PROBLEMS FROM GROUP THEORY

In the problems below, G, H, K, and N generally denote groups. We use p to stand for a positive prime integer. $\mathrm{Aut}(G)$ denotes the group of automorphisms of G. $Z(G)$ denotes the center of G. S_n denotes the group of all permutations of a set with n elements—the so called symmetric group. A_n denotes the alternating group on n elements. $H \leq G$ means that H is a subgroup of G, while $H \triangleleft G$ means that H is a normal subgroup of G. $H \otimes K$ denotes the internal direct product of H and K. $H \times K$ denotes the (external) direct product of H and K.

Problem 0.
Show that any group which has only finitely many subgroups must be finite.

Problem 1.
If $H \leq G$ and the product of any two right cosets of H is again a right coset of H, then $H \triangleleft G$.

Problem 2.
Show that the group $\mathbb{Q}$ of rational numbers under addition is a directly indecomposable group. Show that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.

PROBLEM 3.
Suppose that $G = H \otimes K$ and $N \lhd G$. Prove that if $N \cap H = \{e\} = N \cap K$, then $N \leq Z(G)$.

PROBLEM 4.
Let p be the smallest prime dividing the order of the finite group G. Prove that any subgroup of index p in G is a normal subgroup.

PROBLEM 5.
Prove that every group which has a proper subgroup of finite index must have a proper normal subgroup of finite index.

PROBLEM 6.
Prove that there is not finite group G so that $G/Z(G)$ has exactly 143 elements.

PROBLEM 7.
Prove that there is no group G so that $G/Z(G) \cong \mathbb{Z}$, where $\mathbb{Z}$ denotes the group of integers under addition.

PROBLEM 8.
Let G be a non-Abelian group. Prove that $G/Z(G)$ is not cyclic.

PROBLEM 9.
Let $\mathbb{C}^{\times}$ be the multiplicative group of nonzero complex numbers and let n be a positive integer. How many subgroups of order n does $\mathbb{C}^{\times}$ have?

Problem 10.
Let G be the multiplicative group of complex numbers of modulus 1, let $\mathbb{R}$ be the additive group of real numbers, and let $\mathbb{Z}$ be the additive group of integers. Prove that $G \cong \mathbb{R}/\mathbb{Z}$.

Problem 11.
Prove that G cannot have four distinct proper subgroups H_0, H_1, H_2 , and H_3 so that $H_0 \leq H_1 \leq H_2 \leq G$, $H_0 = H_2 \cap H_3$, and $H_1 H_3 = G$.

Problem 12.
Let G be a finitely generated group. Prove that every proper normal subgroup of G is included in a normal subgroup of G which is maximal among all proper normal subgroups of G .

Problem 13.
Let G_1 and G_2 be Abelian groups and let $\alpha : G_1 \to G_2$ and $\beta : G_2 \to G_1$ be homomorphisms so that $\beta\alpha(g) = g$, for all $g \in G$. Prove that $G_2 = \alpha(G_1) \otimes \ker \beta$.

Problem 14.
Let G be a group which has exactly one element g of order n , where n is a positive integer. Prove that $n = 2$ and $g \in Z(G)$.

Problem 15.
If G is a finite group and $|\{a : a^n = e\}| \leq n$ for all positive integers n, then G is cyclic.

Problem 16.
Suppose $|G| = 2n$, G has exactly n elements of order 2, and the remaining n elements constitute a subgroup H. Prove that H is an Abelian group of odd order.

Problem 17.
Let H and K be subgroups of G, each of finite index. Prove that $H \cap K$ is also of finite index.

Problem 18.
Let A and B be finite Abelian p-groups of the same order and with exponent p^4. Suppose that

$$|\{a \in A : a^p = e\}| = |\{b \in B : b^p = e\}|$$
$$|\{a \in A : a^{p^2} = e\}| = |\{b \in B : b^{p^2} = e\}|.$$

Is it true that $A \cong B$?

Problem 19.
Classify, up to isomorphism, the Abelian groups of order 144.

Problem 20.
Give an example of two finite Abelian p-groups A and B which are **not** isomorphic but all of the following hold:

a. $|A| = |B|$,
b. The exponents of A and B are the same,
c. $|\{a \in A : a^p = e\}| = |\{b \in B : b^p = e\}|$

Problem 21.
Prove that if G, H, and K are finite Abelian groups such that $G \times H \cong G \times K$, then $H \cong K$.

Problem 22.
Prove that a finite Abelian group fails to be cyclic if and only if it has a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, for some prime p.

Problem 23.
Let G be a finite Abelian group and $\mathbb{C}^\times$ be the multiplicative group of nonzero complex numbers, and let $\varphi : G \to \mathbb{C}^\times$ be a nontrivial homomorphism. Prove that $\displaystyle\sum_{x \in G} \varphi(x) = 0$.

Problem 24.
Prove that if G and H are finite Abelian groups and $G \times G \cong H \times H$, then $G \cong H$.

Problem 25.
Prove that if k is a natural number that divides the order of a finite Abelian group G, then G has a subgroup of order k.

Problem 26.
Let G be a finite Abelian group. Prove that if $|G|$ is not divisible by k^2 for any integer $k \geq 2$, then G is cyclic.

Problem 27.
Prove that every group of order 130 has a normal subgroup of order 5.

Problem 28.
Suppose that G is a finite group of odd order and $N \triangleleft G$ with $|N| = 5$. Prove that N is included in the center of G.

Problem 29.
Prove that every group of order 35 is cyclic.

Problem 30.
Let G be a group of order p^n, where p is prime and $n > 0$. Suppose that $\{e\} \neq H \triangleleft G$. Prove that $Z(G) \cap H \neq \{e\}$.

Problem 31.
How many elements of order 7 are there in a simple group of order 168?

Problem 32.
Prove that every group of order 154 has a normal subgroup of order 7.

Problem 33.
Prove that every group of order 30 has a subgroup of order 15.

Problem 34.
Let H be a proper subgroup of a finite p-group G. Prove that G has a subgroup N such that $H \triangleleft N$ and $H \neq N$.

Problem 35.
Describe, up to isomorphism, all the groups of order 325.

Problem 36.
Describe, up to isomorphism, all the groups of order 1225.

Problem 37.
Prove that every finite nontrivial p-group has a nontrivial center.

Problem 38.
Let G be a finite group and let $H \triangleleft G$ so that $|H| = p^k$ for some prime p and some positive integer k. Prove that H is contained in every p-Sylow subgroup of G.

Problem 39.
Prove there is no simple group of cardinality 200.

Problem 40.
Describe, up to isomorphism, all the groups of order $7^2 \cdot 13$.

Problem 41.
Let G be a finite group, let p be a prime number, and let K be a p-subgroup of G. Prove that the number of p-Sylow subgroups of G that contain K is congruent to 1 modulo p.

Problem 42.
Prove that there is no simple group of order 56.

Problem 43.
The $H \triangleleft G$, where G is a finite group. Let K be a Sylow subgroup of H. Prove $G = N_G(K)H$.

Problem 44.
Prove that there is no simple group of order 30.

Problem 45.
Describe, up to isomorphism, all the groups of order 175.

Problem 46.
Describe, up to isomorphism, all the groups of order 99.

Problem 47.
Describe, up to isomorphism, all the groups of order 24.

Problem 48.
Prove that if H is a p-subgroup of a finite group G and P is a p-Sylow subgroup of G, then there is $x \in G$ so that $H \leq xPx^{-1}$. In particular, any two p-Sylow subgroups are conjugate.

Problem 49.
Prove that there is no simple group of order 56.

Problem 50.
Let G be a group of order 45. Prove that every subgroup of G of index 3 is normal.

Problem 51.
Prove that every group of order 25 has a nontrivial automorphism.

Problem 52.
Let G be a group, and denote by $\operatorname{Inn} G$ the group of inner automorphisms of G and by $\operatorname{Aut} G$ the group of all automorphisms of G. Prove that $\operatorname{Inn} G \triangleleft \operatorname{Aut} G$ and that $\operatorname{Inn} G \cong G/Z(G)$.

Problem 53.
Prove that every finite group with at least 3 elements has at least 2 automorphisms.

Problem 54.
Prove that $\mathrm{Aut}\,(S_n) \cong S_n$, for ever natural number n. (Here S_n denotes the group of permutations of a set with n elements.)

Problem 55.
Let $H \leq G$. Prove that $N_G(H)/C_G(H)$ is embeddable into $\mathrm{Aut}\,(H)$.

Problem 56.
Let G be a group of order n. Define $\varphi : G \to G$ by $\varphi(a) = a^{n^2+3n+1}$ for all $a \in G$. Prove that φ is an automorphism of G.

Problem 57.
Let p be a prime number. Prove that if a and b are elements of the symmetric group S_p, where a has order p and b is a transposition, then $\{a, b\}$ generates S_p.

Problem 58.
Prove that a permutation on a finite set cannot be both even and odd.

Problem 59.
Let F be the group freely generated by a countably infinite set.

a. Prove that F has a maximal normal subgroup of index $3!/2$.
b. Prove that F has a maximal normal subgroup of index $n!/2$. for every $n \geq 5$.
c. Prove that F has no maximal normal subgroup of index $4!/2$.

Problem 60.
Suppose $H \triangleleft G$ and $G/H \cong F$, where F is a free group. Prove that G has a subgroup isomorphic to F.

Problem 61.
Is the group of positive rationals under multiplication a free Abelian group?

Problem 62.
Is the gorup of rationals under addition a free Abelian group?

Problem 63.
Let G be the group freely generated by $\{a, b\}$ subject to the relations $b^2 = e, a^{15} = e,$ and $ba = a^{-1}b$.

a. Describe the commutator subgroup of G.
b. Find all the Sylow subgroups of G.
c. Find the center of G.

Problem 64.
Let $Gl(n, F)$ be the multiplicative group of all nonsingular $n \times n$ matrices over a field F. Let $Sl(n, F)$ consist of those $n \times n$ matrices over F with determinant 1. Show that $Sl(n, F)$ is a normal subgroup of $Gl(n, F)$. Compute $[Gl(n, F) : Sl(n, F)]$, when F is finite.

Problem 65.
Let $O_n = \{A \in Gl(n, \mathbb{R}) : A^t A = I\}$ and $SO_n = \{A \in O_n : \det A = 1\}$. Prove $SO_n \triangleleft O_n < Gl(n, \mathbb{R})$ and compute $[O_n : SO_n]$.

Problem 66.
Prove that $Gl(n, F)/Sl(n, F) \cong F^\times$, for any field F where $F^\times$ denotes the multiplicative group of nonzero elements of F.

Problem 67.
Briefly describe examples in each case:

a. A group G with subgroups H and K such that $H \triangleleft K \triangleleft G$ but $H \ntriangleleft G$.
b. A noncyclic Abelian groups that is directly indecomposable.
c. A non-Abelian group G with a normal subgroup H and a subgroup K such that $G/H \cong K$, but $G \ncong H \times K$.

Problem 68.
Prove that every finite p-group is solvable.

Problem 69.
Provide an example of a finite group G abd an integer k so that k divides the order of G, but G has no subgroup of order k.