The Equational Complexity of Višin’s Algebra

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Outline

Varieties of Algebras
Algebra is About Finitary Operations
Equational Complexity of Varieties of Algebras

What Can Be Said About Irreducibles?
Generating Principal Congruences
Subdirectly Irreducible and Finitely Subdirectly Irreducible
Weakening the Finite Residual Bound

What We Found Out
Play It Again
Variations on a Theme
Behind the Scenes
Two Temptations
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Višin’s Weakly Automatic Algebra \(V\)

**Characteristics**

A system \(\langle Q \cup \Sigma \cup \{0\}, \cdot \rangle\) where
- \(Q \cup \Sigma\) and \(\{0\}\) are disjoint sets.
- \(\cdot\) is a two-place operation on \(Q \cup \Sigma \cup \{0\}\).
- \(\cdot\) is determined by the transition function of a weak partial automaton with state set \(Q\) and alphabet \(\Sigma\).
Lyndon’s Automatic Algebra $L$

Characteristics
A system $\langle Q \cup \Sigma \cup \{0\}, \cdot \rangle$ where
- $Q, \Sigma,$ and $\{0\}$ are disjoint sets.
- $\cdot$ is two-place operation on $Q \cup \Sigma \cup \{0\}$.
- $\cdot$ is determined by a partial automaton with alphabet $\Sigma$ and state set $Q$. 
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Deciding Membership for Finite Algebras

Let $\mathcal{V}$ be a variety of finite signature and let $\mathbf{B}$ be a finite algebra of the same signature.
Deciding Membership for Finite Algebras

Let $\mathcal{V}$ be a variety of finite signature and let $\mathcal{B}$ be a finite algebra of the same signature.

How much of the equational theory of $\mathcal{V}$ must be examined to determine whether $\mathcal{B} \in \mathcal{V}$?
Deciding Membership for Finite Algebras

Let $\mathcal{V}$ be a variety of finite signature and let $\mathcal{B}$ be a finite algebra of the same signature.

How much of the equational theory of $\mathcal{V}$ must be examined to determine whether $\mathcal{B} \in \mathcal{V}$?

What if $\mathcal{V}$ is the variety generated by Višin’s algebra $\mathcal{V}$? Or by Lyndon’s algebra $\mathcal{L}$?
We Only Need Finitely Many Equations

If $\mathcal{V}$ is finitely based, then we only need to check a fixed finite set of equations.
We Only Need Finitely Many Equations

Neither $V$ nor $L$ generated finitely based varieties.
We Only Need Finitely Many Equations

But $\mathbf{B}$ is finite, say of size $n$

1. Up to isomorphism there are only finitely many algebras no larger than $\mathbf{B}$.

2. For each one that is not in $\mathcal{V}$ pick an equation, as short as possible, true in $\mathcal{V}$ that fails in the algebra.

3. Let $\beta(n + 1) - 1$ be the length of the longest equation selected in this way.
We Only Need Finitely Many Equations

But \( B \) is finite, say of size \( n \)

1. Up to isomorphism there are only finitely many algebras no larger than \( B \).

2. For each one that is not in \( V \) pick an equation, as short as possible, true in \( V \) that fails in the algebra.

3. Let \( \beta(n + 1) - 1 \) be the length of the longest equation selected in this way.
We Only Need Finitely Many Equations

But $B$ is finite, say of size $n$

1. Up to isomorphism there are only finitely many algebras no larger than $B$.
2. For each one that is not in $\mathcal{V}$ pick an equation, as short as possible, true in $\mathcal{V}$ that fails in the algebra.
3. Let $\beta(n + 1) - 1$ be the length of the longest equation selected in this way.
We Only Need Finitely Many Equations

Definition
Let $\mathcal{V}$ be a variety of finite signature. The *equational complexity* of $\mathcal{V}$ is the function $\beta$ where $\beta(n)$ is the least natural number $\ell$ so that any algebra $B$ of cardinality less than $\ell$ belongs to $\mathcal{V}$ iff each equation of length less than $\ell$ which is true in $\mathcal{V}$ is also true in $B$. 
Some More Things We Know

Theorem (Oates and Powell, 1964)
*Every finite group is finitely based.*

Theorem (McKenzie, 1970)
*Every finite lattice with finitely many operators is finitely based.*

Theorem (Kruse and L'vov, 1973)
*Every finite ring is finitely based.*

Theorem (Perkins, 1966)
*A certain semigroup of six $2 \times 2$ matrices under matrix multiplication is not finitely based.*
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Theorem (Perkins, 1966)
A certain semigroup of six $2 \times 2$ matrices under matrix multiplication is *not* finitely based.
Then We Found Out

Theorem (Baker, 1977)

Let \( \mathcal{V} \) be a variety of finite signature. If

- \( \mathcal{V} \) is congruence distributive.
- \( \mathcal{V} \) has a finite residual bound.

then \( \mathcal{V} \) is finitely based.
Theorem (McKenzie, 1987)

Let \( \mathcal{V} \) be a variety of finite signature. If

- \( \mathcal{V} \) is congruence **modular**.
- \( \mathcal{V} \) has a finite residual bound.

then \( \mathcal{V} \) is finitely based.
Then We Found Out

Theorem (Willard, 2000)

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence meet semidistributive.
- $\mathcal{V}$ has a finite residual bound.

then $\mathcal{V}$ is finitely based.
Anyone’s Guess

Park’s Conjecture/ Jónsson’s Speculation 1974

Let \( \mathcal{V} \) be a variety of finite signature. If

- ..., 
- \( \mathcal{V} \) has a finite residual bound,

then \( \mathcal{V} \) is finitely based.
Good, But . . .

The Group $\mathbb{Q}$ of Quaternions
The variety generated by $\mathbb{Q}$

- Is congruence modular.
- Has no residual bound.

Nevertheless it is finitely based.
Good, But ...

The Flat Graph Algebra $D$

The variety generated by $D$

- Is congruence meet semidistributive.
- Has no residual bound.

Nevertheless it is finitely based.
Bjarni Gets Beyond a Residual Bound

Theorem (Jónsson 1979)

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence distributive.
- $\mathcal{V}_{\text{fsi}}$ is finitely axiomatizable.

then $\mathcal{V}$ is finitely based.
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Translations

Definition
Let $A$ be an algebra. A function $\lambda : A \to A$ is a

Basic translation provided $\lambda$ arises from a basic operation of $A$ by evaluating all but one of its arguments with elements of $A$.

$k$-translation provided $\lambda$ can be realized as the composition of a sequence of $k$ or fewer basic translations.

Translation provided $\lambda$ is a $k$-translation for some natural number $k$. 

Observe that the identity function is the only 0-translation.
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Observe that the identity function is the only 0-translation.
According to Dilworth and Mal’cev

$$\langle a, b \rangle \in Cg^A(c, d)$$
According to Dilworth and Mal’cev

\[ \lambda_0(c) \quad \lambda_0(d) \]

\[ \{ \lambda_0(c), \lambda_0(d) \} = \{ a_0, a_1 \} \]
According to Dilworth and Mal’cev

\[ \langle a, b \rangle \in Cg^A(c, d) \]

\[ \{\lambda_0(c), \lambda_0(d)\} = \{a_0, a_1\} \]

\[ \{\lambda_1(c), \lambda_1(d)\} = \{a_1, a_2\} \]
According to Dilworth and Mal’cev

\[ a_0 = a \]

\[ \langle a, b \rangle \in Cg^A(c, d) \]

\[
\begin{align*}
\{\lambda_0(c), \lambda_0(d)\} &= \{a_0, a_1\} \\
\{\lambda_1(c), \lambda_1(d)\} &= \{a_1, a_2\} \\
\{\lambda_2(c), \lambda_2(d)\} &= \{a_2, a_3\}
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\{\lambda_2(c), \lambda_2(d)\} &= \{a_2, a_3\} \\
\{\lambda_3(c), \lambda_3(d)\} &= \{a_3, a_4\} \\
& \vdots \\
\{\lambda_{n-1}(c), \lambda_{n-1}(d)\} &= \{a_{n-1}, a_n\}
\end{align*}
\]
According to Dilworth and Mal’cev

\[ \langle a, b \rangle \in Cg^A(c, d) \]

We write \( \{c, d\} \overset{n}{\leftrightarrow}_k \{a, b\} \)

provided

\( \lambda_i \) is a \( k \)-translation for each

\( i < n \)

\[ \langle a, b \rangle \in Cg^A(c, d) \]

if and only if

\( \{c, d\} \overset{n}{\leftrightarrow}_k \{a, b\} \)

for some natural numbers \( k \) and

\( n \)
According to Dilworth and Mal’cev

$$\langle a, b \rangle \in Cg^A(c, d)$$

We write $$\{c, d\} \xrightarrow[k]{n} \{a, b\}$$
provided $$\lambda_i$$ is a $$k$$-translation for each $$i < n$$

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According to Dilworth and Mal’cev

$$\langle a, b \rangle \in C^A_g(c, d)$$

We write

$$\{c, d\} \xrightarrow{n_k} \{a, b\}$$

provided

$$\lambda_i$$ is a $k$-translation for each

$$i < n$$

$$\langle a, b \rangle \in C^A_g(c, d)$$

if and only if

$$\{c, d\} \xrightarrow{n_k} \{a, b\}$$

for some natural numbers $k$ and $n$.
According to Dilworth and Mal’cev

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We write \( \{c, d\} \rightsquigarrow_k^n \{a, b\} \)

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for some natural numbers \( k \) and \( n \)
Varieties of Algebras

Algebra is About Finitary Operations
Equational Complexity of Varieties of Algebras

What Can Be Said About Irreducibles?
Generating Principal Congruences
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Weakening the Finite Residual Bound

What We Found Out
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Two Temptations
Definition
An algebra \( A \) is **subdirectly irreducible** provided there is distinct elements \( p, q \in A \) such that for all \( a, b \in A \) with \( a \neq b \)
\( \{a, b\} \xrightarrow[]{}^n_k \{p, q\} \) for some \( n \) and \( k \). In this case \( \langle p, q \rangle \) is called a **critical pair**. \( \mathcal{K}_{si} \) denotes the class of all subdirectly irreducible algebras belonging to the class \( \mathcal{K} \) of algebras.
Definition
An algebra $A$ is finitely subdirectly irreducible if it has at least two elements and for all $a, b, c, d \in A$ with $a \neq b$ and $c \neq d$ there is are two distinct elements $p$ and $q$ of $A$ so that $\{a, b\} \leftrightarrow_k \{p, q\}$ and $\{c, d\} \leftrightarrow_\ell \{p, q\}$ for some natural numbers $k, \ell, m, n$. $\mathcal{K}_{f_{si}}$ denotes the class of all finitely subdirectly irreducible algebras belonging to the class $\mathcal{K}$ of algebras.
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When a Variety $\mathcal{V}$ Has a Finite Residual Bound We Know

- $\mathcal{V}$ is locally finite.
When a Variety $\mathcal{V}$ Has a Finite Residual Bound We Know

- $\mathcal{V}$ is locally finite.
- $\mathcal{V}_{si}$ is a finitely axiomatizable elementary class.
When a Variety $\mathcal{V}$ Has a Finite Residual Bound We Know

- $\mathcal{V}$ is locally finite.
- $\mathcal{V}_{si}$ is a finitely axiomatizable elementary class.
- $\mathcal{V}_{si} = \mathcal{V}_{fsi}$. 
When a Variety $\mathcal{V}$ Has a Finite Residual Bound We Know

- $\mathcal{V}$ is locally finite.
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- $\mathcal{V}_{si} = \mathcal{V}_{fsi}$.
- $\mathcal{V}_{fsi}$ is a finitely axiomatizable elementary class.
Definition of Bounded Critical Depth

**Definition**

A class \( \mathcal{K} \) of algebras of the same finite signature is said to have **bounded critical depth** provided there is a natural number \( \ell \) so that for every \( A \in \mathcal{K}_{\text{si}} \) and all \( a, b, c, d \in A \) such that \( c \neq d \) and \( \langle a, b \rangle \) is a critical pair of \( A \) we have \( \{c, d\} \not\xrightarrow{n} \{a, b\} \) for some natural number \( n \).
When a Variety $\mathcal{V}$ Has a Finite Residual Bound We Know

- $\mathcal{V}$ is locally finite.
- $\mathcal{V}_{si}$ is a finitely axiomatizable elementary class.
- $\mathcal{V}_{si} = \mathcal{V}_{fsi}$.
- $\mathcal{V}_{fsi}$ is a finitely axiomatizable elementary class.
- $\mathcal{V}$ has bounded critical depth.
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Theorem (Willard, 2000)

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence meet semidistributive.
- $\mathcal{V}$ has a finite residual bound.

then $\mathcal{V}$ is finitely based.
Theorem (Jónsson 1979)

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence distributive.
- $\mathcal{V}_{\text{fsi}}$ is finitely axiomatizable.

then $\mathcal{V}$ is finitely based.
A New Theorem

Theorem (With Kirby Baker and Ju Wang)

Let \( \mathcal{V} \) be a variety of finite signature.

If

- \( \mathcal{V} \) is congruence meet semidistributive.
- \( \mathcal{V}_{fsi} \) is finitely axiomatizable.
- \( \mathcal{V} \) is locally finite.
- \( \mathcal{V} \) has bounded critical depth.

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Definition
A class $\mathcal{K}$ of algebras of some finite signature has **term-finite principal congruences** provided there is a natural number $\ell$ so that for every $A \in \mathcal{K}$ and all $a, b, c, d \in A$ we have

$$\langle a, b \rangle \in Cg^A(c, d) \text{ if and only if } \{c, d\} \not\rightarrow^\ell_n \{a, b\} \text{ for some } n.$$
Corollary

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence meet semidistributive.
- $\mathcal{V}_{\text{fsi}}$ is finitely axiomatizable.
- $\mathcal{V}$ is locally finite.
- $\mathcal{V}_{\text{si}}$ has term-finite principal congruences.

then $\mathcal{V}$ is finitely based.
Definition
A class $\mathcal{K}$ of algebras of some finite signature has **bounded critical diameter** provided there is a natural number $\ell$ so that for every $A \in \mathcal{K}_{si}$ and all $a, b, c, d \in A$ such that $\langle a, b \rangle$ and $\langle c, d \rangle$ are critical pairs we have $\{c, d\} \leadsto^n \{a, b\}$ for some $n$. 
Corollary

Let \( \mathcal{V} \) be a variety of finite signature. If

- \( \mathcal{V} \) is congruence meet semidistributive.
- \( \mathcal{V}_{fsi} \) is finitely axiomatizable.
- \( \mathcal{V} \) is locally finite.
- \( \mathcal{V}_{si} \) is an elementary class.
- \( \mathcal{V} \) has bounded critical diameter.

then \( \mathcal{V} \) is finitely based.
Example
Let $\mathcal{D}$ be the variety generated by the flat graph algebra $\mathcal{D}$. Then

- $\mathcal{D}$ is congruence meet semidistributive.
- $\mathcal{D}$ is locally finite.
- $\mathcal{D}_{\text{si}}$ is finitely axiomatizable.
- $\mathcal{D}_{\text{fsi}}$ is finitely axiomatizable.
- $\mathcal{D}$ has bounded critical diameter.
- $\mathcal{D}$ is residually large.
- $\mathcal{D}$ is finitely based.
Example

Let $\mathcal{D}$ be the variety generated by the flat graph algebra $\mathcal{D}$. Then

- $\mathcal{D}$ is congruence meet semidistributive.
- $\mathcal{D}$ is locally finite.
- $\mathcal{D}_{si}$ is finitely axiomatizable.
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- $\mathcal{D}$ is residually large.
- $\mathcal{D}$ is finitely based.
Example

Let \( D \) be the variety generated by the flat graph algebra \( D \). Then

- \( D \) is congruence meet semidistributive.
- \( D \) is locally finite.
- \( D_{si} \) is finitely axiomatizable.
- \( D_{fsi} \) is finitely axiomatizable.
- \( D \) has bounded critical diameter.
- \( D \) is residually large.
- \( D \) is finitely based.
Example

Let $\mathcal{D}$ be the variety generated by the flat graph algebra $\mathcal{D}$. Then

- $\mathcal{D}$ is congruence meet semidistributive.
- $\mathcal{D}$ is locally finite.
- $\mathcal{D}_{\text{si}}$ is finitely axiomatizable.
- $\mathcal{D}_{\text{fsi}}$ is finitely axiomatizable.
- $\mathcal{D}$ has bounded critical diameter.
- $\mathcal{D}$ is residually large.
- $\mathcal{D}$ is finitely based.

Most of these items were pointed out by Zoltán Székely (1998) and Dejan Delić (2001). In particular, Delić showed that $\mathcal{D}$ is finitely based by a different method.
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Definition

An elementary formula \( \Phi(x, y, z, w) \) is said to define nontrivial principal meets in an algebra \( A \) provided for all \( a, b, c, d \in A \) with \( a \neq b \) and \( c \neq d \):

\[
Cg^A(a, b) \cap Cg^A(c, d) \text{ is nontrivial}
\]

if and only if

\[
A \models \Phi(a, b, c, d)
\]
Definition

An elementary formula $\Phi(x, y, z, w)$ is said to define nontrivial principal meets in a class $\mathcal{K}$ of algebras if and only if $\Phi(x, y, z, w)$ defines nontrivial principal meets in every algebra belonging to $\mathcal{K}$. 
We regard $\{x, y\} \rightarrow_k^n \{u, v\}$ as an elementary formula with free variables $x, y, u$ and $v$.

A more elaborate elementary formula is $\{x, y\} \rightarrow_k^n \circ \rightarrow_k^m \{u, v\}$. 
The elementary formula

$$\exists u, v [u \neq v \land \{x, y\} \vdash^m \circ \vdash^n \{u, v\} \land \{z, w\} \vdash^m \circ \vdash^n \{u, v\}]$$

is built from $$\{x, y\} \vdash^m \circ \vdash^n \{u, v\}$$.
Definition
We say \( \{x, y\} \dashv \lor^m \circ \vdash^n \{u, v\} \) supports nontrivial principal meets provided

\[ \exists u, v [u \neq v \land \{x, y\} \dashv \lor^m \circ \vdash^n \{u, v\} \land \{z, w\} \dashv \lor^m \circ \vdash^n \{u, v\}] \]

defines nontrivial principal meets.
Defining Nontrivial Principal Meets

Theorem (With Kirby Baker and Ju Wang)

Let $\mathcal{V}$ be a variety of finite signature.
If

- $\mathcal{V}$ is congruence meet semidistributive.
- $\mathcal{V}$ is locally finite.
- $\mathcal{V}$ has critical depth bounded by $\ell$.

then there are natural numbers $k$ and $n$ such that

$\{x, y\} \quad \dashv \vdash_{\ell} \quad \dashv \vdash_{k}^{n} \quad \{u, v\}$ supports nontrivial principal meets in $\mathcal{V}$. 
Supporting Nontrivial Principal Meets
Theorem (with Kirby Baker and Ju Wang)

Let $\mathcal{V}$ be a variety with a finite signature. If there are natural numbers $\ell, n, \text{ and } k$ and an elementary sentence $\Psi$ such that

- $\mathcal{V}_{\text{fsi}}$ is finitely axiomatizable;
- $\Psi$ is true in $\mathcal{V}$;
- $\{x, y\} \rightarrow_{\ell}^{1} \circ \rightarrow_{k}^{n} \{u, v\}$ supports nontrivial principal meets in $\mathcal{V}$.

- For all algebras $B \models \Psi$ and all $a, b, c, d \in B$ if $Cg^{B}(a, b) \cap Cg^{B}(c, d)$ is nontrivial, then there are $q, r \in B$ with $q \neq r$ and there is a natural number $m$ so that $\{a, b\} \rightarrow_{m}^{1} \circ \rightarrow_{k}^{n} \{q, r\}$ and $\{c, d\} \rightarrow_{m}^{1} \circ \rightarrow_{k}^{n} \{q, r\}$;

then $\mathcal{V}$ is finitely based.
Outline

Varieties of Algebras
  Algebra is About Finitary Operations
  Equational Complexity of Varieties of Algebras

What Can Be Said About Irreducibles?
  Generating Principal Congruences
  Subdirectly Irreducible and Finitely Subdirectly Irreducible
  Weakening the Finite Residual Bound

What We Found Out
  Play It Again
  Variations on a Theme
  Behind the Scenes
  Two Temptations
The Congruence Modular Temptation

Let $\mathcal{V}$ be a variety of finite signature. If

- $\mathcal{V}$ is congruence modular.
- $\mathcal{V}_{\text{fsi}}$ is finitely axiomatizable.
- $\mathcal{V}$ is locally finite.
- $\mathcal{V}$ has bounded critical depth.
- ... then is $\mathcal{V}$ is finitely based?
The Blatant Temptation

Let $\mathcal{V}$ be a variety of finite signature.
If

- $\mathcal{V}_{fsi}$ is finitely axiomatizable.
- $\mathcal{V}$ is locally finite.
- $\mathcal{V}$ has bounded critical depth.
- . . . .

then is $\mathcal{V}$ is finitely based?