.

(d) With the aid of (c) compute $z_1 z_2$.

(e) Check that (b) and (d) give the same point.

17.[R] Let z_1 have magnitude 2 and argument $\pi/4$, and let z_2 have magnitude 3 and argument $3\pi/4$.

(a) Plot z_1 and z_2.

(b) Find $z_1 z_2$ using the polar form.

(c) Write z_1 and z_2 in the form $x + yi$.

(d) With the aid of (c) compute $z_1 z_2$.

(e) Check that (b) and (d) give the same point.

18.[R] The complex number z has argument $\pi/3$ and magnitude 1. Find and plot (a) z^2, (b) z^3, and (c) z^4.

19.[R] Find and plot (a) i^3, (b) i^4, (c) i^5, and (d) i^{73}.

20.[R] Let z have magnitude 2 and argument $\pi/6$.

(a) What are the magnitude and argument of z^2, z^3, and z^4.

(b) Sketch z, z^2, z^3, and z^4.

(c) What are the magnitude and argument of z^n?

21.[R] Let z have magnitude 0.9 and argument $\pi/4$.

(a) Find and plot z^2, z^3, z^4, z^5, and z^6.

(b) What happens to z^n as $n \to \infty$?

22.[R] Find and plot all solutions of the equation $z^5 = 32\left(\cos(\pi/4) + i\sin(\pi/4)\right)$.

23.[R] Find and plot all solutions of the equation $z^4 = 8+8\sqrt{3}i$. HINT: First draw $8+8\sqrt{3}i$.

24.[R] Let z have magnitude r and argument θ. Let w have magnitude $1/r$ and argument $-\theta$. Show that $zw = 1$. NOTE: w is called the **reciprocal of** z, and denoted z^{-1} or $1/z$.

25.[R] Find z^{-1} if $z = 4+4i$. NOTE: See Exercise 24.

26.[R]

(a) By substitution, verify that $2+3i$ is a solution of the equation $x^2-4x+13 = 0$.

(b) Use the quadratic formula to find all solutions of the equation $x^2-4x+13 = 0$.

27.[R] Write in polar form

(a) $5+5i$,

(b) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$,

(c) $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$,

(d) $3+4i$.

28.[R] Write in rectangular form as simply as possible:

(a) $3\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$,

(b) $2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$,

(c) $10\left(\cos(\pi) + i\sin(\pi)\right)$,

(d) $\frac{1}{5}\left(\cos(22^\circ) + i\sin(22^\circ)\right)$ HINT: Express the answer to at least three decimal places.

29.[R] Let z_1 have magnitude r_1 and argument θ_1, and let z_2 have magnitude r_2 and argument θ_2.

(a) Explain why the magnitude of z_1/z_2 is r_1/r_2.

(b) Explain why the argument of z_1/z_2 is $\theta_1 - \theta_2$.

30.[R] Compute

$$\frac{\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)}$$

by two ways: (a) by the result of Exercise 29, (b) by rationalizing the denominator.

31.[R] Compute

(a) $(2+3i)(1+i)$

(b) $\frac{2+3i}{1+i}$

(c) $(7-3i)(\overline{7-3i})$

(d) $3(\cos(42^\circ) + i\sin(42^\circ)) \cdot 5(\cos(168^\circ) + i\sin(168^\circ))$

(e) $\frac{\sqrt{8}(\cos(147^\circ)+i\sin(147^\circ)}{\sqrt{2}(\cos(57^\circ)+i\sin(57^\circ))}$

(f) $1/(3-i)$

(g) $((\cos(52^\circ) + i\sin(52^\circ))^{-1}$

(h) $\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^{12}$

32.[R] Compute

(a) $(4+3i)(4-3i)$

(b) $\frac{3+5i}{-2+i}$

(c) $\frac{1}{2+i}$

(d) $\left(\cos(\left(\frac{\pi}{12}\right) + i\sin(\left(\frac{\pi}{12}\right)\right)^{20}$

(e) $(r(\cos(\theta) + i\sin(\theta))^{-1}$

(f) $\mathrm{Re}\left((r(\cos(\theta) + i\sin(\theta)))^{10}\right)$

(g) $\frac{3\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)}{5-12i}$

33.[R] Find and plot all solutions of $z^3 = i$.

34.[R] Sketch all complex numbers z such that (a) $z^6 = 1$, (b) $z^6 = 64$, (c) $z^6 = -1$.

35.[R]

(a) Why is the symbol $\sqrt{-4}$ ambiguous?

(b) Draw all solutions of $z^2 = -4$.

36.[R] If z_k has argument θ_k and magnitude r_k, $k = 1, 2$, write each of the following in the form $r(\cos(\theta) + i\sin(\theta))$.

(a) z_1^2

(b) $1/z_1$

(c) $\bar{(}z_1)$

(d) $z_1 z_2$

(e) z_1/z_2

(f) $1/\overline{z_1}$

37.[R] Draw the six sixth roots of

(a) 1

(b) 64

(c) i

(d) -1

(e) $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$

38.[M] Using the fact that

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

find formulas for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

39.[M]

(a) If $|z_1| = 1$ and $|z_2| = 1$, how large can $|z_1 + z_2|$ be? HINT: Draw some pictures.

(b) If $|z_1| = 1$ and $|z_2| = 1$, what can be said about $|z_1 z_2|$?

40.[M] Show that (a) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, (b) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

41.[M] Let $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

(a) Compute z^2 algebraically.

(b) Compute z^2 by putting z into polar form.

(c) Sketch the numbers z, z^2, z^3, z^4, and z^5.

42.[M] Let a, b, and c be complex numbers such that $a \neq 0$ and $b^2 - 4ac \neq 0$. Show that $ax^2 + bx + c = 0$ has two distinct roots.

43.[M] Find and plot the roots of $x^2 + ix + 3 - i = 0$.

44.[M] Compute the roots of the following equation and plot them relative to the same axes:

(a) $x^2 - 3x + 2 = 0$

(b) $x^2 - 3x + 2.25 = 0$

(c) $x^2 - 3x + 2.5 = 0$

(d) $x^2 - 3x + 1.5 = 0$

45.[M] The complex number $z = t + i$ (t a real number) lies on the line $y = 1$.

(a) Plot z^2 for $t = 0$, 1, -1, and for at least two other values of t.

(b) Find the equation of the curve on which z^2 lies.

46.[M] The complex number $z = x + i/x$, $x > 0$, lies on the curve $y = 1/x$.

(a) Plot z^2 for $x = 1$, 2, 3, and for at least two other (positive) values of x.

(b) Determine the curve on which z^2 lies.

47.[M] The complex number $z = t + i$ (t a real number) lies on the line $y = 1$.

(a) Plot z^2 for, at least, $x = 0$, 1, and -1.

(b) Determine the curve on which z^2 lies.

48.[M] The complex number $z = 1 + ti$ (t a real number) lies on the line $x = 1$.

(a) Plot the points $1/z$ for $t = 0$, 1, -1, and 2.

(b) Determine the curve on which $1/z$ lies.

49.[M]

(a) Draw the curve on which $z = t + 2ti$ lies.

(b) Draw the curve on which z^2 lies.

50.[M]

(a) Let $z = 1 + \sqrt{3}i$. Plot z, $\overline{z}$, and $1/z$ on the same set of axes.

(b) Let $z = (1 + i)/\sqrt{2}$. Plot z, $\overline{z}$, and $1/z$ on the same set of axes.

(c) Let $z = 3$. Plot z, $\overline{z}$, and $1/z$ on the same set of axes.

(d) Let $z = 2i$. Plot z, $\overline{z}$, and $1/z$ on the same set of axes.

(e) For an arbitrary complex number z, give a verbal explanation (no equations and no graphjs) of the relationships among z, $\overline{z}$, and $1/z$.

51.[C] For which complex numbers z is $\overline{z} = 1/z$?

52.[C] Let z be a point on the line $x + y = 1$.

(a) On what curve does z^2 lie?

(b) On what curve does $1/z$ lie?

HINT: In each case, plot a few points. See also Exercise 50.

53.[C] Let $z = \frac{1}{2} + \frac{i}{2}$.

(a) Sketch the numbers z^n for $n = 1$, 2, 3, 4, and 5.

(b) What happens to z^n as $n \to \infty$?

54.[C] Let $z = 1 + i$.

(a) Sketch the numbers $z^n/n!$ for $n = 1$, 2, 3, 4, and 5.

(b) What happens to $z^n/n!$ as $n \to \infty$?

55.[C]

(a) Graph $r = \cos(\theta)$ in polar coordinates.

(b) Pick five points on the curve in (a). Viewing each as a complex number z, plot z^2.

(c) As z runs through the curve in (a), what curve does z^2 sweep out? HINT: Give its polar equation.

56.[C] The partial-fraction representation of a rational function is much simpler when we have complex numbers available. No second-degree polynomial ax^2+bx+c is needed. This exercise indicates why this is the case.
Let z_1 and z_2 be the roots of $ax^2 + bx + c = 0$, $a \neq 0$.

(a) Using the quadratic formula (or by other means), show that $z_1 + z_2 = -b/a$ and $z_1 z_2 = c/a$.

(b) From (a) deduce that

$$ax^2 + bx + c = a(x - z_1)(x - z_2).$$

(c) With the aid of (b) show that

$$\frac{1}{ax^2 + bx + c} = \frac{1}{a(z_1 - z_2)}\left(\frac{1}{x - z_1} - \frac{1}{x - z_2}\right).$$

Part (c) shows that the theory of partial fractions, described in Section 8.4, becomes much simpler when complex numbers are allowed as the coefficients of the polynomials. Only partial fractions of the form $k/(ax+b)^n$ are needed.

57.[C] Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, where each coefficient is real.

(a) Show that if c is a root of $f(x) = 0$, then so is $\overline{c}$.

(b) Show that if c is a root of f and is not real, then $(x - c)(x - \overline{c})$ divides $f(x)$.

(c) Using the fundamental theorem of algebra, show that any fourth-degree polynomial with real coefficients can be expressed as the product of polynomials of degree at most 2 with real coefficients.

Exercise 58 is related to Exercise 90 on page 781. (See also Exercises 5 and 6 at the end of this chapter.)
58.[C] Let a point **0** be a distance $a \neq 1$ from the center of a unit circle.

(a) Show that the average value of the (natural) logarithm of the distance from **0** to points on the circumference is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\ln(1 + a^2 - 2a\cos(\theta))\, d\theta.$$

(b) Spend at least three minutes, but at most 5 minutes, trying to evaluate the integral in (a).

Note: The problem was first seen in Exercise 90 on page 781 and continues in Exercises 4 to 6 in Section 12.7.

12.6 The Relation Between the Exponential and the Trigonometric Functions

With the aid of complex numbers Leonard Euler discovered in 1743 that the trigonometric functions can be expressed in terms of the exponential function e^z, where z is complex. This section retraces his discovery. In particular, it will be shown that

Expressing $\sin(x)$ and $\cos(x)$ in terms of the exponential function.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Complex Series

In order to relate the exponential function to the trigonometric functions, we will use infinite series such as $\sum_{k=0}^{\infty} z_k$, where the z_k's are complex numbers. Such a series is said to converge to S if its n^{th} partial sum S_n approaches S in the sense that $|S - S_n| \to 0$ as $n \to \infty$. It is shown in Exercise 37 that if $\sum_{k=0}^{\infty} |z_k|$ (a series with real-valued terms) converges, so does $\sum_{k=0}^{\infty} z_k$, and the series $\sum_{k=0}^{\infty} z_k$ is said to *converge absolutely*. If a series converges absolutely, we may rearrange the terms in any order without changing the sum.

Here, $|\cdot|$ refers to the magnitude of a complex number.

Let $z_k = x_k + iy_k$, where x_k and y_k are real. If $\sum_{k=0}^{\infty} z_k$ converges, so do $\sum_{k=0}^{\infty} x_k$ and $\sum_{k=0}^{\infty} y_k$. If $\sum_{k=0}^{\infty} z_k = S = a + bi$, then $\sum_{k=0}^{\infty} x_k = a$ and $\sum_{k=0}^{\infty} y_k = b$. $\sum_{k=0}^{\infty} x_k$ is called the *real part* of $\sum_{k=0}^{\infty} z_k$ and $\sum_{k=0}^{\infty} y_k$ is the *imaginary part* of $\sum_{k=0}^{\infty} z_k$.

$\mathrm{Re}\left(\sum_{k=0}^{\infty} z_k\right) = \sum_{k=0}^{\infty} x_k$
$\mathrm{Im}\left(\sum_{k=0}^{\infty} z_k\right) = \sum_{k=0}^{\infty} y_k$

EXAMPLE 1 Determine for which complex numbers z, $\sum_{k=0}^{\infty} z^k/k!$ converges.

SOLUTION We will examine absolute convergence, that is, the convergence of $\sum_{k=0}^{\infty} |z^k|/k!$. This series has real terms. In fact, it is the Maclaurin series for $e^{|z|}$, which converges for all real numbers. Since $\sum_{k=0}^{\infty} z^n/n!$ converges absolutely for all z, it converges for all z. ◇

$|z|$ is a real number

Defining e^z

The Maclaurin series for e^x when x is real suggests the following definition:

In some treatments of exponentials e^z is defined as a power series and e is defined as the value of the series when $z = 1$.

DEFINITION *(e^z for complex z.)* Let z be a complex number. Define e^z to be the sum of the convergent series $\sum_{k=0}^{\infty} z^k/k!$.

Observe that when z happens to be real, $z = x$, e^z is our familiar real-valued exponential function: e^x. It can be shown by multiplying the series for e^{z_1} and e^{z_2} that $e^{z_1+z_2} = e^{z_1}e^{z_2}$ in accordance with the basic law of exponents.

Figure 12.6.1: The license plate of mathematician Martin Davis, whose e-mail signature is "eipye, add one, get zero."

When the expression for z is complicated, we sometimes write e^z as $\exp(z)$. For example, in exp notation the law of exponents becomes $\exp(z_1 + z_2) = (\exp(z_1))(\exp(z_2))$.

Euler's Formula: The Link between $e^{i\theta}$, $\cos(\theta)$, and $\sin(\theta)$

The following theorem of Euler provides the key link between the exponential function e^z and the trigonometric functions $\cos(\theta)$ and $\sin(\theta)$.

Euler's Formula

Theorem 12.6.1 (Euler's Formula)**.** *Let θ be a real number. Then*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Proof

Recall that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, ...

By definition of e^z for any complex number,

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots. \\
&= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots. \\
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots. \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right) \quad \text{(rearranging)} \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}$$

Figure **??** shows $e^{i\theta}$, which lies on the standard unit circle. •

Theorem 12.6.1 asserts, for instance, that

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i \cdot 0 = -1.$$

The equation $e^{i\pi} = -1$ is remarkable in that it links e (the fundamental number in calculus), π (the fundamental number in trigonometry), i (the fundamental complex number), and the negative number -1. The history of that short equation would recall the struggles of hundreds of mathematicians to create the number system that we now take for granted. It is as important in mathematics as $F = ma$ or $E = mc^2$ in physics.

There is an old saying: "God created the complex numbers; anything less is the work of man."

With the aid of Theorem 12.6.1, both $\cos(\theta)$ and $\sin(\theta)$ may be expressed in terms of the exponential function.

Theorem 12.6.2. *Let θ be a real number. Then*

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \textit{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Proof
We begin with Euler's formula (Theorem 12.6.1),

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{12.6.1}$$

Replacing θ by $-\theta$ in (12.6.1), we obtain

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta). \tag{12.6.2}$$

The sum of (12.6.1) and (12.6.2) yields

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta),$$

hence

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Subtraction of (12.6.2) from (12.6.1) yields

$$e^{i\theta} - e^{-i\theta} = 2i\sin(\theta),$$

hence

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

This establishes the two results in this theorem. •

The hyperbolic functions $\cosh(x)$ and $\sinh(x)$ were defined in terms of the exponential function by

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

sinh and cosh were defined in Section 4.1, see Exercises 49 to 52 on page 301.

Theorem 12.6.2 shows the trigonometric functions could be similarly defined in terms of the exponential function — if complex numbers were available. This means one could bypass right triangles and unit circles when defining $\sin(\theta)$ and $\cos(theta)$.

Just as Maxwell discovered the connection between light and electricity, Euler discovered the connection between the exponential and trigonometric functions.

Indeed, from the complex numbers and e^z we could even obtain the derivative formulas for $\sin(\theta)$ and $\cos(\theta)$. For instance,

$$\frac{d}{d\theta}\sin(\theta) = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)' = \frac{ie^{i\theta} + ie^{-i\theta}}{2i} = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta).$$

(That the familiar rules for differentiation extend to complex-valued functions is justified in a course in complex variables.)

Sketching e^z

Magnitude and argument of e^{x+iy}

If $z = x + iy$, the evaluation of e^z can be carried out as follows:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

The magnitude of e^{x+iy} is e^x and the argument of e^{x+iy} is y.

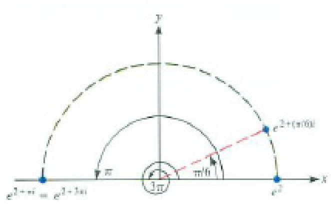


Figure 12.6.3:

EXAMPLE 2 Compute and sketch (a) $e^{2+(\pi/6)i}$, (b) $e^{2+\pi i}$, and (c) $e^{2+3\pi i}$.
SOLUTION (a) $e^{2+(\pi/6)i}$ has magnitude e^2 and argument $\pi/6$. (b) $e^{2+\pi i}$ has magnitude e^2 and argument π; it equals $-e^2$. (c) $e^{2+3\pi i}$ has magnitude e^2 and argument 3π, so is the same number as the number in (b). The results are sketched in Figure 12.6.3. ◇

The next example illustrates a typical computation in alternating currents. Electrical engineers frequently use j as the symbol for i (so they can use i to represent current).

EXAMPLE 3 Find the real part of $100e^{j(\pi/6)}e^{j\omega t}$. Here t refers to time, ω is a real constant related to frequency, and j is the mathematician's i.
SOLUTION

$$\begin{aligned} 100e^{j(\pi/6)}e^{j\omega t} &= 100e^{j(\pi/6)+j\omega t} \\ &= 100e^{j(\pi/6+\omega t)} \\ &= 100\left(\cos\left(\frac{\pi}{6}+\omega t\right) + i\sin\left(\frac{\pi}{6}+\omega t\right)\right). \end{aligned}$$

Thus

$$\mathrm{Re}\left(100e^{j(\pi/6)}e^{j\omega t}\right) = 100\cos\left(\frac{\pi}{6}+\omega t\right).$$

◇

It is often convenient to think of $\cos(\theta)$ as $\mathrm{Re}\left(e^{i\theta}\right)$. The next example exploits this point of view.

EXAMPLE 4 Evaluate $\sum_{k=0}^{\infty}\cos(k\theta)$.
SOLUTION Recall that $e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$. Hence $\cos(k\theta) = \mathrm{Re}\left(e^{ik\theta}\right)$, and we have

Recall the definition of e^z.

$$\sum_{k=0}^{\infty}\frac{\cos(k\theta)}{2^k} = \sum_{k=0}^{\infty}\mathrm{Re}\left(\frac{e^{ik\theta}}{2^k}\right) = \mathrm{Re}\left(\sum_{k=0}^{\infty}\frac{e^{ik\theta}}{2^k}\right) \qquad (12.6.3)$$

To simplify the complex-valued expression inside the parentheses, notice that

$$\frac{e^{ik\theta}}{2^k} = \left(\frac{e^{i\theta}}{2}\right)^k.$$

Now, because $\left|e^{i\theta}/2\right| = 1/2 < 1$, this "geometric" series converges with sum

$$\frac{1}{1-\left(\frac{e^{i\theta}}{2}\right)} = \frac{2}{2-\cos(\theta)-i\sin(\theta)} = \frac{2(2-\cos(\theta)+i\sin(\theta))}{(2-\cos(\theta))^2+(\sin(\theta))^2}. \qquad (12.6.4)$$

Inserting (12.6.4) as the sum of the series in (12.6.3) gives

$$\sum_{k=0}^{\infty}\frac{\cos(k\theta)}{2^k} = \operatorname{Re}\left(\sum_{k=0}^{\infty}\frac{e^{ik\theta}}{2^k}\right) = \operatorname{Re}\left(\frac{2(2-\cos(\theta)+i\sin(\theta))}{5-4\cos(\theta)}\right) = \frac{2(2-\cos(\theta))}{5-4\cos(\theta)}.$$

◇

Summary

Using power series, we obtained the fundamental relation $e^{i\theta} = \cos(\theta)+i\sin(\theta)$ and showed that $\cos(\theta)$ and $\sin(\theta)$ can be expressed in terms of the exponential function. Since $\ln(x)$ is the inverse of e^x, it too is obtained from the exponential function. We may define even x^n, $x > 0$, in terms of the exponential function as $e^{n\ln(x)}$. Similarly, a^x, $a > 0$, can be defined as $e^{x\ln(a)}$. These observations suggest that the most fundamental function in calculus is e^x, where x is real or complex.

EXERCISES for Section 12.6 *Key:* R–routine, M–moderate, C–challenging

In Exercises 1 to 6 sketch the numbers given and state their real and imaginary parts.

1.[R] $e^{5\pi i/4}$

2.[R] $5e^{\pi i/4}$

3.[R] $2e^{\pi i/4} + 3e^{\pi i/6}$

4.[R] e^{2+3i}

5.[R] $e^{\pi i/6}e^{3\pi i/4}$

6.[R] $2e^{\pi i} \cdot 3e^{-\pi i/3}$

In Exercises 7 to 10 express the given numbers in the form $re^{i\theta}$ for a positive real number r and argument θ, where $-\pi < \theta \leq \pi$.

7.[R] $\frac{e^2}{\sqrt{2}} - \frac{e^2}{\sqrt{2}}i$

8.[R] $3\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

9.[R] $5\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) \cdot 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

10.[R] $7\left(\cos\left(\frac{7\pi}{3}\right) + i\sin\left(\frac{7\pi}{3}\right)\right)$

In Exercises 11 to 14 plot $\exp(z)$ for the given values of z:

11.[R] $z = 2$

12.[R] $\pi i/2$

13.[R] $2 - \pi i/3$

14.[R] $-1 + 17\pi i/6$

In Exercises 15 to 18 plot the given complex numbers:

15.[R] $\exp(\pi i/4 + 3\pi i)$

16.[R] $\exp(1 + 9\pi i/4$

17.[R] $\exp(2 - \pi i/3)$

18.[R] $\exp(-1 + 17\pi i/6)$

19.[R] Let $z = e^{a+bi}$. Find (a) $|z|$, (b) $\bar{z}$, (c) z^{-1}, (d) $\mathrm{Re}(z)$, (e) $\mathrm{Im}(z)$, and (f) $\arg(z)$. NOTE: In (f), assume a and b are positive.

20.[R] How far is $\exp(x + iy)$ from the origin?

21.[R] How far is $\exp(x + iy)$ from the x-axis? From the y-axis?

22.[R] For which values of a and b is $\lim_{n\to\infty}\left(e^{a+ib}\right)^n = 0$?

23.[R] Find all complex numbers z such that $e^z = 1$.

24.[R] Find all complex numbers z such that $e^z = -1$.

25.[R]

(a) Find $\left|e^{3+4i}\right|$.

(b) Plot the complex number e^{3+4i}.

26.[R]

(a) Plot all complex numbers of the form e^{x+4i}, x real.

(b) Plot all complex numbers of the form e^{3+yi}, y real.

27.[M] If z lies on the line $y = 1$, where does $\exp(z)$ lie?

28.[M] If z lies on the line $x = 1$, where does $\exp(z)$ lie?

29.[M] In Claude Garrod's *Twentieth Century Physics*, Faculty Publishing, Davis, Calif., p. 107, there is the remark: "Using the fact that

$$\left(e^{-i\omega_0 t}\right)^* \left(e^{-i\omega_0 t}\right) = 1,$$

we can easily evaluate the probability density for these standard waves." Justify this equation. NOTE: In this text, z^* denotes the conjugate of z and ω_0 is real.

30.[M] Use the fact that $1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos((n-1)\theta)$ is the real part of $1 + e^{\theta i} + e^{2\theta i} + \cdots + e^{(n-1)\theta i}$ to find a short formula for that trigonometric sum.

31.[M] Find all z such that $e^z = 3 + 4i$.

32.[M] Assuming that $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for complex numbers z_1 and z_2, obtain the trigonometric identities for $\cos(A+B)$ and $\sin(A+B)$.

33.[M] Evaluate

$$\sum_{k=0}^{\infty} \frac{\cos(k\theta)}{k!}.$$

NOTE: First, show that the series converges (absolutely).

34.[M] Evaluate

$$\sum_{k=0}^{\infty} \frac{\sin(k\theta)}{k!}.$$

NOTE: First, show that the series converges (absolutely).

35.[M] Evaluate

$$\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k}.$$

NOTE: First, show that the series converges (absolutely).

36.[M] Evaluate

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k}.$$

NOTE: First, show that the series converges (absolutely).

37.[M] This problem shows that if $\sum_{k=0}^{\infty} |z_k|$ converges, so does $\sum_{k=0}^{\infty} z_k$.

(a) Let $z_k = x_k + iy_k$. Show that $\sum_{k=0}^{\infty} |x_k|$ and $\sum_{k=0}^{\infty} |x_k|$ both converge. HINT: $|a| \leq \sqrt{a^2 + b^2}$

(b) Show that $\sum_{k=0}^{\infty} x_k$ and $\sum_{k=0}^{\infty} y_k$ both converge.

(c) Show that $\sum_{k=0}^{\infty} (x_k + iy_k)$ converges.

38.[M] Let $f(z)$ be a polynomial with real coefficients.

(a) Show that if $f(a) = 0$, then $f(\overline{a}) = 0$. (This shows that roots of f occur in conjugate pairs.)

(b) Show that $\overline{e^z} = e^{\overline{z}}$.

(c) Show that $\overline{\sin(z)} = \sin(\overline{z})$.

39.[M] When z is real, $|\sin(z)| \leq 1$ and $|\cos(z)| \leq 1$. Do these inequalities hold for all complex z?

40.[M] Does the equation $\cos^2(z) + \sin^2(z) = 1$ hold for complex z?

41.[M] Let

$$z = \frac{1+i}{\sqrt{2}}.$$

(a) Plot z, $z^2/2!$, $z^3/3!$, and $z^4/4!$.

(b) Plot $1 + z + z^2/2! + z^3/3! + z^4/4!$, which is an estimate for $\exp\left((1+i)/\sqrt{2}\right)$.

(c) Plot $\exp\left((1+i)/\sqrt{2}\right)$ on the xy plane.

42.[M] An integral table lists $\int xe^{ax}\,dx = e^{ax}\left(ax-1\right)/a^2$. At first glance, finding $\int xe^{ax}\cos(bx)\,dx$ may appear to be a much harder problem. However, by noticing that $\cos(bx) = \text{Re}\left(e^{ibx}\right)$, we can reduce it to a simpler problem. Following this approach, find $\int xe^{ax}\cos(bx)\,dx$. HINT: The formula for $\int xe^{ax}\,dx$ holds when a is complex.

43.[M] In Section 4.1 we define $\cosh(x) = \left(e^x + e^{-x}\right)/2$ and $\sinh(x) = \left(e^x - e^{-x}\right)/2$. We can use the same definitions when x is complex. In view of Theorem 12.6.2, let us define sine and cosine for complex z by $\sin(z) = \left(e^{iz} - e^{-iz}\right)/(2i)$ and $\cos(z) = \left(e^{iz} + e^{-iz}\right)/2$. Establish the following links between the hyperbolic and trigonometric functions:

(a) $\cosh(z) = \cos(iz)$

(b) $\sinh(z) = -i\sin(iz)$

44.[M] Show that

(a) $\sin(z) = i\sinh(iz)$.

(b) $\cos(z) = \cosh(iz)$.

(c) $\cosh(z)^2 - \sinh(z)^2 = 1$

45.[M] Sam is at it again: "I don't need power series to define e^z. I just write z as $x + iy$ and define e^{x+iy} to be $e^x(\cos(y) + i\sin(y))$. That's all there is to it. If I call this function $E(z)$, then it's easy to check that $E(z_1 + z_2) = E(z_1)E(z_2)$. Moreover, if z is real, then $y = 0$ and $E(z) = e^x$, agreeing with our familiar $\exp(x)$."

(a) Is Sam right?

(b) Does his $E(z)$ obey the basic law of exponents, as he claims?

(c) Jane asks him, "But where did you get the idea for that definition? It seems to float in out of thin air." What is Sam's answer?

46.[C] For which z is

(a) $e^z = e^{-z}$,

(b) $e^{iz} = e^{-ix}$

(c) $\sin(z) = 0$.

47.[C] Let z be a complex number and θ a real number. What is the geometric relationship between z and $e^{i\theta}z$? Experiment, conjecture, and explain.

Exercises 48 and 49 treat the complex logarithms of a complex number. They show that $z = \ln(w)$ is not single-valued.

48.[C] Let w be a nonzero complex number. Show that there are an infinite number of complex numbers z such that $e^z = w$. HINT: Use Euler's formula.

49.[C] (See Exercise 48.) When $e^z = w$, we write $z = \ln(w)$ although $\ln(w)$ is not a uniquely defined number. If b is a nonzero complex number and q is a complex number, define b^q to be $e^{q\ln(b)}$. Since $\ln(b)$ is not unique, b^q is usually not unique. List all possible values of (a) $(-1)^i$, (b) $10^{1/2}$, (c) 10^3,

12.7 Fourier Series

In Section 5.4 we used sums of terms of the form ax^n, where n is a non-negative integer and a is a number, to represent a function. This required a function to have derivatives of all orders. Now, instead, we will use sums of terms of the form $a\cos(kx)$ and $b\sin(kx)$, where a, b, and k are numbers. This method applies to a much broader class of functions, even, for instance, the absolute value function, $f(x) = |x|$, which is not differentiable at 0, and some functions that are not even continuous. The technique, called **Fourier Series**, is used in such varied fields as heat conduction, electric circuits, the theory of sound and mechanical vibrations.

To listen to several tuning forks, go to `http://www.onlinetuningfork.com/`.

At first glance, the use of sine and cosine, which are periodic functions, may seem a surprising choice. However, if you think in terms of sound, it is quite plausible. Every tuning fork produces a pure pitch at a specific frequency. With a collection of such devices, each at a different pitch, struck simultaneously, you can approximate the sound made by a band or an orchestra. Each tuning fork corresponds to $\sin(kt)$ or $\cos(kt)$, where t is time. The one set at concert A vibrates at the rate of 440 cycles per second, that is, 440 Hertz (440 Hz). In this case the acoustic wave is expressed as $\sin(400(2\pi t))$, for, as t increases by $1/400$ second, the argument $400(2\pi t)$ increases by 2π, enabling the function to complete one cycle.

Periodic Functions

The function $\cos(x)$ (and $\sin(x)$) has period 2π, that is, $\cos(x + 2\pi) = \cos(x)$. Changing the input by 2π does not change the output. It follows that $\cos(x - 2\pi) = \cos(x)$, $\cos(x + 4\pi) = \cos(x)$, and, more generally, for any integer n, $\cos(x)$ has $n(2\pi)$ as a period. When we say "$\cos(x)$ has period 2π" we are emphasizing the smallest period. Its other periods are all integer multiples of that period.

EXAMPLE 1 Find the period of (a) $\cos(3\pi x)$, (b) $\cos(k\pi x/L)$, where k is a positive integer and L is a positive number.

SOLUTION In each case we ask, "How much must x change in order for the argument (the input) to change by 2π?"

(a) For $3\pi x$ to change by 2π, we solve the equation $3\pi x = 2\pi$, obtaining $x = 2/3$. Thus $\cos(3\pi x)$ has period $2/3$.

(b) For $\cos(k\pi x/L)$ the reasoning used in (a) leads us to conclude the period is $2L/k$.

Note that in (b) the larger L is, the longer the period. Also, the larger k is, the shorter the period. For each k, $2L$ is among its periods. ◇

Fourier Series for Functions with Period 2π

We first treat the familiar case of functions that have period 2π. Then we consider the general case, where the period is $2L$, for any positive number L.

Let $f(x)$ have period 2π. Its values are determined by its values on any interval of length 2π. We choose the interval $(-\pi, \pi]$ rather than $[0, 2\pi)$ to simplify some computations that we will encounter momentarily.

Let $f(x)$ be a function of period 2π. The Fourier Series associated with this function is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right) \tag{12.7.1}$$

The formulas for a_k and b_k are known as "**Euler's formulas**." Euler published them in 1777, but Fourier was unaware of them.

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \qquad k = 0, 1, 2, \ldots \tag{12.7.2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx \qquad k = 1, 2, \ldots. \tag{12.7.3}$$

(This assumes the integrals in (12.7.2) and (12.7.3) exist.)

After we compute two Fourier series, we will show why the coefficients are given by the integrals in (12.7.2) and (12.7.3).

Constant term is $a_0/2$

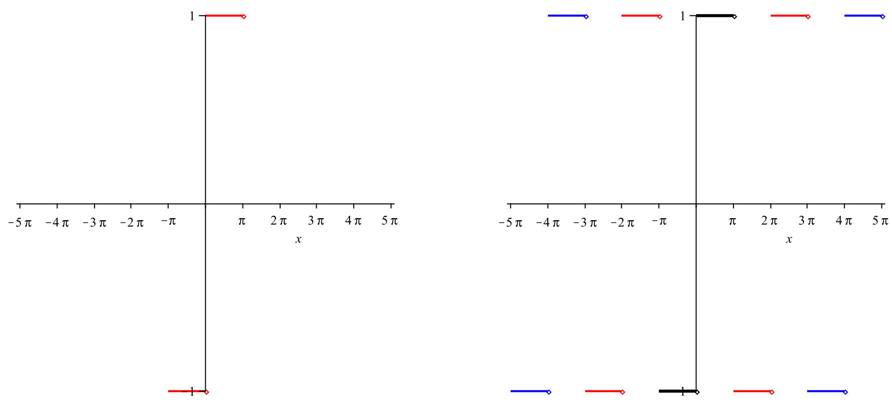
The numbers a_k and b_k are called the **Fourier coefficients** for $f(x)$. The formula for a_0 reduces to $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$. This means that the constant term $a_0/2$ is the average value of the function $f(x)$ over one period. Note that the formula for a_k in (12.7.2) also holds for $k = 0$ because the constant term in (12.7.1) is $a_0/2$. (The 2 was included in (12.7.1) so (12.7.2) would hold when $k = 0$.)

EXAMPLE 2 Find the Fourier series associated with the function defined by

$$f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x \le \pi. \end{cases}$$

Because $f(x)$ is (almost) an odd function, we expect only sines to appear in its Fourier series.

To make $f(x)$ have period 2π, just repeat the graph on every interval of the form $[-\pi + 2n\pi, \pi + 2n\pi)$. The graph of $f(x)$ is shown in Figure 12.7.1(a) and the extension of $f(x)$ is shown in Figure 12.7.1(b).



(a) (b)

Figure 12.7.1:

SOLUTION

$$
\begin{aligned}
a_0 &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f(x)\ dx \\
&= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{0} f(x)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} f(x)\ dx \\
&= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{0} -1\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} 1\ dx \\
&= \tfrac{1}{\pi}(-\pi) + \tfrac{1}{\pi}(\pi) = 0.
\end{aligned}
$$

Similarly, for $k \geq 1$,

$$
\begin{aligned}
a_k &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \\
&= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{0} f(x)\cos(kx)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\ dx \\
&= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{0}(-\cos(kx))\ dx + \tfrac{1}{\pi}\int_{0}^{\pi}\cos(kx)\ dx \\
&= \tfrac{1}{\pi}\ \tfrac{-\sin(kx)}{k}\Big|_{-\pi}^{0} + \tfrac{1}{\pi}\ \tfrac{\sin(kx)}{k}\Big|_{0}^{\pi} = 0 + 0 = 0
\end{aligned}
$$

and

$$
\begin{aligned}
b_k &= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx \\
&= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{0} f(x)\sin(kx)\ dx + \tfrac{1}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\ dx \\
&= \tfrac{1}{\pi}\textstyle\int_{-\pi}^{0}(-\sin(kx))\ dx + \tfrac{1}{\pi}\int_{0}^{\pi}\sin(kx)\ dx \\
&= \tfrac{1}{\pi}\ \tfrac{\cos(kx)}{k}\Big|_{-\pi}^{0} + \tfrac{1}{\pi}\ \tfrac{-\cos(kx)}{k}\Big|_{0}^{\pi} = \tfrac{1}{\pi}\left(\tfrac{1-\cos(-k\pi)}{k}\right) + \tfrac{1}{\pi}\left(\tfrac{-\cos(k\pi)+1}{k}\right)
\end{aligned}
$$

Because $\cos(-k\pi) = \cos(k\pi)$, we have

$$b_k = \tfrac{1}{k\pi}\left((1-\cos(k\pi)) + (1-\cos(k\pi))\right) = \tfrac{2(1-\cos(k\pi))}{k\pi}.$$

When k is even, $1-\cos(k\pi) = 1-1 = 0$. And, when k is odd, $1-\cos(k\pi) = 1-(-1) = 2$. Thus

$$b_k = \begin{cases} 0 & \text{when } k \text{ is even} \\ \frac{4}{k\pi} & \text{when } k \text{ is odd.} \end{cases}$$

The Fourier Series (12.7.1) in this case has only terms involving $\sin(kx)$

with k odd. It is

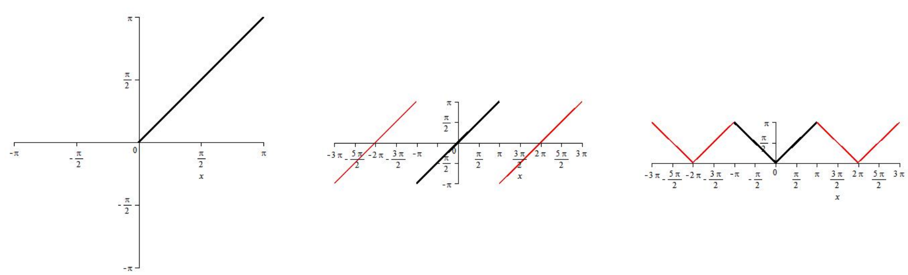
$$\frac{4}{\pi}\sin(x)+\frac{4}{3\pi}\sin(3x)+\frac{4}{5\pi}\sin(5x)+\dots.$$

In particular, when $x=\pi/2$, $f(x)=1$ and we have

$$\begin{aligned} 1 &= \tfrac{4}{\pi}\sin\left(\tfrac{(\pi}{2}\right)+\tfrac{4}{3\pi}\sin\left(\tfrac{3\pi}{2}\right)+\tfrac{4}{5\pi}\sin\left(\tfrac{5\pi}{2}\right)+\dots \\ 1 &= \tfrac{4}{\pi}-\tfrac{4}{3\pi}+\tfrac{4}{5\pi}-\dots \\ \text{Thus}\qquad \tfrac{\pi}{4} &= 1-\tfrac{1}{3}+\tfrac{1}{5}-\dots. \end{aligned}$$

This result was obtained previously in Exercise 22 in Section 12.4 with the aid of the Maclaurin series for $\arctan(x)$. ◇

The fact that the function $f(x)$ in Example 2 is defined on a full period is quite convenient. In many applications the function is given only on one half of the period. For example,$f(x)=x$ for $0\le x<\pi$ (see Figure 12.7.2(a)). Because $f(x)$ is not periodic, the first step is to replace $f(x)$ with a function $g(x)$ that has period 2π and coincides with $f(x)$ on its domain, that is, on $[0,\pi)$. Two possible periodic extensions of $f(x)$ are shown in Figure 12.7.2(b) and (c). Both have period 2π; one is odd, the other even.



(a) (b) (c)

Figure 12.7.2:

The Fourier series for a function of period 2π has the form

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos(kx)+b_k\sin(kx)\right) \qquad (12.7.4)$$

Note that the formula for a_k includes the case for a_0.

with coefficients given by

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \qquad k=0,1,2,\dots \qquad (12.7.5)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \qquad k=1,2,\dots. \qquad (12.7.6)$$

EXAMPLE 3 Find the Fourier series of the triangular wave with period 2π shown in Figure 12.7.2(c).

SOLUTION Let $T(x)$ denote the triangular wave. To compute the Fourier series of $T(x)$ we need to know the definition of $T(x)$ on an interval with length 2π.

$T(x) = |x|$ for x in $[-\pi, \pi)$

$$T(x) = \begin{cases} x & \text{for } 0 \le x \le \pi \\ -x & \text{for } -\pi \le x < 0 \end{cases} .$$

Because $T(x)$ is an even function, $b_k = 0$ for $k = 1, 2, \ldots$. Then

If $T(x) = T(-x)$, then $\int_{-\pi}^{\pi} T(x)\ dx = 2\int_0^{\pi} T(x)\ dx$.

$$a_0 \ = \ \frac{1}{\pi}\int_{-\pi}^{\pi} T(x)\ dx = \frac{2}{\pi}\int_0^{\pi\pi} x\ dx = \frac{1}{\pi}\ x^2\Big|_0^{\pi} = \pi.$$

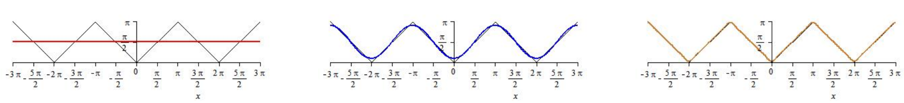
The coefficients of the cosine terms are

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} T(x)\cos(kx)\ dx = \frac{2}{\pi}\int_0^{\pi} x\cos(kx)\ dx && \text{because } T(x)\cos(kx) \text{ is even} \\ &= \frac{2}{\pi}\left(\frac{x}{k}\sin(kx)\Big|_0^{\pi} - \frac{1}{k}\int_0^{\pi} \sin(kx)\ dx \right) && \text{integrate by parts} \\ &= 2\left(0 + \frac{1}{k^2}\cos(kx)\Big|_0^{\pi}\right) && \sin(k\pi) = 0 \text{ for all integers } k \\ &= \frac{2}{k^2\pi}(\cos(k\pi) - 1) = \frac{2((-1)^k - 1)}{k^2} \end{aligned}$$

When k is an even integer, $a_k = 2((-1)^k - 1)/(k^2\pi) = 0$. And, when k is an odd integer, $a_k = 2((-1)^k - 1)/(k\pi)^2 = -4/(k^2\pi)$.

Then, the Fourier series for the triangular wave is

$$T(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \ldots\right). \qquad (12.7.7)$$



(a) (b) (c)

Figure 12.7.3:

◇

Figure 12.7.3 shows the partial Fourier sums for the triangular wave with 1, 2, and 5 terms. In an advanced calculus course it is proved that the partial sums converge to the function for every real number. As is easy to check, replacing x by 0 in (12.7.7) showed the sum of the reciprocals of the squares of all the positive odd integers is $\pi^2/8$.

The Origins of the Formulas for a_k and b_k

We will derive the formulas for the Fourier coefficients in the special case when the period is 2π. Exercises 12 and 13 outline the similar argument for the general case when the period is $2L$.

These integrals were evaluated in Exercises 17 to 19.

The keys are the following three integrals:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(kx)\sin(mx)\;dx &= \begin{cases}\pi & \text{if } m = k,\ k = 1, 2, \ldots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \ldots\end{cases}\\ \int_{-\pi}^{\pi} \cos(kx)\cos(mx)\;dx &= \begin{cases}\pi & \text{if } m = k,\ k = 1, 2, \ldots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \ldots\end{cases}\\ \int_{-\pi}^{\pi} \sin(kx)\cos(mx)\;dx &= 0 \quad \text{for any } m = 1, 2, \ldots \text{ and any } k = 1, 2, \ldots.\end{aligned}$$

The third one is immediate, for the integrand, being the product of an odd function and an even function, is an odd function. The other two depend on trigonometric identities, and were developed in Exercises 17 to 19 inSection 8.5.

We are assuming it's legal to switch the order, integrate term-by-term, then sum: "$\int_{-\pi}^{\pi}\sum_{k=1}^{\infty} = \sum_{k=1}^{\infty}\int_{-\pi}^{\pi}$".

The formula for a_m, $m = 1, 2, \ldots$, is found by multiplying $f(x)$ by $\cos(mx)$ and integrating term-by-term over one period of length 2π:

$$\begin{aligned}&\int_{-\pi}^{\pi} f(x)\cos(mx)\;dx\\ &= \int_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(kx) + b_k\sin(kx)\right)\right)\cos(mx)\;dx\\ &= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)\;dx\\ &\quad + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos(kx)\cos(mx)\;dx + b_k\int_{-\pi}^{\pi}\sin(kx)\cos(mx)\;dx\right).\end{aligned}$$

Each integral in this last expression is zero — except the coefficient of a_m. This gives the equation

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx \quad = \quad a_m \int_{-\pi}^{\pi} (\cos(kx))^2\ dx = a_m\pi.$$

Solving for a_m, we find that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\ dx.$$

The derivation of the formulas for a_0 and for b_k are similar. (See Exercises 12 and 13.)

Remarks on the Underlying Theory

Just as a Taylor series associated with a function may not represent the function, the Fourier series associated with a function may not represent it, even if the function is continuous. However, there are several theorems that assure us that for many functions met in applications the series does converge to the function. First, a couple of definitions.

Recall that the right-hand limit of $f(x)$ at a is defined as the limit of $f(x)$ as x approaches a through values larger than a, and is denoted $\lim_{x\to a^+} f(x)$. Similarly, the left-hand limit, denoted $\lim_{x\to a^-} f(x)$, is defined as the limit of $f(x)$ as x approaches a through values smaller than a. If both these limits exist at a and are different, we say that the function has a "jump discontinuity at a."

Theorem. *Let $f(x)$ have period $2L$. Assume that in the interval $[-L, L)$ (a) $f(x)$ is differentiable exept at a finite number of points, where there are jump discontinuities, and (b) at L the right-hand limit of $f(x)$ exists and at $-L$ the left-hand limit of $f(x)$ exists. Then,*

I. if the function is continuous at a, its associated Fourier series converges to $f(a)$.

II. if $f(x)$ has a jump discontinuity at a, then the series converges to the average of the left- and right-hand limits at a.

III. at the endpoints, L and $-L$, the Fourier series converges to the average of $\lim_{x\to -L^+} f(x)$ and $\lim_{x\to L^-} f(x)$.

Note that there is no mention of the existence of any second-order, or higher-order derivatives.

The name Joseph Fourier (1768—1830) is attached to trigonometric series because he explored and applied them in his classic Analytic Theory of Heat, published in 1822. He came upon the formulas for the coefficients by an indirect route, starting with the Maclaurin series for $\sin(x)$ and $\cos(x)$. For the details, see Morris Kline's *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972 (especially pages 671–675, but see further references in its index). In the nineteenth and twentieth centuries mathematicians developed a variety of conditions that implied the series converges to the function. The most recent is due to Lenart Carleson (1928–) in 1966, which settled a famous conjecture.

Summary

While Taylor Series are useful for dealing with a function that is very smooth (having derivatives of all orders), Fourier series can represent a function that is not even continuous. While the coefficients in Taylor series are expressed in terms of derivatives, those in Fourier series are expressed in terms of integrals. Even non-periodic functions can be represented by Fourier series. For instance, to deal with x^2 on, say, $[0, 100)$ just extend its domain to the whole x-axis by defining a function of period 100 that agrees with x^2 on $[0, 100)$.

EXERCISES for Section 12.7 *Key:* R–routine, M–moderate, C–challenging

SHERMAN: Check this set of exercises very closely.
The following table of integrals will be helpful in evaluating some of the integrals in these exercises.

$$\begin{aligned}
\int x\sin(ax)\,dx &= \tfrac{1}{a^2}\sin(ax) - \tfrac{x}{a}\cos(ax) + C\\
\int x\cos(ax)\,dx &= \tfrac{1}{a^2}\cos(ax) + \tfrac{x}{a}\sin(ax) + C\\
\int x^2\sin(ax)\,dx &= \tfrac{2}{a^3}\cos(ax) + \tfrac{2x}{a^2}\sin(ax) - \tfrac{x^2}{a}\cos(ax) + C\\
\int x^2\cos(ax)\,dx &= \tfrac{-2}{a^3}\sin(ax) + \tfrac{2x}{a^2}\cos(ax) + \tfrac{x^2}{a}\sin(ax) + C\\
\int \sin(x)\sin(ax)\,dx &= \tfrac{1}{2(a-1)}\sin((a-1)x) - \tfrac{1}{2(a+1)}\sin((a+1)x) + C\\
\int \sin(x)\cos(ax)\,dx &= \tfrac{1}{2(a-1)}\cos((a-1)x) - \tfrac{1}{2(a+1)}\cos((a+1)x) + C\\
\int \cos(x)\sin(ax)\,dx &= \tfrac{-1}{2(a-1)}\cos((a-1)x) - \tfrac{1}{2(a+1)}\cos((a+1)x) + C\\
\int \cos(x)\cos(ax)\,dx &= \tfrac{1}{2(a-1)}\sin((a-1)x) + \tfrac{1}{2(a+1)}\sin((a+1)x) + C\\
\int e^x\sin(ax)\,dx &= \tfrac{1}{1+a^2}e^x\sin(ax) - \tfrac{a}{1+a^2}e^x\cos(ax) + C\\
\int e^x\cos(ax)\,dx &= \tfrac{a}{1+a^2}e^x\sin(ax) + \tfrac{1}{1+a^2}e^x\cos(ax) + C
\end{aligned}$$

In Exercises 1 to 8 give the period of the function

1.[R] $\tan(x)$

2.[R] $2/\cos^2(x)$

3.[R] $\sin(3x)$

4.[R] $\sin(2\pi x)$

5.[R] $\sin(x/5)$

6.[R] $\cos(2\pi x/5)$

7.[R] $\sin(\pi x/3)$

8.[R] $\sin(x/3)$

9.[R] Let $f(x) = x^2$ for x in $[-\pi, \pi)$ and have period 2π.

(a) Find $f(\pi)$, $f(2\pi)$, $f(3\pi)$, $f(-\pi)$, $f(-2\pi)$, and $f(-3\pi)$.

(b) Graph $f(x)$ for x in $[-4\pi, 4\pi]$.

(c) Why will the Fourier series for $f(x)$ have no sine terms?

(d) Find the Fourier series for $f(x)$.

10.[R] Let $f(x) = -x^2$ for x in $[-\pi, 0)$ and x^2 for x in $[0, \pi)$ and have period 2π.

(a) Find $f(\pi)$, $f(2\pi)$, $f(-\pi)$, and $f(-2\pi)$.

(b) Graph $f(x)$ for x in $[-4\pi, 4\pi]$.

(c) Show that f is "almost" an odd function. For what x is $f(x) \neq -f(x)$?

(d) Show that the Fourier series of $f(x)$ is

$$2\frac{\pi^2-4}{\pi}\sin(x)-\pi\sin(2x)+2\frac{9\pi^2-4}{27\pi}\sin(3x)-\frac{\pi}{2}\sin(4x)+2\frac{25\pi^2-4}{125\pi}\sin(5x)-\frac{\pi}{3}\sin(6x)$$

(e) Why are there no cosine terms in the series?

11.[R] Let $f(x) = x$ for x in $[-\pi, \pi)$ and have period 2π. NOTE: This function is known as a sawtooth function.

(a) Find $f(\pi)$, $f(2\pi)$, $f(-\pi)$, and $f(-2\pi)$.

(b) Graph $f(x)$ for x in $[-4\pi, 4\pi]$.

(c) Show that the Fourier series of $f(x)$ is

$$2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x).$$

(d) What does the series converge to at the discontinuities of $f(x)$?

Exercises 12 and 13 complete the derivation of the Fourier series associated with a function with period 2π:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right) \qquad (12.7.8)$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \qquad k = 0, 1, 2, \ldots \qquad (12.7.9)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx \qquad k = 1, 2, \ldots. \qquad (12.7.10)$$

12.[M] Derive (12.7.9).

13.[M] Derive (12.7.10).

Exercises 14 to 16 develop the formulas for the Fourier Series for a function with period $2L$ (instead of 2π).

14.[M] Show that

$$\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\, dx = \begin{cases} L & \text{if } m = k,\ k = 1, 2, \dots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \dots \end{cases}.$$

HINT: Use the trigonometric identity $\sin(u)\sin(v) = \frac{1}{2}\left(\cos(u-v) - \cos(u+v)\right)$.

15.[M] Show that

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx = \begin{cases} L & \text{if } m = k,\ k = 1, 2, \dots \\ 0 & \text{if } m \neq k,\ k = 1, 2, \dots \end{cases}.$$

HINT: Use the trigonometric identity $\cos(u)\cos(v) = \frac{1}{2}\left(\cos(u-v) + \cos(u+v)\right)$.

16.[M] Show that

$$\int_{-L}^{L} \sin\left(\frac{k\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right)\, dx = 0.$$

HINT: While you could use the trigonometric identity $\sin(u)\cos(v) = \frac{1}{2}\left(\sin(u-v) + \sin(u+v)\right)$ this exercise can be completed without finding any integrals.

17.[R] Find the Fourier series of $f(x) = \sin(x)$, viewed as a function of period 2π.

18.[R] Find the Fourier series of $f(x) = \sin(x)$, viewed as a function of period 4π.

19.[R] Find the Fourier series of $f(x) = cos(2x)$, viewed as a function of period π.

20.[R] Find the Fourier series of $f(x) = \cos(2x)$, viewed as a function of period 4π.

In Exercises 21 to 30, compute the Fourier series of the indicated function. Sketch at least two periods of the function corresponding to the Fourier series. NOTE: In each case assume the function is periodic.

21.[R] $f(x) = x^2$, $-1 \leq x < 1$ (period 2)

22.[R] $f(x) = x^2$, $-2 \leq x < 2$ (period 4)

23.[R] $f(x) = \begin{cases} 0 & \text{for } -1 \leq x < 0 \\ 1 & \text{for } 0 \leq x < 1 \end{cases}$

24.[R] $f(x) = \begin{cases} 1 & \text{for } -1 \leq x < 0 \\ 0 & \text{for } 0 \leq x < 1 \end{cases}$

25.[R] $f(x) = \begin{cases} 0 & \text{for } -1 \leq x < 0 \\ x & \text{for } 0 \leq x < 1 \end{cases}$

26.[R] $f(x) = \begin{cases} 1 & \text{for } -1 \leq x < 0 \\ x & \text{for } 0 \leq x < 1 \end{cases}$

27.[R] $f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < 0 \\ \sin(x) & \text{for } 0 \leq x < \pi \end{cases}$

28.[R] $f(x) = \begin{cases} 1 & \text{for } -\pi \leq x < 0 \\ \cos(x) & \text{for } 0 \leq x < \pi \end{cases}$

29.[R] $f(x) = \begin{cases} 0 & \text{for } -2\pi \leq x < 0 \\ \sin(x) & \text{for } 0 \leq x < 2\pi \end{cases}$

30.[R] $f(x) = \begin{cases} 1 & \text{for } -2\pi \leq x < 0 \\ \cos(x) & \text{for } 0 \leq x < 2\pi \end{cases}$

In Exercises 31 to 36, (a) extend the given function to be an odd periodic function with period $2L$, (b) compute the Fourier series of the function found in (a), (c) graph at least two periods of the first three non-zero terms of the Fourier series found in (b).

31.[R] $f(x) = 1$, $0 \leq x \leq 1$ $(L = 1)$

32.[R] $f(x) = x$, $0 \leq x \leq 1$ $(L = 1)$

33.[R] $f)x) = x^2$, $0 \leq x \leq 1$ $(L = 1)$

34.[R] $f)x) = |x - 1|$, $0 \leq x \leq 2$ $(L = 2)$

35.[R] $f)x) = \sin(x)$, $0 \leq x \leq \pi$ $(L = \pi)$

36.[R] $f)x) = \cos(x)$, $0 \leq x \leq \pi$ $(L = \pi)$

In Exercises 37 to 42, (a) extend the given function to be an even periodic function with period $2L$, (b) compute the Fourier series of the function found in (a), (c) graph at least two periods of the function corresponding to the Fourier series found in (b).

37.[R] $f(x)$ from Exercise 31

38.[R] $f(x)$ from Exercise 32

39.[R] $f(x)$ from Exercise 33

40.[R] $f(x)$ from Exercise 34

41.[R] $f(x)$ from Exercise 35

42.[R] $f(x)$ from Exercise 36

43.[M] Use the properties of even and odd functions to justifiy that:

(a) the product of two even functions is even.

(b) the product of two odd functions is even.

(c) the product of an even function and an odd function is odd.

44.[M] Determine which of the statements in Exercise 43 is true if the word "product" is replaced with "sum".

45.[M] Show that any function, $f(x)$, can be written as the sum of an even function (f_{even}) and an odd function (f_{odd}). HINT: Write $f(x) = f_{even}(x) + f_{odd}(x)$. Use the properties of f_{even} and f_{odd} to express $f(-x)$ in terms of $f_{even}(x)$ and $f_{odd}(x)$.

46.[] Write each of the following functions as the sum of an even function and an odd function.

(a) $f(x) = x^2 + 2x$

(b) $f(x) = x^3 - 2x$

(c) $f(x) = x^3 + 3x^2 - 2x + 1$

(d) $f(x) = \sin(4x) - 3x^3$

(e) $f(x) = |x| \sin(x)$

(f) $f(x) = |x| \cos(x)$

(g) $f(x) = (\sin(x) + 1)^3$

(h) $f(x) = (\cos(x) + 1)^3$

47.[R] Let $f(x) = x$ for x in $[-1, 1)$ and have period 2. NOTE: This function is known as a sawtooth function.

(a) Find $f(1)$, $f(2)$, $f(-1)$, and $f(-2)$.

(b) Graph $f(x)$ for x in $[-4, 4]$.

(c) Find the Fourier series of $f(x)$.

(d) Why are there no sine terms in the Fourier series?

(e) What is the average value of $f(x)$ over any interval of length 2π?

(f) What does the series converge to at the jump discontinuities?

(g) How does this Fourier series compare with the one in Exercise 11?

48.[M] In Section 11.6, Example 3, it is claimed that the series

$$\frac{\cos(x)}{1^2} + \frac{\cos(2x)}{2^2} + \frac{\cos(3x)}{3^2} + \cdots + \frac{\cos(kx)}{k^2} + \cdots$$

converges to $\frac{1}{12}(3x^2 - 6\pi x + 2\pi^2)$ for $0 \le x \le 2\pi$. Use Fourier series to verify this claim.

49.[C] Let $f(x)$ be a periodic function with period $2L$.

(a) Show that $\int_0^{2L} f(x)\ dx = \int_{-L}^{L} f(x)\ dx$.

(b) Show that $\int_{-2L}^{0} f(x)\ dx = \int_{-L}^{L} f(x)\ dx$.

(c) Show that $\int_a^{a+2L} f(x)\ dx = \int_{-L}^{L} f(x)\ dx$ for any number a.

Exercise 50 Just as the complex numbers revealed a close tie between the exponential and trigonometric functions, they also reveal a relation between power series and Fourier series. Exercise 50 helps to make this connection.

50.[C] A Taylor series $\sum_{k=0}^{\infty} a_k z^k$ does not look like a Fourier series. However, when a_k is written as $b_k + ic_k$ and z is expressed as $r(\cos(\theta) + i\sin(\theta))$, where r is constant, the connection becomes clear. To check that this is so, write the series in the form $A + Bi$ where A and B are real. What two Fourier series appear as the real and imaginary parts arise from these manipulations?

12.S Chapter Summary

The Taylor polynomials first encountered in Section 5.4 suggested the powerful power series associated with a function that has derivatives of all orders at a, namely

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{12.S.1}$$

which certainly converges when x is a. It may even converge for other values of x, but not necessarily to $f(x)$. For the common functions e^x, $\sin(x)$, and $\cos(x)$ the corresponding power series does converge to the function for all values of x.

The error in using a front end up through the power $(x-a)^n$ to estimate $f(x)$ is given by Lagrange's formula,

$$f(x)-\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } x \text{ and } a. \tag{12.S.2}$$

For some functions, such as $\tan(x)$, it is not easy to find the k^{th} derivative. So, we should be glad that e^x, $\sin(x)$, and $\cos(x)$ have such convenient higher derivatives.

Replace x by $-x^2$.

One can obtain a few terms of the Maclaurin series for $\tan(x)$ by dividing the series for $\sin(x)$ by the series for $\cos(x)$. The series for $1/(1+x^2)$ is easily found by massaging the sum of the geometric series $1/(1-x) = 1+x+x^2+\dots$. Integration of that series yields painlessly the Maclaurin series for $\arctan(x)$.

Each power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ has a radius of convergence, R. For $|x-a| < R$, the series converges absolutely and for $|x-a| > R$ the series does not converge. If it converges for all x, then $R = \infty$. For $|x-a| < R$, one may safely differentiate and integrate a series, producing new series.

Estimating an integrand $f(x)$ by the front end of a power series, we can then estimate $\int_a^b f(x)\,dx$. Also, power series are of use in finding indeterminate limits of the type zero-over-zero. that is, $\lim_{x\to 0} \frac{f(x)}{g(x)}$, where both $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$.

Maclaurin series, combined with complex numbers, exposed a fundamental relation between exponential and trigonometric functions:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Other important truths, not covered in this chapter, are revealed with the aid of complex numbers. For instance, if we allow complex coefficients, every polynomial can be written as the product of first-degree polynomials, thus simplifying the partial fractions of Section 8.4. Complex numbers can also help us find the radius of convergence. For instance, what is the radius of

Function	Maclaurin Series	R	How Found?
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	∞	Taylor's Theorem
$\sin(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	∞	Taylor's Theorem
$\cos(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	∞	Taylor's Theorem
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	1	Geometric Series
$\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$	1	Integrate Geometric Series
$\arctan(x)$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$	1	Integrate Geometric Series
$\arcsin(x)$	$x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}$ $+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$	1	Integrate Geometric Series
$(1+x)^r$	$1 + rx + \frac{r(r-1)}{2!}x^2$ $+ \frac{r(r-1)(r-2)}{3!}x^3 + \cdots$	1	Taylor's Theorem
$\frac{1}{(1-x)^2}$	$\sum_{k=0}^{\infty} kx^{k-1}$	1	Differentiate Geometric Series

Table 12.S.1:

convergence of the Taylor series in powers of $x-3$ associated with $1/(1+x^2)$? Answer: it is the distance from the point $(3,0)$ to the nearest complex number at which $1/(1+x^2)$ "blows up," that is, when $1+x^2=0$. This occurs when x is i or $-i$, both of which, by the Pythagorean Theorem, are at a distance $\sqrt{1^2+3^2}=\sqrt{10}$ from $(3,0)$. So, $R=\sqrt{10}$.

The final section introduced Fourier series. In contrast to Taylor series, its coefficients are given by integrals, rather than by derivatives. Consequently, Fourier series apply to a larger class of functions. However, this method applies directly only to periodic functions. In the case of a non-periodic function, one restricts the domain to an interval $(-L,L)$ and extends the function to have period $2L$

EXERCISES for 12.S *Key:* R–routine, M–moderate, C–challenging

Exercise 1 provides additional detail for the historical discussion (see page 58) about Newton's calculation of the area under a hyperbola to more than 50 decimal places. (See also Exercises 29 and 30 in Section 6.5.)

1.[R] Let c be a positive constant.

(a) Show that the area under the curve $y=1/(1+x)$ above the interval $[0,c]$ is
$$-\sum_{k=1}^{\infty} \frac{(-c)^k}{k}.$$

(b) Show that the area under the curve $y=1/(1+x)$ above the interval $[-c,0]$ is
$$\sum_{k=1}^{\infty} \frac{c^k}{k}.$$

2.[M] Assume that a Maclaurin series $M(x)$ is associated with $f(x)$ for x in $(-a, a)$. Show that $M(x^2)$ is the Maclaurin series associated with $g(x) = f(x^2)$ for x in $(-\sqrt{a}, \sqrt{a})$.

3.[M] The integral $\int_0^{2\pi} (1 - \cos(x))/x dx$ occurs in the theory of antennas.

(a) Show that it is an improper integral.

(b) Show that there is a continuous function whose domain is $[0, 2\pi]$ that coincides with the integrand when x is not 0.

(c) The integrand does not have an elementary antiderivative. Why is the power series technique of approximation inconvenient here?

Exercises 4 to 6 use complex numbers to find the average value of the logarithm of a function. Exercise 4 is related to Exercise 90 on page 781 and to Exercise 58 on page 1060.

4.[C] Let a point Q be a distance $a \neq 1$ from the center of a unit circle.

(a) Show that the average value of the (natural) logarithm of the distance from Q to points on the circumference is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \ln(1 + a^2 - 2a\cos(\theta))\, d\theta.$$

(b) Spend at least three minutes, but at most 5 minutes, trying to evaluate the integral in (a).

5.[C] This algebraic exercise is needed in Exercise 6. Let z_0, z_1, $\dots$, z_{n-1} be the n n^{th} roots of 1. Then it is shown in an algebra course that

$$(z - z_0)(z - z_1)(z - z_2) \cdots (z - z_{n-1}) = z^n - 1.$$

Check that this equation holds when n is (a) 2, (b) 3, (c) 4.

6.[C] Let z_0, z_1, $\dots$, z_{n-1} be the n n^{th} roots of 1.

(a) Why is $\displaystyle\frac{1}{n}\sum_{i=0}^{n-1} \ln|a - z_i|$ an estimate of the average distance?

(b) Show that the average in (a) equals

$$\frac{1}{n} \ln|a^n - 1|. \tag{12.S.3}$$

(c) If $0 < a < 1$, show that the limit of (12.S.3) as $n \to \infty$ is 0.

(d) The case when $a = 1$ is not covered by parts (c) and (d). In this case, choose Q to be a point on the unit circlel whose polar angle is not a rational multiple of π. (So no z_i coincides with Q.) Then argue as in parts (c) or (e).

(e) If $a > 1$, show that the limit of (12.S.3) as $n \to \infty$ is $\ln(a)$.

(f) Use the results in (c) and (d) to evaluate the integral in Exercise 4(a).

7.[C] Find

(a) $\lim_{x\to\infty} \frac{xe^x}{e^{x^2}}$

(b) $\lim_{x\to 0} \frac{x(e^{\sqrt{x}}-1)}{e^{x^2}-1}$

8.[C] Does $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$ converge or diverge? Explain.

9.[C] Assume that $f(x)$ has a continuous fourth derivative. Let M_4 be the maximum of $\left|f^{(4)}(x)\right|$ for x in $[-1, 1]$. Show that

$$\left|\int_{-1}^{1} f(x)\ dx - f\left(\frac{1}{\sqrt{3}}\right) - f\left(\frac{-1}{\sqrt{3}}\right)\right| \leq \frac{7M_4}{270}.$$

HINT: Use the representation $f(x) = f(0) + f'(0)x + f''(0)x^2/2 + f^{(3)}(0)x^3/6 + f^{(4)}(c)x^4/24$, where c depends on x.

10.[C] Justify this statement, found in a biological monograph:

Expanding the equation

$$a \cdot \ln(x + p) + b \cdot \ln(y + q) = M,$$

we obtain

$$a\left(\ln(p) + \frac{x}{p} - \frac{x^2}{2p^2} + \frac{x^3}{3p^3} - \cdots\right) + b\left(\ln(q) + \frac{y}{q} - \frac{y^2}{2q^2} + \frac{y^3}{3q^3} - \cdots\right) = M.$$

11.[R] Estimate $\int_1^3 e^{-x^2}\ dx$ using a Taylor series at $x = 2$ associated with e^{-x^2}.

12.[M] Explain why both $\cos(x)$ and $\sin(x)$ can be expressed in terms of the exponential function e^z.

13.[M] State some of the advantages of complex numbers over real numbers.

14.[M] Why is the "radius of convergence" called "the radius of convergence" rather than the "interval of convergence."

15.[M]

(a) What is the Maclaurin series for $\sin(x)$? $\sin(2x)$? and $\cos(x)$?

(b) The identity $\sin(2x) = 2\sin(x)\cos(x)$ implies that the product of 2. the Maclaurin series for $\cos(x)$, and the Maclaurin series for $\sin(x)$ should equal the Maclaurin series for $\sin(2x)$.

16.[M] Starting with $1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}$ obtain the Maclaurin series for

(a) $1/(1-x)^2$

(b) $1/(1+x)$

(c) $\frac{1}{1+x^2}$

(d) $\ln(1+x)$

(e) $\arctan(x)$

17.[M] Find the radius of convergence for each series in Exercise 16

18.[M] Show that each series in Exercise 16 converges to the given function.

19.[M] Sam says, "According to their book, if I multiply the Maclaurin series for e^x by the one for e^{-x} I should get the Maclaurin series for $e^x e^{-x}$, which is just 1. I don't believe that the product could be that simple." Multiply enough terms of the two series to calm Sam down.

20.[M] The function $f(z) = 1/\overline{z}$ maps the part of the hyperbola $xy = 1$ into a curve $\mathcal{C}$.

(a) Find and sketch at least four points on $\mathcal{C}$. Then, sketch $\mathcal{C}$.

(b) What is the polar equation for the hyperbola $xy = 1$?

(c) What is the polar equation of $\mathcal{C}$?

(d) Check that the image of the point (x, y) is $(x/(x^2 + y^2), y/(x^2 + y^2))$.

(e) Check that if (x, y) is on the hyperbola, its image satisfies the equation found in (d).

21.[M] The function $f(z) = 1/\overline{z}$ maps the part of the parabola $y = x^2$ into a curve $\mathcal{C}$.

(a) Find and sketch at least four points on $\mathcal{C}$. Then, sketch $\mathcal{C}$.

(b) What is the polar equation for the parabola $y = x^2$?

(c) What is the polar equation of $\mathcal{C}$?

(d) Find the rectangular equation of $\mathcal{C}$.

(e) Check that if (x, y) is on the parabola, its image satisfies the equation found in (d).

22.[M]

(a) Graph the circle $r = \sqrt{2}\cos(\theta)$.

(b) Show that the function $f(z) = z^2$ maps the circle in (a) into the cardioid $r = 1 + \cos(\theta)$.

23.[C] Suppose f is a function with the property that $f^{(n)}(x)$ is "small" in the sense that $|f^{(n)}(x)| \leq |(x + 100)^n|$ for all x. Show that the Maclaurin series represents $f(x)$ for all x.

Calculus is Everywhere # 15

Sparse Traffic

Customers arriving at a checkout counter, cars traveling on a one-way road, raindrops falling on a street and cosmic rays entering the atmosphere all illustrate one mathematical idea — the theory of sparse traffic involving independent events. We will develop the mathematics, which is the basis of the study of waiting time – whether customers at the checkout counter or telephone calls at a switchboard.

First we sketch briefly a bit of probability theory.

Some Probability Theory

The probability that an event occurs is measured by a number p, which can be anywhere from 0 up to 1; $p = 1$ implies the event will certainly occur with negligible exceptions and $p = 0$ that it will not occur with negligible exceptions. The probability that a penny turns up heads is $p = 1/2$ and that a die turns up 2 is $p = 1/6$. (The phrase "certainly occurs with negligible exceptions" means, roughly, that the times the event does not occur are so rare that we may disregard them. Similarly, the phrase "certainly will not occur with negligible exceptions" means, roughly, that the times the event does not occur are so rare that we may disregard them.)

The probability that two events that are independent of each other both occur is the product of their probabilities. For instance, the probability of getting heads when tossing a penny and a 2 when tossing the die is $p = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{12}$.

The probability that exactly one of several mutually exclusive events occurs is the sum of their probabilities. For instance, the probability of getting a 2 or a 3 with a die is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

With that thumbnail introduction, we will analyze sparse traffic on a one-way road. We will assume that the cars enter the traffic independently of each other and travel at the same speed. Finally, to simplify matters, we assume each car is a point.

The Model

To construct our model we introduce the functions $P_0, P_1, P_2, \ldots, P_n, \ldots$ where $P_n(x)$ shall be the probability that any interval of length x contains exactly n

cars (independently of the location of the interval). Thus $P_0(x)$ is the probability that an interval of length x is empty. We shall assume that

$$P_0(x) + P_1(x) + \cdots + P_n(x) + \cdots = 1 \qquad \text{for any } x.$$

We also shall assume that $P_0(0) = 1$ ("the probability is 1 that a given point contains no cars").

For our model we make the following two major assumptions:

(a) The probability that exactly one car is in any fixed short section of the road is approximately proportional to the length of the section. That is, there is some positive number k such that

$$\lim_{\Delta x \to 0} \frac{P_1(\Delta x)}{\Delta x} = k.$$

(b) The probability that there is more than one car in any fixed short section of the road is neglible, even when compared to the length of the section. That is,

$$\lim_{\Delta x \to 0} \frac{P_2(\Delta x) + P_3(\Delta x) + P_4(\Delta x) + \cdots}{\Delta x} = 0. \qquad \text{(C.15.1)}$$

We shall now put assumptions (a) and (b) into more useful forms. If we let

$$\epsilon = \frac{P_1(\Delta x)}{\Delta x} - k \qquad \text{(C.15.2)}$$

where ϵ depends on Δx, assumption (a) tells us that $\lim_{\Delta x} \epsilon = 0$. Thus, solving (C.15.2) for $P_1(\Delta x)$, we see that assumption (a) can be phrased as

$$P_1(\Delta x) = k\Delta x + \epsilon \Delta x \qquad \text{(C.15.3)}$$

where $\epsilon \to 0$ as $\Delta x \to 0$.

Since $P_0(\Delta x) + P_1(\Delta x) + \cdots + P_n(\Delta x) + \cdots = 1$, assumption (b) may be expressed as

$$\lim_{\Delta x \to 0} \frac{1 - P_0(\Delta x) - P_1(\Delta x)}{\Delta x} = 0. \qquad \text{(C.15.4)}$$

In light of assumption (a), equation (C.15.4) is equivalent to

$$\lim_{\Delta x \to 0} \frac{1 - P_0(\Delta x)}{\Delta x} = k. \qquad \text{(C.15.5)}$$

In the manner in which we obtained (C.15.3), we may deduce that

$$1 - P_0(\Delta x) = k\Delta x + \delta \Delta x,$$

where $\delta \to 0$ as $\Delta x \to 0$. Thus

$$P_0(\Delta x) = 1 - k\Delta x - \delta\Delta x, \qquad \text{(C.15.6)}$$

where $\delta \to 0$ as $\Delta x \to 0$. *On the basis of (a) and (b), expressed in* (C.15.3) *and* (C.15.6)*, we shall obtain an explicit formula for each* P_n.

Let us determine P_0 first. Observe that a section of length $x+\Delta x$ is vacant if its left-hand part of length x is vacant and its right-hand part of length Δx is also vacant. Since the cars move independently of each other, the probability that the whole interval of length $x + \Delta x$ being empty is the product of the probabilities that the two smaller intervals of lengths x and Δx are both empty. (See Figure C.15.1.) Thus we have

Figure C.15.1: No cars in a section of length $x + \Delta x$.

$$P_0(x + \Delta x) = P_0(x)P_0(\Delta x). \qquad \text{(C.15.7)}$$

Recalling (C.15.6), we write (C.15.7) as

$$P_0(x + \Delta x) = P_0(x)(1 - k\Delta x - \delta\Delta x)$$

which a little algebra transforms to

$$\frac{P_0(x + \Delta x) - P_0(x)}{\Delta x} = -(k + \delta)P_0(x). \qquad \text{(C.15.8)}$$

Taking limits on both sides of (C.15.8) as $\Delta x \to 0$, we obtain

$$P_0'(x) = -kP_0(x). \qquad \text{(C.15.9)}$$

(Recall that $\delta \to 0$ as $\Delta x \to 0$.) From (C.15.9) it follows that there is a constant A such that $P_0(x) = Ae^{-kx}$. Since $1 = P_0(0) = Ae^{-k0} = A$, we conclude that $A = 1$, hence

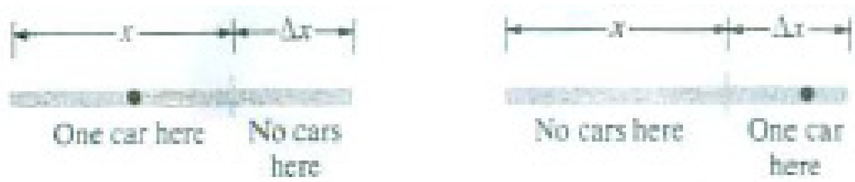
$$P_0(t) = e^{-kx}.$$

This explicit formula for P_0 is reasonable; e^{-kx} is a decreasing function of x, so that the larger an interval, the less likely that it is empty.

Now let us determine P_1. To do so, we examine $P_1(x + \Delta x)$ and relate it to $P_0(x)$, $P_0(\Delta x)$, $P_1(x)$, and $P_1(\Delta x)$, with the goal of finding an equation involving the derivative of P_1.

Again, imagine an interval of length $x + \Delta x$ cut into two intervals, the left-hand subinterval of length x and the right-hand subinterval of length Δx. Then there is precisely one car in the whole interval if *either* there is exactly one car in the left-hand interval and none in the right-hand subinterval *or* there is none in the left-hand subinterval and exactly one in the right-hand subinterval. (See Figure C.15.2.) Thus we have

$$P_1(x + \Delta x) = P_1(x)P_0(\Delta x) + P_0(x)P_1(\Delta x). \qquad \text{(C.15.10)}$$



(a) (b)

Figure C.15.2: The two ways to have exactly one care in an interval of length $x + \Delta x$.

In view of (C.15.3) and (C.15.6), we may write (C.15.10) as

$$P_1(x + \Delta x) = P_1(x)(1 - k\Delta x - \delta\Delta x) + P_0(x)(k\Delta x + \epsilon\Delta x)$$

which a little algebra changes to

$$\frac{P_1(x + \Delta x) - P_1(x)}{\Delta x} = -(k + \delta)P_1(x) + (k + \epsilon)P_0(x). \tag{C.15.11}$$

Letting $\Delta x \to 0$ in (C.15.11) and remembering that $\delta \to 0$ and $\epsilon \to 0$ as $\Delta x \to 0$, we obtain $P_1'(x) = -kP_1(x) + kP_0(x)$; recalling that $P_0(x) = e^{-kx}$, we deduce that

$$P_1'(x) = -kP_1(x) + ke^{-kx}. \tag{C.15.12}$$

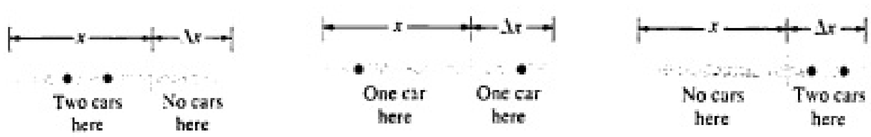
From (C.15.12) we shall obtain an explicit formula for $P_1(x)$. Since $P_0(x)$ involves e^{-kx} and so does (C.15.12). it is reasonable to guess that $P_1(x)$ involves e^{-kx}. Therefore let us express $P_1(x)$ as $g(x)e^{-kx}$ and determine the form of $g(x)$. (Since we have the identity $P_1(x) = \left(P_1(x)e^{kx}\right)e^{-kx}$, we know that $g(x)$ exists.)

According to (C.15.12) we have $\left(g(x)e^{-kx}\right)' = -kg(x)e^{-kx} + ke^{-kx}$; hence

$$g(x)(ke^{-kx}) + g'(x)e^{-kx} = -kg(x)e^{-kx} + ke^{-kx}$$

from which it follows that $g'(x) = k$. Hence $g(x) = kx + c_1$, where c_1 is some constant: $P_1(x) = (kx + c_1)e^{-kx}$. Since $P_1(0) = 0$, we have $P_1(0) = (k \cdot 0 + c_1)e^{-k \cdot 0} = c_1$ and hence $c_1 = 0$. Thus we have shown that

$$P_1(x) = kxe^{-kx} \tag{C.15.13}$$



(a) (b) (c)

Figure C.15.3: The three ways to have exactly two cars in an interval of length $x + \Delta x$.

and P_1 is completely determined.

To obtain P_2 we argue as we did in obtaining P_1. Instead of (C.15.10) we have

$$P_2(x + \Delta x) = P_2(x)P_0(\Delta x) + P_1(x)P_1(\Delta x) + P_0(x)P_2(\Delta x) \qquad \text{(C.15.14)}$$

an equation that records the three ways in which two cars in a section of length $x + \Delta x$ can be situated in a section of length x and a section of length Δx. (See Figure C.15.3.)

Similar reasoning shows that

See Exercise 8.

$$P_2(x) = \frac{k^2x^2}{2}. \qquad \text{(C.15.15)}$$

Then, applying the same reasoning inductively leads to

See Exercises 9 and 10.

$$P_n(x) = \frac{(kx)^n}{n!}e^{-kx}. \qquad \text{(C.15.16)}$$

We have obtained in (C.15.16) the formulas on which the rest of our analysis will be based. Note that these formujlas refer to a road section of any length, though the assumptions (a) and (b) refer only to short sections. What has enabled us to go from the "microscopic" to the "macroscopic" is the additional assumption that the traffic in any one section is independent of the traffic in any other section. The formulas (C.15.16) are known as the **Poisson formulas**.

The Meaning of k

The constant k was defined in terms of arbitrarily short intervals, at the "microscopic level". How would we compute k in terms of observable data, at the "macroscopic level"? It turns out that k records the traffic density: the average number of events during an interval of length x is kx.

The average number of events in a section of length x is defined as $\sum_{n=0}^{\infty} nP_n(x)$. This weights each possible number of events (n) with it's likelihood of occurring ($P_n(x)$). This average is

$$\begin{aligned}\sum_{n=0}^{\infty} nP_n(x) &= \sum_{n=1}^{\infty} nP_n(x) = \sum_{n=1}^{\infty} n\frac{(kx)^n e^{-kx}}{n!} \\ &= kxe^{-kx}\sum_{n=1}^{\infty}\frac{(kx)^{n-1}}{(n-1)!} \\ &= kxe^{-kx}\sum_{n=0}^{\infty}\frac{(kn)^n}{n!} = kxe^{-kx}e^{kx} = kx.\end{aligned}$$

Thus the expected number of cars in a section is proportional to the length of the section. This shows that the k appearing in assumption (a) is the measure of traffic density, the number of cars per unit length of road.

To estimate k, in the case of traffic for instance, divide the number of cars in a long section of the road by the length of that section.

EXAMPLE 4 (Traffic at a checkout counter.) Customers arrive at a checkout counter at the rate of 15 per hour. What is the probability that exactly five customers will arrive in any given 20-minute period?

SOLUTION We may assume that the probability of exactly one customer coming in a short interval of time is roughly proportional to the duration of that interval. Also, there is only a negligible probability that more than one customer may arrive in a brief interval of time. Therefore conditions (a) and (b) hold, if we replace "length of section" by "length of time". Without further ado, we conclude that the probability of exactly n customers arriving in a period of x minutes is given by (C.15.16). Moreover, the "customer density" is one per 4 minutes; hence $k = 1/4$, and thus the probability that exactly five customers arrive during a 20-minute period, $P_5(20)$, is

$$\left(\frac{1}{4}\cdot 20\right)^5 \frac{e^{-(1/4)\cdot 20}}{5!} = \frac{5^5 e^{-5}}{120} \approx 0.17547.$$

◇

Modeling of the type within this section is of use in predicting the length of waiting lines (or times) or the waiting time to cross. This is part of the theory of queues. See, for instance, Exercises 2 and 3. (See also Exercise 65 in the Summary Exercises in Chapter 4.)

EXERCISES

1.[R]

(a) Why would you expect that $P_0(a+b) = P_0(a) \cdot P_0(b)$ for any a and b?

(b) Verify that $P_0(x) = e^{-kx}$ satisfies the equation in (a).

2.[R] A cloud chamber registers an average of four cosmic rays per second.

(a) What is the probability that no cosmic rays are registered for 6 seconds?

(b) What is the probability that exactly two are registered in the next 4 seconds?

3.[R] Telephone calls during the busy hour arrive at a rate of three calls per minute. What is the probability that none arrives in a period of (a) 30 seconds, (b) 1 minute, (c) 3 minutes?

4.[R] In a large continually operating factory there are, on the average, two accidents per hour. Let $P_n(x)$ denote the probability that there are exactly n accidents in an interval of time of length x hours.

(a) Why is it reasonable to assume that there is a constant k such that $P_0(x)$, $P_1(x)$, ... satisfy 1 and 2 on page 1092?

(b) Assuming that these conditions are satisfied, show that $P_n(x) = (kx)^n e^{-kx}/n!$.

(c) Why must $k = 2$?

(d) Compute $P_0(1)$, $P_1(1)$, $P_2(1)$, $P_3(1)$, and $P_4(1)$.

5.[R] A typesetter makes an average of one mistake per page. Let $P_n(x)$ be the probability that a section of x pages (x need not be an integer) has exactly n errors.

(a) Why would you expect $P_n(x) = x^n e^{-x}/n!$?

(b) Approximately how many pages would be error-free in a 300-page book?

6.[R] In a light rainfall you notice that on one square foot of pavement there are an average of 3 raindrops. Let $P_n(x)$ be the probability that there are n raindrops on an area of x square feet.

(a) Check that assumptions 1 and 2 are likely to hold.

(b) Find the probability that an area of 3 square feet has exactly two raindrops.

(c) What is the most likely number of raindrops to find on an area of one square foot?

7.[R] Write x^2 in the form $g(x)e^{-kx}$.

8.[R] Show that $P_2(x) = \frac{k^2x^2}{2}e^{-kx}$.

9.[R] Show that $P_3(x) = \frac{(kx)^3}{3!}e^{-kx}$.

10.[M] Show that $P_n(x) = \frac{(kx)^n}{n!}e^{-kx}$.

11.[R]

(a) Why would you expect $P_3(a+b) = P_0(a)P_3(b) + P_1(a)P_2(b) + P_2(a)P_1(b) + P_3(a)P_0(b)$?

(b) Do functions defined in (C.15.16) satisfy the equation in (a)?

12.[R]

(a) Why would you expect $\lim_{n\to\infty} P_n(x) = 0$?

(b) Show that the functions defined in (C.15.16) have the limit in (a).

13.[R]

(a) Why would you expect $\lim_{x\to 0} P_1(x) = 1$ and, for all $n \geq 1$, $\lim_{x\to 0} P_n(x) = 0$?

(b) Show that the functions defined in (C.15.16) satisfy the limit in (a).

14.[R] We obtained $P_0(x) = e^{-kx}$ and $P_1(x) = kxe^{-kx}$. Verify that $\lim_{\Delta x\to 0} P_1(\Delta x)/\Delta x = k$, and $\lim_{\Delta x\to 0} P_0(\Delta x)/\Delta x = 1-k$. Hence show that $\lim_{\Delta x\to 0}(P_2(\Delta x) + P_3(\Delta x) + \cdots +)/\Delta x = 0$, and that assumptions 1 and 2 on page 1092 are indeed satisfied.

15.[R]

(a) Obtain assumption 1 from equation (C.15.3).

(b) Obtain equation (C.15.3) from assumption 2.

(c) Obtain assumption 2 from equation (C.15.6).

16.[M] What length of road is most likely to contain exactly one car? That is, what x maximizes $P_1(x)$?

17.[M] What length of road is most likely to contain three cars?

18.[M] For any $x \geq 0$, $\sum_{n=0}^{\infty} P_n(x)$ should equal 1 because it is certain that some number of cars is in a given section of length x (maybe 0 cars). Check that $\sum_{n=0}^{\infty} P_n(x) = 1$. NOTE: This provides a probabilistic argument that $e^u = \sum_{n=0}^{\infty} u^n/n!$ for $n \geq 0$.

19.[M] Planes arrive randomly at an airport at the rate of one per 2 minutes. What is the probability that more than three planes arrive in a 1-minute interval?