Residual Finiteness and Finite Equational Bases:
Undecidable Properties of Finite Algebras

Lectures on Some Recent Work of Ralph McKenzie and Ross Willard

Prepared By
George F. McNulty
Preamble

In 1993 Ralph McKenzie resolved several of the most intriguing and challenging problems concerning varieties generated by finite algebras with only finitely many basic operations. These accomplishments can be summarized as follows.

ACCOMPLISHMENT 1. There is a finite algebra generating a residually countable inherently non-finitely based variety.

This refutes the R-S Conjecture that every finitely generated residually small variety should be residually very finite. It also refutes an old and provocative speculation that the finite algebras of finite type which generate residually small varieties might all be finitely based.

ACCOMPLISHMENT 2. There is no algorithm for deciding which finite algebras generate residually finite varieties.

ACCOMPLISHMENT 3. There is no algorithm for deciding which finite algebras are finitely based.

So Ralph McKenzie settled Tarski’s celebrated Finite Basis Problem. Until McKenzie’s breakthrough, no algebraically reasonable property of finite algebras was known to be undecidable. Indeed, a number of properties (generating a minimal variety, generating a congruence modular variety, etc.) were long known to be decidable.

Ralph McKenzie invented a robust technique for interpreting an arbitrary Turing machine $T$ into a finite algebra $A(T)$ so that the machine computations would be available in the variety generated by $A(T)$. It seems likely that it can be used to demonstrate the undecidability of a wide assortment of properties of varieties generated by finite algebras. Indeed, further undecidability results have already been obtained by Charles Latting, Ralph McKenzie, and Ross Willard.

These lectures are intended to provide a path to these accomplishments. There are only two main differences between McKenzie’s exposition and the one found here. First, I organized the material into lectures which are each, more or less, amenable to presentation in fifty-minutes. The second and more significant deviation is that I followed the work of Ross Willard to prove that $A(T)$ is finitely based when $T$ halts. I added no new results (and I hope no errors either). This presentation is intended to be concrete, and to reveal how the ideas develop toward the ultimate results. See Ross Willard’s work for a more abstract perspective.

These notes arose from three occasions on which I gave series of lectures on this material. Brian Davey, Zsolt Lengyvarszky, Marton Nagy, Zoltan Szekely, Ralph Freese, Bill Lampe, and JB Nation all endured my struggles to talk reasonably about these results. Their criticisms and ideas have become part of these notes. Ralph McKenzie and Ross Willard both shared early drafts of their work with me. Of course, these lectures owe a large debt to Ralph McKenzie (say about 98%) who originated these spectacular results.
LECTURE 0

An Interesting Locally Finite Algebra

During the course of developing this material we will deal with algebras that have many basic operations. Among them will always be three denoted by $\cdot$, $\land$, and $0$. $\land$ will always denote the least element with respect to $\leq$, the underlying semilattice order. Meet semilattices of height one are referred to as flat semilattices. $0$ is an absorbing element for the product $\cdot$ in the sense that $0 \cdot x \approx x \cdot 0 \approx 0$ always holds. The product also satisfies $(x \cdot y) \cdot z \approx 0$. Therefore, only right associated products can produce results other than 0.

The remaining operations will mostly be monotone with respect to the semilattice order, they typically commute with $\land$, and $0$ will usually be an absorbing element for these operations. For this lecture we assume that the remaining operations are term operations built up from $\cdot$, $\land$, and $0$.

Let $Q = \{0\} \cup \{a_p : p \in \mathbb{Z}\} \cup \{b_p : p \in \mathbb{Z}\}$, where all the $a_p$'s and $b_q$'s are distinct and different from 0. The algebra $(Q, \land, 0)$ is a height 1 semilattice with least element 0. The product in $Q$ is defined so that $a_p \cdot b_{p+1} = b_p$ for all $p \in \mathbb{Z}$, with all other products 0. Here is a picture that might help:

```
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  $a_{-3}$ & $a_{-2}$ & $a_{-1}$ & $a_0$ & $a_1$ & $a_2$ & $a_3$ \\
  $b_{-3}$ & $b_{-2}$ & $b_{-1}$ & $b_0$ & $b_1$ & $b_2$ & $b_3$ \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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  & & & & & & $0$
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The operation $\cdot$ could be referred to as an edge operation and the algebra $(Q, \cdot, 0)$ as an edge algebra. Any directed labelled graph gives rise to an edge algebra provided no two edges directed away from the same vertex have the same label. Specifically, for such a directed graph, $\cdot$ is an edge operation on $A$ provided the elements of $A$ fall into three disjoint sets—vertex elements, edge labels, and a default element 0—and $b = a \cdot d$ holds when $a$ labels an edge directed from vertex $d$ to vertex $b$, with all other $\cdot$ products producing the default element 0. Such edge algebras are related to Shallon’s graph algebras. They might also be called “automatic algebras” since they are clearly related to finite automata. However, the term “automatic group” is already in use. In later lectures more complicated ternary operations rooted in digraphs with doubly labelled edges will be used to encode Turing machines and their computations.

$Q_\omega$ denotes the subalgebra of $Q$ with universe $\{0\} \cup \{a_p : p \in \omega\} \cup \{b_p : p \in \omega\}$. Likewise, for each natural number $n$, $Q_n$ denotes the subalgebra with universe $\{0\} \cup \{a_p : 0 \leq p < n\} \cup \{b_p : 0 \leq p \leq n\}$. This algebra has $2n + 2$ elements.

**Theorem 0.** $Q_\omega$ is subdirectly irreducible, as is each $Q_n$.

**Proof:** Indeed $(0, b_0)$ belongs to every nontrivial congruence. To see this, let $\theta$ be a nontrivial congruence. First, suppose that $b_0 \theta c$ with $b_0 \neq c$. Then $b_0 = b_0 \land b_0 \theta b_0 \land c = 0$. Next, suppose
that $0 < p$ and that $\theta$ collapses $b_p$ to $c$ where $b_p \neq c$. We obtain $b_{p-1} = a_{p-1} \cdot b_p \theta a_{p-1} \cdot c = 0$. So, inductively we have $b_0 \theta 0$. Finally, suppose that $\theta$ collapses $a_p$ to $c$ where $a_p \neq c$. Then we obtain $b_p = a_p \cdot b_{p+1} \theta c \cdot b_{p+1} = 0$, and so also $b_0 \theta 0$.

**Theorem 1.** \(Q_Z\) generates a locally finite variety.

**Proof:** Note that no $a_p$ results from the product operation and that $b_p$ can only result from the product $a_p \cdot b_{p+1}$. So if $S$ is any subset of $Q_Z$, then $S \cup \{0\} \cup \{b_p : a_p \in S\}$ is a subuniverse of $Q_Z$. Thus the subuniverse of $Q_Z$ generated by a set of $n$ elements will have no more than $2n + 1$ elements, and usually a lot less. Hence $Q_Z$ generates a locally finite variety.

**Theorem 2.** \(Q_Z\) is inherently nonfinitely based.

**Proof:** For each large enough natural number $N$ we build an algebra $Q^{(N)}_Z$. The algebra $Q^{(6)}_Z$ is pictured below:

![Diagram](image)

The product $\cdot$ in $Q^{(N)}_Z$ is defined differently, while the meet and 0 retain their old meanings (and the remaining operations are still defined by the same terms). The universe $Q^{(N)}_Z = \{0\} \cup \ldots$
The algebra $Q_{Z}^{(N)}$ is infinite but finitely generated. Let $B$ be a subalgebra of $Q_{Z}^{(N)}$ generated by fewer than $N$ elements. It follows that some $a_p$ is not in $B$. Let $C$ be the subalgebra of $Q_{Z}^{(N)}$ whose universe consists of all elements except $a_p$. Below is a picture of $C$ where $N = 6$ and $a_{-4}$ is the omitted element.

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Thus $Q_{Z}^{(N)}$ is made from $C$ by a helical wrapping and the addition of a new element.

Now in $C$ the $b_p$’s are arranged in rows. Select a row. Let $\theta$ be the equivalence relation that isolates each $b_p$ on the selected row, as well as each $a_q$, but collapses all the other $b_p$’s to 0. Evidently, $\theta$ is a congruence relation of $C$. It is also clear that $C/\theta$ is isomorphic to a subalgebra of $Q_{Z}$. Since by selecting different rows we arrive at a family of congruences that separates the points of $C$, we conclude that $C$ belongs to the variety generated by $Q_{Z}$. Hence, every subalgebra of $Q_{Z}^{(N)}$ generated by fewer than $N$ elements belongs to the variety generated by $Q_{Z}$. Since this variety is locally finite, we have that it is inherently nonfinitely based. $\square$

**Theorem 3.** There is a finite algebra $A$ so that $Q_{Z}$ belongs to the variety generated by $A$.

**Proof:** Our approach is to let $A$ denote an unknown finite algebra, set up the obvious conditions based on Birkhoff’s HSP Theorem, and then try to solve for $A$.

So we want $B \subseteq A^{Z}$ and also $\theta \in \text{Con}B$ so that $Q_{Z} \cong B/\theta$. To make this as easy as possible we would like $B$ to be very much like $Q_{Z}$. We need to have in $A^{Z}$ elements like $a_p$ and $b_p$. So we want
\( \alpha_p, \beta_p \in A^\mathbb{Z} \) so that \( \beta_p = \alpha_p \beta_{p+1} \) for all \( p \in \mathbb{Z} \). Writing this coordinatewise, we obtain conditions on \( A \):

\[
\alpha_p(i) \beta_{p+1}(i) = \beta_p(i) \quad \text{for all } i \in \mathbb{Z}
\]

An easy way to satisfy this is to provide \( A \) with five elements \( 1, H, 2, C, \) and \( D \). Then we can let

\[
\alpha_p := \ldots \ 1 \ 1 \ 1 \ H \ 2 \ 2 \ 2 \ \ldots \\
\beta_p := \ldots \ C \ C \ C \ D \ D \ D \ \ldots
\]

where the change is taking place at the \( p \)th position. Then our condition looks like:

\[
\begin{array}{cccccccc}
\alpha_p: & \ldots & 1 & 1 & 1 & H & 2 & 2 & 2 & \ldots \\
\beta_p: & \ldots & C & C & C & D & D & D & D & \ldots \\
\beta_{p+1}: & \ldots & C & C & C & C & D & D & D & \ldots
\end{array}
\]

This imposes conditions on the product in \( A \). Indeed,

\[
1 \cdot C = C \\
H \cdot C = D \\
2 \cdot D = D
\]

To complete the description of \( A \) we make all the other products \( 0 \), a new element, and insist that \( \wedge \) make it into a flat semilattice. \( A \) has (at least) six elements. It is an edge algebra for the digraph below:

```
  2
 / \   \\
D -- H -- C
  \ /   \\
  1
```

To finish, we take \( B \) to be the subalgebra of \( A^\mathbb{Z} \) generated by all the \( \alpha \)'s and the \( \beta \)'s. Let \( \theta \) collapse all the \( \mathbb{Z} \)-tuples except the \( \alpha \)'s and the \( \beta \)'s. Then everything works. \( \square \)

It would be ideal if we could arrange for a finitely generated variety whose subdirectly irreducibles were exactly \( Q_\omega \) and the \( Q_n \)'s. The surprising thing is that this is almost possible. With this in mind, the next lecture develops properties of finite subdirectly irreducible algebras in varieties generated by algebras like \( A \) above. However, by Lecture 2, we will see that it is convenient to add more elements and even more operations to \( A \). Additional elements do not interfere much with the construction above, but care is needed with the additional operations—we must ensure that in \( Q_\mathbb{Z} \) they turn out to be term operations built up from \( \cdot, \wedge, \) and \( 0 \). Thus, in Lecture 3 we will revisit the constructions above.
Finite Subdirectly Irreducibles Generated by Finite Flat Algebras

In this lecture we will suppose that $A$ is a finite flat algebra (that is, an algebra among whose operations $\wedge$ and $0$ can be found which provide the algebra with the structure of a meet-semilattice of height one with least element $0$) and that $S$ is a finite subdirectly irreducible algebra in the variety generated by $A$. Of course, we have in mind for $A$ the flat algebra described at the end of Lecture 0. And we hope to show that $S$ is one of the $Q_n$'s.

Now $S$ will always arise as a quotient of some $B$, which is in turn a subalgebra of $A^T$ for some $T$. Since $S$ is subdirectly irreducible, we know that there is a strictly meet irreducible $\theta \in \text{Con }B$ such that $S \cong B/\theta$. It is more convenient to work with $B$ than with $S$. Since $S$ is finite, we can choose $T$ to be finite. Indeed, in this lecture we assume the following:

- $B \subseteq A^T$
- $\theta, \bar{\theta} \in \text{Con }B$
- $\bar{\theta}$ covers $\theta$ in $\text{Con }B$.
- $S \cong B/\theta$
- $T$ is as small as possible for representing $S$ in this way.

In particular this last condition entails that if $t \in T$, then there must be $u, v \in B$ so that $(u, v) \notin \theta$ but $u(s) = v(s)$ for all $s \in T - \{t\}$. Our effort at understanding the finite subdirectly irreducible $S$ is largely focussed on the covering of $\theta$ by $\bar{\theta}$.

First, we locate an element in $B$ which is like the element $b_0$ in $Q_n$. Since $B$ is a semilattice, there are elements $u, v \in B$ with $u < v$ and $(u, v) \in \bar{\theta} - \theta$. Using the finiteness of $B$ pick $p$ to be minimal among all those $v \in B$ such that $(u, v) \in \bar{\theta} - \theta$ for some $u < v$.

**FACT 0.** If $w < p$, then $(w, p) \notin \bar{\theta}$.

**PROOF:** Suppose $w < p$ but $w \theta p$. Pick $u < p$ with $(u, p) \in \bar{\theta} - \theta$. Then $w = p \wedge w \bar{\theta} u \wedge w$. So by the minimality of $p$, we have $u \wedge w \theta w$. But then $u = u \wedge p \theta u \wedge w \theta w p$, contradicting $(u, p) \notin \bar{\theta}$. □

Now for each $t \in T$ pick $(x, y) \in B^2 - \theta$ so that $x(t) \neq y(t)$ but $x(s) = y(s)$ for all $s \in T - \{t\}$. So $\bar{\theta} \subseteq \theta \cup \text{Cg}_B(x, y)$. Pick $u < p$ so that $(u, p) \in \bar{\theta} - \theta$. Then according to Mal'cev there is a unary polynomial $g$ of $B$ such that $p \in \{p \wedge g(x), p \wedge g(y)\}$ but $p \wedge g(x) \neq p \wedge g(y)$. Let $\chi_t$ be the element of $\{p \wedge g(x), p \wedge g(y)\}$ that is different from $p$. From this construction we obtain:

- $\chi_t(s) = p(s)$ for all $s \in T - \{t\}$.
- $\chi_t(t) < p(t)$ for all $t \in T$.
- $\chi_t(t) = 0$ and $0 < p(t)$ for all $t \in T$.

Thus, $\chi_t$ agrees with $p$ at all coordinates with the exception of $t$, where $\chi_t$ is $0$ while $p$ is not $0$. So $\chi_t$ is uniquely determined by $p$ and $t$ (and is independent of the choices of $x, y$, and $g$ made above).

We will eventually see—once enough is specified about $A$—that $p$ is also uniquely determined.
Fix $t_0 \in T$ so that $u \leq \chi_{t_0}$ for some $u < p$ for which $(u, p) \in \bar{\theta} - \theta$. Let $q = \chi_{t_0}$.

**Fact 1.** $p$ is a maximal element of $A^T$. $\chi_t \in B$ and $p$ covers $\chi_t$ in $A^T$ for all $t \in T$. Finally, if $u \in A^T$ and $u < p$, then $u \in B$.

**Proof:** Fact 0 gathers the conclusions we drew above. The elements of $A^T$ less than or equal to $p$ form a Boolean algebra in which every element is a meet of the coatoms $\chi_t$.

**Fact 2.** If $p \theta x$, then $p = x$

**Proof:** Suppose $p \theta x$. Meeting both sides with $p$ we also get $p \theta p \land x$. From Fact 0, we conclude that $p \neq p \land x$. Thus $p \leq x$. But since $p$ is a maximal element, we arrive at $p = x$.

**Fact 3.** $(q, p) \in \bar{\theta} - \theta$.

**Proof:** $(q, p) \notin \theta$ according to Fact 0. Let $u \leq q < p$ with $(u, p) \in \bar{\theta}$. Then we have $p \bar{\theta} u = q \land u \bar{\theta} q \land p = q$.

**Fact 4.** If $x, y < p$ and $(x, y) \in \bar{\theta}$, then $(x, y) \in \theta$.

**Proof:** The minimality of $p$ entails that $(x, x \land y), (x \land y, y) \in \theta$.

**Fact 5.** If $x < p$, then $(x, x \land q) \in \theta$.

**Proof:** $p \theta q$ by Fact 3, so $x = x \land p \theta x \land q$. The result follows by Fact 4.

**Fact 6.** $x \theta y$ if and only if $f(x) = p \Leftrightarrow f(y) = p$ for all unary polynomials $f$ of $B$.

**Proof:** In the forward direction the result follows from Fact 2. In the converse direction, suppose $(x, y) \notin \theta$ and pick the unary polynomial $g$ as we did in defining $\chi_t$ and let $f = p \land g$.

**Fact 7.** Let $f$ be a monotone unary polynomial of $B$. If $x < p$ and $f(x) = p$, then $f(q) = p$.

**Proof:** By Fact 5 we have $x \theta x \land q$. This entails $p = f(x) \theta f(x \land q)$. So by Fact 2 we get $p = f(x \land q)$. But then $p \leq f(q)$ by the monotonicity of $f$. Thus $p = f(q)$ by the maximality of $p$.

**Fact 8.** $S \in HSA$ or $(x \land y) \lor (x \land z)$ is not a polynomial of $B$.

**Proof:** Suppose $S \notin HSA$. Then $T$ has at least two elements. Let $t_1 \in T$ with $t_0 \neq t_1$. Let $q' = \chi_{t_1}$. Since $q' < p$ we have by Fact 5 that $q' \theta q' \land q$. But then, were $(x \land y) \lor (x \land z)$ a polynomial of $B$, we would have $p = (p \land q) \lor (p \land q') \theta (p \land q) \lor (p \land q \land q') = q$. Since $(p, q) \notin \theta$, we conclude that $(x \land y) \lor (x \land z)$ is not a polynomial.

Fact 8 reveals that our investigation of (finite) subdirectly irreducible algebras can be split in two. Since $A$ is finite, a complete description of the subdirectly irreducible algebras is $HSA$ can be devised given a description of $A$. We only note the obvious upper bound on their cardinality. Most of our effort will concern the alternative case when $(x \land y) \lor (x \land z)$ is not a polynomial of $B$. We will call such subdirectly irreducible algebras *joinless*. It is these algebras we want to show must be isomorphic to our $Q_n$’s.
The Eight Element Algebra $A$

The six element algebra which was constructed at the end of Lecture 0 must be augmented in order to get a better grip on the joinless subdirectly irreducible algebras in the generated variety. We will force $S$ to be isomorphic to some $Q_n$ by arranging for $\theta$ to isolate all the factors of $p$ with respect to product $\cdot$, while lumping everything else into the congruence class of 0. Key to this plan is a unique factorization property for the product $\cdot$, a property evidently holding in $Q_Z$ and its subalgebras. We are led to an algebra with eight elements and eight basic operations.

The universe is $A = \{0\} \cup \{1, H, 2\} \cup \{C, \bar{C}, D, \bar{D}\}$. We set $U = \{1, H, 2\}$ and $W = \{C, \bar{C}, D, \bar{D}\}$. We regard $\sim$ as an involution on $W$. The basic operations of $A$ are denoted by $0, \wedge, \cdot, J, J', T, S_1,$ and $S_2$. $(A, \wedge, 0)$ is a flat semilattice with least element 0. The operation $\cdot$ is defined to give the default value 0 except when

$$
1 \cdot C = C \quad 1 \cdot \bar{C} = \bar{C} \\
H \cdot C = D \quad H \cdot \bar{C} = \bar{D} \\
2 \cdot D = D \quad 2 \cdot \bar{D} = \bar{D}
$$

This is an edge operation. Ordinarily, we represent the product $\cdot$ simply by juxtaposition.

The operations $J$ and $J'$ are ternary; they are defined below.

$$
J(x, y, z) = \begin{cases} 
x & \text{if } x = y \neq 0, \\
x \wedge z & \text{if } x = \bar{y} \in W, \\
0 & \text{otherwise.} 
\end{cases} \\
J'(x, y, z) = \begin{cases} 
x \wedge z & \text{if } x = y \neq 0, \\
x & \text{if } x = \bar{y} \in W, \\
0 & \text{otherwise.} 
\end{cases}
$$

The operations $S_1$ and $S_2$ are 5-ary; they are defined as follows.

$$
S_1(u, v, x, y, z) = \begin{cases} 
(x \wedge y) \lor (x \wedge z) & \text{if } u \in \{1, 2\}, \\
0 & \text{otherwise.} 
\end{cases} \\
S_2(u, v, x, y, z) = \begin{cases} 
(x \wedge y) \lor (x \wedge z) & \text{if } u = \bar{v} \in W, \\
0 & \text{otherwise.} 
\end{cases}
$$

Finally, $T$ is the 4-ary operation defined below.

$$
T(x, y, z, w) = \begin{cases} 
xy & \text{if } xy = zw \neq 0 \text{ and } x = z \text{ and } y = w, \\
\bar{xy} & \text{if } xy = zw \neq 0 \text{ and either } x \neq z \text{ or } y \neq w, \\
0 & \text{otherwise.} 
\end{cases}
$$
As we shall see, the first four of these new operations ensure that the algebra $B$ cannot have certain kinds of elements. $T$ is crucial for the unique factorization property we desire.

Now we continue to develop facts about $B$ and its congruences $\theta$ and $\bar{\theta}$. Denote by $B_1$ the set consisting of $p$ and all its factors with respect to the product $\cdot$. That is $u \in B_1$ if and only if $u = p$ or $u = c_i$ for some factorization $p = c_0c_1\ldots c_m$ (where this latter product is associated to the right). Let $B_0$ denote that set of those tuples in $B$ which contain at least one $0$. Plainly $B_0 \subseteq B - B_1$. It is also clear that in the joinless case the ranges of the operations $S_1$ and $S_2$ are contained in $B_0$ and hence in $B - B_1$.

**PROVISO:** The facts below are established under the assumption that the ranges of $S_1$ and $S_2$ are contained in $B - B_1$.

**FACT 9.** If $v \in B$ so that for all $s \in T$ either $p(s) = v(s)$ or $p(s) = \overline{v(s)} \in W$, then $p = v$.

**PROOF:** Let $Y = \{s : p(s) = \overline{v(s)}\}$.

**CLAIM:** $Y$ is empty.

Proof of the Claim: Since the range of the operation $S_2$ is disjoint from $B_1$, it follows that $T \neq Y$. Pick $t' \in T - Y$ and let $q' = \chi_{t'}$. So for each $s \in T$ we have

$$J(p(s), v(s), q'(s)) = \begin{cases} p(s) & \text{if } s \notin Y, \\ p(s) \land q'(s) & \text{if } s \in Y. \end{cases}$$

But this entails $J(p, v, q') = p$, since $q'(s) = p(s)$ for all $s \in Y$ because $t \notin Y$. Therefore, by Fact 7 and the monotonicity of $J$, we have $J(p, v, q) = p$. But then the definition of $J$ gives us $q(s) = p(s)$ for all $s \in Y$. Since $q(t_0) = 0$, it follows that $t_0 \notin Y$. Now observe that $J'(p(t_0), v(t_0), q(t_0)) = p(t_0) \land 0 = 0$. Hence, $J'(p, v, q) \neq p$. So by Fact 7 and the monotonicity of $J'$, we conclude that $J'(p, v, \chi_t) \neq p$ for all $t \in T$. But for all $s, t \in T$

$$J'(p(s), v(s), \chi_t(s)) = \begin{cases} p(s) \land \chi_t(s) & \text{if } s \notin Y, \\ p(s) & \text{if } s \in Y. \end{cases}$$

It follows that $t \notin Y$ for all $t \in Y$. This means $Y$ is empty. So the Claim is established.

Since $Y$ is empty, we also know that $p(s) = v(s)$ for all $s \in T$. Hence $u = v$ as desired. 

**FACT 10.** If $u \in B_1$ and $v \in B$ so that for all $s \in T$ either $u(s) = v(s)$ or $u(s) = \overline{v(s)} \in W$, then $u = v$.

**PROOF:** Fact 9 settles the matter when $u = p$. There are two kinds of elements in $B_1 - \{p\}$—those in $U^T$ and those in $W^T$. Clearly, we can restrict our attention to the case when $u \in W^T$. So pick $d_0, \ldots, d_{m-1} \in B_1$ with $p = d_0 \ldots d_{m-1}u$. Let $p' = d_0 \ldots d_{m-1}v$. Thus for all $s \in T$ either $p(s) = p'(s)$ or $p(s) = \overline{p'(s)}$. It follows from Fact 9 that $p = p'$. Now a straightforward induction on $m$ entails that $u = v$.

We can now establish the unique factorization property for the product that we require.

**FACT 11.** If $ab = cd \in B_1$, then $a = c$ and $b = d$. 

2. THE EIGHT ELEMENT ALGEBRA $\mathbf{A}$

**Proof:** Let $u = ab$ and $v = T(a, b, c, d)$. From the definition of the operation $T$, we see that $u$ and $v$ satisfy the hypotheses of Fact 10. Hence, $ab = T(a, b, c, d)$. But then the definition of $T$ gives $a = c$ and $b = d$. \qed

**Fact 12.** No factorization of $p$ has repeated factors.

**Proof:** It is clear that if $d_0d_1 \ldots d_{m-1}e = p$ then $e \in W^T$ while $d_0, \ldots, d_{m-1} \in U^T$. Suppose that $d_i = d_j$ with $i < j$. Since the range of the operation $S_1$ is disjoint from $B_1$, we conclude that $B$ contains no elements from $\{1, 2\}^T$. So pick $s \in T$ so that $d_i(s) = d_j(s) = H$. Now we see

$$p(s) = d_0(s) \ldots d_{i-1}(s)Hd_i(s) \ldots d_{j-1}(s)Hd_j(s) \ldots d_{m-1}(s)e(s)$$

So $p(s) = 0$, violating the maximality of $p$. \qed

**Fact 13.** If $u \in B$ and $f(u) \in B_1$ for some nonconstant unary polynomial $f$, then $u \in B_1$.

**Proof:** The proof is by induction on the complexity of $f$. The initial step of the induction is obvious, since the identity function is the only simplest nonconstant unary polynomial. The inductive step breaks down into seven cases, one for each basic operation of positive rank.

**Case $\wedge$:** $f(x) = g(x) \wedge h(x)$.

We have $f(u) \leq g(u), h(u)$. But every element of $B_1$ is maximal with respect to the semilattice order. So $f(u) = g(u) = h(u) \in B_1$. Now at least one of $g$ and $h$ must be nonconstant. Invoking the induction hypothesis, we get $u \in B_1$.

**Case $\lor$:** $f(x) = g(x)h(x)$.

We have $g(u)h(u) = f(u) \in B_1$. So $g(u), h(u) \in B_1$. Since at least one of $g$ and $h$ must be nonconstant, we can invoke the induction hypothesis to conclude that $u \in B_1$.

**Case $J$:** $f(x) = J(g(x), h(x), k(x))$.

We have $f(u) = J(g(u), h(u), k(u)) \leq g(u)$. By the maximality of $f(u)$ we get

$$f(u) = J(g(u), h(u), k(u)) = g(u) \in B_1.$$ 

Moreover, the definition of $J$ implies that $g(u)$ and $h(u)$ fulfill the hypothesis of Fact 10. So $g(u) = h(u)$. Now $g$ and $h$ cannot both be constant, since then they would have to be the same constant (namely $f(u)$), forcing $f$ to be constant by the definition of $J$. Hence we can invoke the inductive hypothesis (on $g$ or $h$) to conclude that $u \in B_1$.

**Case $J'$: $f(x) = J'(g(x), h(x), k(x))$.**

This case is easier than the last one and is omitted.

**Cases $S_1$ and $S_2$:** Too easy.

**Case $T$:** $f(x) = T(g(x), h(x), k(x), \ell(x))$.

We have $f(u) = T(g(u), h(u), k(u), \ell(u)) \in B_1$. Evidently, $f(u)$ and $g(u)h(u)$ satisfy the hypotheses of Fact 10. So $f(u) = g(u)h(u) = k(u)\ell(u)$ follows from the definition of $T$. Since $f(u) \in B_1$, we know that $g(u), h(u), k(u), \ell(u) \in B_1$ by the definition of $B_1$. Now at least one of $g, h, k, \ell$ is nonconstant. So $u \in B_1$ by the inductive hypothesis. \qed

**Fact 14.** If $u \in B_1$ and $u \not= v$, then $u = v$. 
2. THE EIGHT ELEMENT ALGEBRA

Proof: Let \( f(u) = p \) for some unary polynomial \( f \) built just using \( . \) It follows that \( f(v) = p \) by Fact 6. By the obvious inductive extension of Fact 11, our unique factorization property, we conclude that \( u = v \).

\textbf{Fact 15.} \( B - B_1 = 0/\theta \).

\textbf{Proof:} Fact 13 says that \( B - B_1 \) is closed with respect to nonconstant unary polynomials. Since \( p \in B_1 \), we have that \( f(u) \neq p \) for all \( u \in B - B_1 \) and all nonconstant unary polynomials \( f \). Hence, by Fact 6, \( B - B_1 \) is collapsed by \( \theta \). But \( B_1 \) is the union of (singleton) \( \theta \)-classes according to Fact 14. Hence \( B - B_1 \) is a \( \theta \)-class. Clearly, \( 0 \in B - B_1 \).

The following facts reveal how the remaining operations work in the subdirectly irreducible algebra \( S \).

\textbf{Fact 16.} If \( x, y \in B_1 \) and \( x \neq y \), then \( x \land y \in B - B_1 \).

\textbf{Proof:} Since \( x \neq y \) there is \( t \in T \) with \( x(t) \neq y(t) \). But then \( (x \land y)(t) = 0 \). So \( x \land y \in B - B_1 \).

\textbf{Fact 17.} \( T(x, y, z, w) \theta (xy) \land (zw) \) for all \( x, y, z, w \in B \).

\textbf{Proof:} Since \( B - B_1 \) is a \( \theta \)-class, Fact 10 forces \( T(x, y, z, w) \in B - B_1 \) unless \( xy = zw \in B_1 \). In the latter case, \( T(x, y, z, w) = xy = zw = (xy) \land (zw) \in B_1 \). But also, \( (xy) \land (zw) \in B - B_1 \) unless \( xy = zw \in B_1 \). In the latter case, \( (xy) \land (zw) \in B_1 \). Therefore, \( T(x, y, z, w) \theta (xy) \land (zw) \).

\textbf{Fact 18.} \( J(x, y, z) \theta x \land y \) for all \( x, y, z \in B \).

\textbf{Proof:} Again using that \( B - B_1 \) is a \( \theta \)-class and Fact 10, \( J(x, y, z) \in B - B_1 \), unless \( x = y \in B_1 \). In the latter case, \( J(x, y, z) = x = y = x \land y \in B_1 \). But also, \( x \land y \in B - B_1 \), unless \( x = y \in B_1 \). In the latter case, \( x \land y = x \in B_1 \). Therefore, \( J(x, y, z) \theta x \land y \).

\textbf{Fact 19.} \( J'(x, y, z) \theta x \land y \land z \) for all \( x, y, z \in B \).

\textbf{Proof:} This is too easy.

Thus we arrive at the conclusion that either \( S \in HSA \) or else that \( S_1 \) and \( S_2 \) are constantly 0 in \( S \), and that it is isomorphic to some \( Q_n \). When \( S \notin HSA \), the following equations hold in \( S \):
LECTURE 3

A is Inherently Nonfinitely Based and Has Residual Character $\omega_1$

The algebra $\mathbb{Q}_Z$ and its subalgebras $\mathbb{Q}_\omega$, and $\mathbb{Q}_n$ for each $n \in \omega$, were introduced in Lecture 0. The operations $0$, $\land$, and $\cdot$ were examined in detail, but the only stipulation about any remaining operations was that they must be defined as term operations of these first three. In Lecture 2, five more operation symbols were introduced: $T, J, J', S_1,$ and $S_2$. In the algebras $\mathbb{Q}_Z, \mathbb{Q}_\omega,$ and $\mathbb{Q}_n$ these five further basic operations are defined so that the following equations are true:

$$T(x, y, z, w) \approx (xy) \land (zw)$$
$$J(x, y, z) \approx x \land y \quad S_1(u, v, x, y, z) \approx 0$$
$$J'(x, y, z) \approx x \land y \land z \quad S_2(u, v, x, y, z) \approx 0$$

The whole discussion of these algebras in Lecture 0 goes through in this expanded setting, with the exception of the last phase. The five new operations were not defined on the six element algebra in Lecture 0. We now want to replace that algebra with the eight element algebra $A$ introduced in Lecture 2. What we need is the following theorem to replace Theorem 3 of Lecture 0.

**Theorem 4.** $\mathbb{Q}_Z$ belongs to the variety generated by $A$.

**Proof:** We retrace the proof of Theorem 3. First, for each $p \in \mathbb{Z}$ we designate elements $\alpha_p$ and $\beta_p$ of $A^Z$ as before:

$$\begin{align*}
\alpha_p := & \ldots \ 1 \ 1 \ 1 \ H \ 2 \ 2 \ 2 \ \ldots \\
\beta_p := & \ldots \ C \ C \ C \ D \ D \ D \ D \ \ldots
\end{align*}$$

where the change is taking place at the $p^{th}$ position. Next we let $B_1 = \{\alpha_p : p \in \mathbb{Z}\} \cup \{\beta_p : p \in \mathbb{Z}\}$ and $B_0$ be the set of all $\mathbb{Z}$-tuples of elements of $A$ in which 0 occurs. Then set $B = B_0 \cup B_1$.

We need to check that $B$ is a subuniverse of $A^Z$. Plainly, $0 \in B$ and checking that $B$ is closed under $\land$ and $\cdot$ present no trouble. For $J$ and $J'$, observe that to produce an output not in $B_0$, the inputs $x$ and $y$ must satisfy either $x(s) = y(s) \neq 0$ or $x(s) = y(s) \in W$, for all $s \in \mathbb{Z}$. In $B$, this can only happen if $x = y \in B_1$. So $B$ is closed under both $J$ and $J'$. Similar reasoning shows that $S_1$ and $S_2$ always produce outputs in $B_0$, and thus $B$ is closed under both $S_1$ and $S_2$. Finally, if $T(x, y, z, w) \notin B_0$, then $x(s)y(s) = z(s)w(s) \neq 0$ for all $s \in \mathbb{Z}$. In $B$ this can only happen if $x = z = \alpha_p$ and $y = w = \beta_{p+1}$ for some $p \in \mathbb{Z}$. Thus $B$ is closed under $T$.

Now let $\Phi$ be the map defined from $B$ to $\mathbb{Q}_Z$ via

$$\Phi(x) = \begin{cases} 
\alpha_p & \text{if } x = \alpha_p \text{ for some } p \in \mathbb{Z}, \\
\beta_p & \text{if } x = \beta_p \text{ for some } p \in \mathbb{Z}, \\
0 & \text{otherwise}
\end{cases}$$
We contend that $\Phi$ is a homomorphism from $B$ onto $Q_Z$. That $\Phi$ preserves $0, \land$, and $\cdot$ is easy. $\Phi$ preserves $S_1$ and $S_2$ since it takes any output of either of these operations to 0. Now suppose $J(x, y, z) \notin B_0$. Then $J(x, y, z) = x = y \in B_1$. So $\Phi(J(x, y, z)) = \Phi(x) = \Phi(y) \neq 0$. So in $Q_Z$ we have $J(\Phi(x), \Phi(y), \Phi(z)) = \Phi(x) = \Phi(J(x, y, z))$. On the other hand, suppose $J(x, y, z) \in B_0$. This can happen only if $y \neq x \in B_1$ or $x \in B_0$. Thus $\Phi(J(x, y, z)) = 0$ and either $\Phi(y) \neq \Phi(x)$ or $\Phi(x) = 0$. But in $Q_Z$ this means $J(\Phi(x), \Phi(y), \Phi(z)) = \Phi(x) \land \Phi(y) = 0$. Hence $\Phi$ preserves $J$. By a similar argument $\Phi$ also preserves $J'$. Finally consider $T$. Suppose $T(x, y, z, w) \notin B_0$. As pointed out above, for some $p \in Z$ we must have $x = z = \alpha_p, y = w = \beta_{p+1}$, and so $T(x, y, z, w) = (xy) \land (zw)$. On the other hand, suppose $T(x, y, z, w) \in B_0$. This entails that either at least one of $xy$ and $zw$ belongs to $B_0$ or that $xy \neq zw$ while $xy, zw \in B_1$. Thus, in $Q_Z$ we get $\Phi(T(x, y, z, w)) = 0$ and either $\Phi(xy) = 0$ or $\Phi(zw) = 0$ or $\Phi(xy) \neq \Phi(zw)$. This means $\Phi(xy) \land \Phi(zw) = 0$ in $Q_Z$. Therefore, $\Phi$ preserves $T$. We conclude that $\Phi$ is a homomorphism and that $Q_Z$ is in the variety generated by $A$.

At this point we know that the eight element algebra $A$, which has eight basic operations, is inherently nonfinitely based, that the finite subdirectly irreducible algebras in the variety generated by $A$ are the si homsubs of $A$ and the algebras $Q_n$ for each $n \in \omega$, and that $Q_\omega$ is a countably infinite subdirectly irreducible member of the variety. To demonstrate that our variety has no other infinite subdirectly irreducible algebras, we use the following theorem, due independently to Quackenbush and Dziobiak.

**Theorem 5.** Let $\mathcal{V}$ be a locally finite variety, let $S \in \mathcal{V}$ be subdirectly irreducible, and let $B$ be a finite subalgebra of $S$. Then $B$ is embeddable into a finite subdirectly irreducible algebra in $\mathcal{V}$.

**Proof:** Let $(a, b)$ be a critical pair for $S$. Since $\mathcal{V}$ is locally finite, at the cost of enlarging $B$ if necessary, we can suppose that $a, b \in B$. Now for each two element subset $\{c, d\} \subseteq B$ we choose a finite $F_{c, d} \subseteq S$ which consists of all the new constants used along some Mal'cev chain witnessing that $(a, b) \in C(B)(c, d)$. Let $C$ be the subalgebra of $S$ generated by $B \cup \bigcup_{c \neq d} F_{c, d}$. This subalgebra is finite since the variety is locally finite. Now let $\theta$ be a congruence of $C$ maximal with respect to not collapsing $a$ and $b$. Hence $\theta$ is a meet irreducible congruence, so $C/\theta$ is subdirectly irreducible. But, by construction, $\theta$ cannot collapse any two distinct elements of $B$. Therefore the quotient map embeds $B$ into the finite subdirectly irreducible algebra $C/\theta$.

Now let $\mathcal{V}$ be the variety generated by our eight element algebra $A$. Let $S$ be any infinite subdirectly irreducible algebra in $\mathcal{V}$. Any finite subalgebra of $S$ can be embedded into arbitrarily large finite subdirectly irreducible algebras in $\mathcal{V}$, i.e. into $Q_n$ for all large enough $n$. This means that every finitely generated (= finite) subalgebra of $S$ is embeddable into $Q_\omega$. Consequently, every universal sentence true in $Q_\omega$ must be true in $S$.

Here are some interesting properties of $Q_\omega$ which can be expressed with universal sentences:

- Any equation true in $Q_Z$. For example: $T(x, y, z, w) \approx (xy) \land (zw)$.
- The height is no bigger than 1: $x \neq y \rightarrow x \land y \approx 0$.
- $xy \approx zw \neq 0 \rightarrow (x \approx z \land y \approx w)$.
- $xy \neq 0 \neq xz \rightarrow y \approx z$.
- $xy \neq 0 \neq zy \rightarrow x \approx z$.
- $xy \neq 0 \rightarrow zx \approx 0 \approx yw$. 

Consequently, in $S$, the operations $T, J, J', S_1$, and $S_2$ are term functions (using the same terms as in $Q_\omega$) in $0, \wedge$, and $\cdot$. We ignore them from now on. With respect to $\wedge$ and $0$, $S$ is a height 1 meet-semilattice with least element $0$. So the balance of our analysis depends primarily on the product $\cdot$. Since $(xy)z \approx 0$ is true in $Q_\omega$, we see that in $S$, just as in $Q_\omega$, only right-associated products can differ from 0. The last four properties itemized above put further and severe restrictions on the product in $S$.

We make $S - \{0\}$ into a labelled directed graph as follows. We take as the vertex set those elements which are right factors, outputs or do not occur in nonzero products. We take as the set of labels those elements which are left factors in nonzero products. Our itemized properties entail that the set of vertices and the set of labels are disjoint. We put an edge from $b$ to $c$ and label it with $a$ provided $ab = c$ in $S$. Our itemized assertions ensure that a vertex can have outdegree at most 1, indegree at most 1, and that every edge has a uniquely determined label which occurs as a label of exactly one edge in the whole graph.

Let $C$ be a connected component of our graph. Let $\theta_C$ be the equivalence relation that collapses all the vertices and labels in $C$ to $0$, but which isolates every other point. $\theta_C$ is a congruence of $S$. Since $S$ is subdirectly irreducible, it follows that our graph has only one component. This already implies that $S$ is countably infinite. But more is true. There are only three possible countable connected graphs of this kind: the one associated with $\mathbb{Z}$ (and then we would have $S \cong Q_\mathbb{Z}$), the one associated with $\omega$ (and then we would have $S \cong Q_\omega$), and the one associated with the set of nonnegative integers (and then $S$ would be isomorphic to an algebra we might as well call $Q_{\omega}$).

But neither $Q_{\mathbb{Z}}$ nor $Q_{\omega}$ is subdirectly irreducible. So $S$ must be isomorphic to $Q_\omega$.

We summarize the results in the following theorem.

**Theorem 6.** The eight element algebra $A$, which has only eight basic operations, is inherently nonfinitely based. The subdirectly irreducible algebras in the variety generated by $A$ are, up to isomorphism, exactly the subdirectly irreducible homomorphic images of subalgebras of $A$, the algebra $Q_\omega$, and the algebra $Q_n$ for each $n \in \omega$.

This theorem settles in the negative some outstanding problems. We will say that a variety is finitely generated provided it is generated by a finite algebra with only finitely many fundamental operations. It is residually small if there is an upper bound on the cardinalities of its subdirectly irreducible algebras. It is residually finite if all its subdirectly irreducible algebras are finite. It is residually very finite if there is a finite upper bound on the cardinalities of its subdirectly irreducible algebras.

**The R-S Conjecture:** Every finitely generated residually small variety is residually very finite.

**The Broader Finite Basis Speculation:** Every finitely generated residually small variety is finitely based.

Theorem 6 is a counterexample to both of these. However, the two problems below are closely related and still open.

**The Quackenbush Conjecture:** Every finitely generated residually finite variety is residually very finite.

**The Narrower Finite Basis Speculation:** Every finitely generated residually finite variety is finitely based.
LECTURE 4

How A(T) Encodes The Computations of T

In this lecture we describe, in part, McKenzie’s machine algebras and show how they capture the computations of Turing machines. Turing machines are finite objects, but the computations that they produce can be endless. So it is reasonable to expect to use a finite algebra to convey the information of any particular Turing machine. However, finite algebras are too small to hold arbitrary computations. The algebra \( Q \), however, suggests a way to grapple with arbitrary computations. The idea is to designate certain elements of the algebra as configurations of a Turing machine and draw labeled directed edges between configurations to represent the transitions of the machine computation. Then we try to realize these directed edges by new operations applied to certain elements. Next we try to find a finite algebra so that the whole thing is happening coordinatewise inside a big direct power. Finally, we will have to add further operations to control all the finite subdirectly irreducible algebras.

For a Turing machine \( T \), we devise a finite algebra \( A(T) \) which enlarges \( A \) (in order to have enough distinct elements to code configurations) by adding finitely many elements and which expands \( A \) by adjoining operations to emulate the transitions between configurations, as well as to keep control of the finite subdirectly irreducible algebras. But the analysis of computation itself will go on in \( A(T)^X \) for some large set \( X \) [think of \( X = \mathbb{Z} \)].

We conceive of a Turing machine \( T \) as having finitely many internal states \( 0, 1, \ldots, m - 1 \). The machine is always launched in state 1 and we take 0 to be the unique halting state. The Turing machine \( T \) has a tape alphabet consisting of the symbols 0 and 1. The Turing machine itself is a finite collection of 5-tuples each of the form:

\[
[i, \gamma, \delta, M, j]
\]

This 5-tuple is the instruction, “If you are in state \( i \) and you are examining a tape square containing the symbol \( \gamma \), then write the symbol \( \delta \) on that square, move one square in the direction \( M \) (\( M \) must be either \( L \) for left or \( R \) for right), and pass into internal state \( j \)”. We insist that no 5-tuple begin with 0 and that otherwise the machine must have exactly one instruction which begins \([i, \gamma, \ldots]\) for each state \( i \) other than the halting state 0 and each tape symbol \( \gamma \).

We say \( Q \) is a configuration for a Turing machine \( T \) provided \( Q = (t, n, i) \) where \( t \in \{0, 1\}^\mathbb{Z} \), \( n \in \mathbb{Z} \), and \( i \) is one of the states of \( T \). The idea is that at some stage of a computation, the tape of the machine looks like \( t \), the machine is focussed on square \( n \) and is itself in state \( i \).

A significant problem we have to resolve comes from the fact that machine computations, at any given stage, happen at a particular location on the tape, and that these locations are arranged in a sequence with only the adjacent locations available for the next step in the computation. Thus
some elements of our “computation algebra” which are used to label those directed edges must also fall into a sequence of “tape locations”. To make short work of this point we take the elements \(a_p\) of \(Q_z\) as a model of how elements fall into sequence. Looking at what we had to have in \(A\) to get these \(a_p\)’s we recall:

\[
\begin{align*}
\alpha_p : & \quad \ldots \quad 1 \quad 1 \quad 1 \quad H \quad 2 \quad 2 \quad 2 \quad \ldots \\
\alpha_{p+1} : & \quad \ldots \quad 1 \quad 1 \quad 1 \quad 1 \quad H \quad 2 \quad 2 \quad \ldots \\
\alpha_{p+2} : & \quad \ldots \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad H \quad 2 \quad \ldots
\end{align*}
\]

So in all our machine algebras we want a subset \(U = \{1, H, 2\}\) making elements like the ones above available in direct powers. To impose the precedence above in the direct power, we impose \(2 < 2 < H < 1 < 1\) on \(U\). We also use \(<\) to denote the coordinatewise relation in any direct power of a machine algebra. Suppose \(B = A(T)^X\). A subset \(F \subseteq B\) is **sequentiable** provided

- \(F \subseteq U^X\),
- \(H\) occurs at least once in \(f\), for each \(f \in F\), and
- \(<\) gives \(F\) a structure isomorphic to some convex substructure of the ordered set of integers.

Since \(H\) may occur at several places in such an \(f\), sequentiable sets can be more complex than \(\{\alpha_p : p \in Z\}\). For a fixed sequentiable set \(F\) the index set \(X\) falls into natural pieces that help us see the structure. Look at the following display of the four element sequentiable set \(F = \{f_0, f_1, f_2, f_3\}\).

\[
\begin{align*}
f_0 : & \quad 1 \quad 1 \quad H \quad 2 \quad 2 \quad 2 \quad H \quad 2 \quad 2 \quad 2 \quad 1 \quad H \quad 1 \\
f_1 : & \quad 1 \quad 1 \quad 1 \quad H \quad 2 \quad 2 \quad 1 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \\
f_2 : & \quad 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 1 \quad H \quad 2 \quad H \quad 1 \quad 1 \quad 1 \\
f_3 : & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad H \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 1
\end{align*}
\]

Examining the 13 columns, we see that several are exactly the same. In this example the set \(X\) has 13 elements and some unspecified arrangement of these thirteen elements underlies the display above. But the particular arrangement of \(X\) is immaterial from the point of view of the algebra \(A(T)^X\). Thus we are free to rearrange \(X\) to make the precedence on \(F\) more transparent. Below is the result of such a rearrangement:

\[
\begin{align*}
f_0 : & \quad 1 \quad 1 \quad 1 \quad 1 \quad H \quad H \quad H \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \\
f_1 : & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad H \quad 2 \quad 2 \quad 2 \quad 2 \\
f_2 : & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad H \quad H \quad 2 \quad 2 \\
f_3 : & \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad H \quad 2 \quad 2
\end{align*}
\]

We have put all the columns consisting entirely of 1’s to the left. Next we put all the columns beginning with \(H\) in position 0, then all columns with \(H\) in position 1, and so on. At the right we have placed all columns consisting entirely of 2’s. Doing this, we see that there are only \(6 = 4 + 2\) different kinds of columns possible:

\[
\begin{align*}
1 \quad H \quad 2 \quad 2 \quad 2 \quad 2 \\
1 \quad 1 \quad H \quad 2 \quad 2 \quad 2 \\
1 \quad 1 \quad 1 \quad H \quad 2 \quad 2 \\
1 \quad 1 \quad 1 \quad 1 \quad H \quad 2
\end{align*}
\]

This means our sequentiable set \(F\) partitions the index set \(X\) into 6 blocks. The blocks can be labeled \(X_L\) for the set of all indices of columns that are constantly 1, \(X_R\) for the set of all indices of columns that are constantly 2, and \(X_n\) for the set of all indices where the necessarily unique \(H\) occurs at the \(n^{th}\) position.
4. HOW $A(T)$ ENCODES THE COMPUTATIONS OF $T$

To simplify the presentation a bit and make the pictures understandable, once a sequentiable set $F$ has been specified, we will assume that $X$ is arranged in such a line so that the set $X_L$ is an initial (or left) segment, $X_R$ is a final segment (or right) segment, and the each $X_n$ is placed at the obvious position on the line. Since at its biggest, $F$ can be indexed only by $\mathbb{Z}$, we can accommodate such a line like picture if we are willing to place $X_L$ at $-\infty$ and $X_R$ at $+\infty$.

Now let $F$ be the four element sequentiable set above but with the columns collapsed to 6 and arranged as in the last display, and let $Q = \langle t, 2, i \rangle$ be a configuration. We code $Q$ by

$$
\beta: C_{i,t(2)}^0, C_{i,t(2)}^{t(0)}, C_{i,t(2)}^{t(1)}, M_{i,t(2)}^{t(2)}, D_{i,t(2)}^{t(3)}, D_{i,t(2)}^0
$$

block: $X_L, X_0, X_1, X_2, X_3, X_R$.

This gives a real forest of superscripts and subscripts and the truth is that we will need a few more to get to full generality. However, we can decode it a bit. The $C$’s mean “left of the reading head”. The $D$’s mean “to the right of the reading head”. $M$ locates where the machine reading head is. The index $i$ specifies the state of the machine. The subscript $t(2)$ tells what symbol is written on the tape square scanned by the reading head. Finally, the indices $t(j)$ tell us what is printed on the corresponding square of the tape, unless it is too far off to the left (in $X_L$) or too far off to the right (in $X_R$), in which case we have used 0 as a default value (other choices would be okay). So reading across the superscripts is like reading across the tape. In this way, each component of $\beta$ carries a lot of information about the configuration.

Now $X$ in this example had 13 elements rather than 6, so the $\beta$ above is too short. However, by duplicating the entries in $\beta$ the correct number of times (e.g. the first entry $C_{i,t(2)}^0$ should occur 4 times while the last entry $D_{i,t(2)}^0$ should occur twice) we would get a $\beta$ of the correct length. That $|X| = 13$ is immaterial. But our particular sequentiable set had only four elements, it was indexed with the convex set $\{0, 1, 2, 3\}$, and we took $n = 2$ in our configuration. To get the general case, let $I$ be any convex subset of $\mathbb{Z}$ and suppose that $F$ is a sequentiable set indexed by $I$. Let $n \in I$ and let $Q = \langle t, n, i \rangle$ be a configuration. Then we use the $\beta$ below as a code for $Q$ and we say that $\beta$ codes $Q$ over $F$.

$$
\beta(x) =
\begin{cases}
    C_{i,t(n)}^0 & \text{if } x \in X_L, \\
    C_{i,t(n)}^{t(j)} & \text{if } x \in X_j \text{ and } j < n \text{ and } j \in I, \\
    M_{i,t(n)}^t & \text{if } x \in X_j \text{ and } j = n \in I, \\
    D_{i,t(n)}^{t(j)} & \text{if } x \in X_j \text{ and } n < j \in I, \\
    D_{i,t(n)}^0 & \text{if } x \in X_R.
\end{cases}
$$

CAPTURING THE TRANSITIONS BETWEEN CONFIGURATIONS

To get a grip on how to handle the transition between configurations let $B = A(T)^\mathbb{Z}$ and let $F = \{\alpha_p : p \in \mathbb{Z}\}$. Then $F$ is a sequentiable set indexed by $\mathbb{Z}$, and the partition imposed on $\mathbb{Z}$ by $F$ consists of singleton sets $\{p\}$. Let $Q = \langle t, n, i \rangle$ be a configuration of $T$, let $t(n) = \gamma$, and suppose that $[i, \gamma, \delta, L, j]$ is an instruction in $T$. It also proves convenient to let $t(n - 1) = \varepsilon$. Then $T(Q) = \langle s, n - 1, j \rangle$ is the configuration following $Q$ in the computation of $T$, where

$$
s(k) =
\begin{cases}
    \delta & \text{if } k = n, \\
    t(k) & \text{otherwise}.
\end{cases}
$$
The configuration $Q$ is coded over $F$ by

$$
\beta = \ldots C_{i,\gamma}^{t(n-3)} C_{i,\gamma}^{t(n-2)} C_{i,\gamma}^{e} M_{i}^{\gamma} D_{i,\gamma}^{t(n+1)} D_{i,\gamma}^{t(n+2)} D_{i,\gamma}^{t(n+3)} \ldots
$$

whereas the configuration $T(Q)$ is coded over $F$ by

$$
T(\beta) = \ldots C_{j,\varepsilon}^{t(n-3)} C_{j,\varepsilon}^{t(n-2)} M_{j}^{\varepsilon} D_{j,\varepsilon}^{t(n+1)} D_{j,\varepsilon}^{t(n+2)} D_{j,\varepsilon}^{t(n+3)} \ldots
$$

$T(\beta)$ differs from $\beta$ in several ways. First, the two positions indexed by $n - 1$ and $n$ undergo a change of character from $C$ to $M$ and from $M$ to $D$. Second, the remaining changes amount to changing $\gamma$ to $\varepsilon$ and $i$ to $j$ in various subscripts and superscripts. The idea is to effect this transition with a new operation for the machine instruction $[i, \gamma, \delta, L, j]$. Changes of the first kind have to do with two tape locations. Our new operation must combine the two location elements, $\alpha_{n-1}$ and $\alpha_{n}$, with the configuration element $\beta$ to produce the new configuration element $T(\beta)$—our “instruction” operation should be ternary. To see what is needed to accomplish this, look at

$$
\alpha_{n-1} = \ldots 1 1 1 H 2 2 2 2 \ldots
$$

$$
\alpha_{n} = \ldots 1 1 1 1 H 2 2 2 \ldots
$$

$$
\beta = \ldots C_{i,\gamma}^{t(n-3)} C_{i,\gamma}^{t(n-2)} C_{i,\gamma}^{e} M_{i}^{\gamma} D_{i,\gamma}^{t(n+1)} D_{i,\gamma}^{t(n+2)} D_{i,\gamma}^{t(n+3)} \ldots
$$

$$
T(\beta) = \ldots C_{j,\varepsilon}^{t(n-3)} C_{j,\varepsilon}^{t(n-2)} M_{j}^{\varepsilon} D_{j,\varepsilon}^{t(n+1)} D_{j,\varepsilon}^{t(n+2)} D_{j,\varepsilon}^{t(n+3)} \ldots
$$

The instruction $[i, \gamma, \delta, L, j]$ makes no reference to $\varepsilon$ (the symbol written on square $n - 1$ of the tape). Since our operation must act coordinatewise, we will build $\varepsilon$ into the operation itself. So to each machine instruction we will associate two ternary operations, one for each of the two possible values of $\varepsilon$. Since the machine instructions for a fixed Turing machine $T$ are determined by their first two components we will denote the operations corresponding to the machine instruction above by $F_{i,\gamma,\varepsilon}$. What must happen in $A(T)$ to accomplish the transition above is

$$
F_{i,\gamma,\varepsilon}(1, 1, C_{i,\gamma}^{\nu}) = C_{j,\varepsilon}^{\nu}
$$

$$
F_{i,\gamma,\varepsilon}(2, 2, D_{i,\gamma}^{\varepsilon}) = D_{j,\varepsilon}^{\nu}
$$

$$
F_{i,\gamma,\varepsilon}(H, 1, C_{i,\gamma}^{e}) = M_{j}^{\varepsilon}
$$

$$
F_{i,\gamma,\varepsilon}(2, H, M_{i}^{\gamma}) = D_{j,\varepsilon}^{\delta}
$$

We would like to declare that in $A(T)$ the operation $F_{i,\gamma,\varepsilon}$ results in the default value 0 except in the cases above. Ultimately, this won’t do since we will find it necessary to introduce barred versions of all those $C$’s, $D$’s, and $M$’s with all the attached subscripts and superscripts in order to control the finite subdirectly irreducible algebras. So we will have to revisit the definition of $F_{i,\gamma,\varepsilon}$. For the present, it is no great distortion to think that all the other values are 0.

A similar analysis of right-moving instructions leads the ternary operations $F_{i,\gamma,\varepsilon}$ being defined (with caveats about barred elements) in $A(T)$ via

$$
F_{i,\gamma,\varepsilon}(1, 1, C_{i,\gamma}^{\nu}) = C_{j,\varepsilon}^{\nu}
$$

$$
F_{i,\gamma,\varepsilon}(2, 2, D_{i,\gamma}^{\varepsilon}) = D_{j,\varepsilon}^{\nu}
$$

$$
F_{i,\gamma,\varepsilon}(H, 1, C_{i,\gamma}^{e}) = M_{j}^{\varepsilon}
$$

$$
F_{i,\gamma,\varepsilon}(2, H, M_{i}^{\gamma}) = D_{j,\varepsilon}^{\delta}
$$

With this definition, in $A(T)^Z$
4. HOW $A(T)$ ENCODES THE COMPUTATIONS OF $T$

\[ F_{i\gamma\varepsilon}(\alpha_n, \alpha_{n+1}, \beta) = \mathcal{T}(\beta) \]

provided $\beta$ is as above, $\varepsilon$ is the symbol on tape square $n+1$, and $[i, \gamma, \delta, R, j]$ is an instruction of $T$. For a given Turing machine $T$, the definition of $F_{i\gamma\varepsilon}$ is unambiguous, since whether $F_{i\gamma\varepsilon}$ should be left or right moving can be determined from $T$, $i$, and $\gamma$.

These operations can be envisioned as edge operations, where, however, the edges representing a particular operation now have two labels.

On the basis of these definitions, we obtain the following very useful conclusion.

**The Key Coding Lemma:** Let $T$ be a Turing machine, and let $X$ be a set. Let $F$ be a sequentiable set for $A(T)^X$ and let $i$ be a nonhalting state of $T$. Finally, let $\gamma, \varepsilon \in \{0, 1\}$ and let $f, g$, and $\beta$ be any elements of $A(T)^X$.

Then $F_{i\gamma\varepsilon}(f, g, \beta) = \mathcal{T}(\beta)$ if

- $\beta$ codes a configuration $Q$ over $F$,
- $i$ and $\gamma$ are the first two components of the $T$ instruction determined by $Q$,
- $f, g \in F$ with $f < g$ and these two elements refer to the two adjacent tape squares involved in the motion called for in the instruction,
- $\varepsilon$ is the symbol in the square to which the reading head is being moved, and
- $\mathcal{T}(\beta)$ codes the configuration $\mathcal{T}(Q)$ over $F$;

Otherwise 0 occurs in $F_{i\gamma\varepsilon}(f, g, \beta)$. 

\[ \square \]
LECTURE 5

A(T) and What Happens If T Doesn’t Halt

The basic plan is to do for A(T) what we did for A. We were able to prove for A three crucial things:
1. QZ is in the variety generated by A (and hence that variety was inherently nonfinitely based and had a countably infinite subdirectly irreducible member).
2. Any finite subdirectly irreducible in the variety, except possibly a few very small ones, had a very well determined structure (in fact they were all embeddable into QZ).
3. There were no other infinite subdirectly irreducible algebras in the variety.

It was the second point that compelled us to adjoin additional elements and operations to our original 6-element algebra. Having done that, we had to revisit the first point to assure ourselves that the new elements and operations were innocuous. The third point depended on the first two and Quackenbush’s Theorem.

Proceeding along the same lines with A(T) we are able to do the following:
1. QZ is in the variety generated by A(T), provided T does not halt.
2. In the event that T halts, the cardinality of any finite subdirectly irreducible can be bounded by a function of the size of T and the number of tape squares it visits before halting.
3. In the event that T halts, the variety generated by A(T) has no infinite subdirectly irreducible algebras.
4. In the event that T halts, the variety generated by A(T) is finitely based.

In the second point, at the cost of adding more elements and more operations to our 8-element algebra A, we can ensure that any sequentiable set arising in the construction of a finite subdirectly irreducible cannot be large enough to accommodate the full halting computation. (The idea is that being able to reach a “halting configuration” would force the forbidden \( (x \land y) \lor (x \land z) \) to be a polynomial.) Then we need to argue that bounding the size of sequentiable sets entails a bound on the subdirectly irreducible algebra itself. In the first point, after making an inessential modification to QZ to make it into an algebra of the correct similarity type, it is the inaccessibility of the codes of halting configurations that ensures that the extra operations we had to add to accomplish the second point are innocuous. The third point is an immediate consequence of Quackenbush’s Theorem. The fourth point requires a tough proof due, on various parts, to Ross Willard as well as Ralph McKenzie.

The Algebra A(T)

Let T be a Turing machine with states 0, 1, ..., m. The universe of the algebra A(T) is easiest to describe in pieces. For each of the 4m + 4 choices of \( i = 0, 1, ..., m \) and \( \gamma, \delta \in \{0, 1\} \), we need four distinct elements denoted by \( C^\delta_{i,\gamma}, C^\delta_{i,\gamma}, D^\delta_{i,\gamma}, \) and \( D^\delta_{i,\gamma} \). For each of the 2m + 2 choices of \( i = 0, 1, ..., m \) and \( \gamma \in \{0, 1\} \), we need two elements denoted by \( M^\gamma_i \) and \( M^\gamma_i \). The unbarred versions
were needed to code configurations. The barred versions help us control the finite subdirectly irreducible algebras. Let \( V \) be the set comprised of all \( 20m + 20 \) of these elements. We also let \( V_i \) denote the set of 20 elements of \( V \) whose first lower index is \( i \). In particular, \( V_0 \) contains all the elements used in coding halting configurations. The universe of \( \mathbf{A}(T) \) is just

\[
\mathbf{A}(T) = \{0\} \cup U \cup W \cup V
\]

where \( U = \{1, H, 2\} \) and \( W = \{C, \bar{C}, D, \bar{D}\} \). Thus the size of \( \mathbf{A}(T) \) is \( 20m + 28 \) where \( m \) is the number of nonhalting states of \( T \).

The old algebra \( \mathbf{A} \) will be a subreduct of \( \mathbf{A}(T) \). Indeed, we insist that \( \wedge \) make \( \mathbf{A}(T) \) into a height 1 meet-semilattice with least element 0, and that any product involving a new element results in 0. The definitions of the remaining old operations are changed little or not at all. Here are the \( J \)'s:

\[
J(x, y, z) = \begin{cases} 
  x & \text{if } x = y \neq 0 \\
  x \wedge z & \text{if } x = \bar{y} \in V \cup W \\
  0 & \text{otherwise.}
\end{cases}
\]

\[
J'(x, y, z) = \begin{cases} 
  x \wedge z & \text{if } x = y \neq 0 \\
  x & \text{if } x = \bar{y} \in V \cup W \\
  0 & \text{otherwise.}
\end{cases}
\]

Along with the old \( S \)'s we insert one more:

\[
S_0(u, v, x, y, z) = \begin{cases} 
  (x \wedge y) \lor (x \land z) & \text{if } u \in V_0, \\
  0 & \text{otherwise.}
\end{cases}
\]

\[
S_1(u, v, x, y, z) = \begin{cases} 
  (x \wedge y) \lor (x \land z) & \text{if } u \in \{1, 2\}, \\
  0 & \text{otherwise.}
\end{cases}
\]

\[
S_2(u, v, x, y, z) = \begin{cases} 
  (x \wedge y) \lor (x \land z) & \text{if } u = \bar{v} \in V \cup W, \\
  0 & \text{otherwise.}
\end{cases}
\]

Along with the old \( T \) we insert two new operations \( U^1_{i \gamma \varepsilon} \) and \( U^2_{i \gamma \varepsilon} \) for each of the \( 4m + 4 \) choices of \( i, \gamma, \) and \( \varepsilon \):
5. \(A(T)\) AND WHAT HAPPENS IF \(T\) DOESN’T HALT

\[ T(x, y, z, w) = \begin{cases} 
xy & \text{if } xy = zw \neq 0 \text{ and } x = z \text{ and } y = w \\
xy & \text{if } xy = zw \neq 0 \text{ and } x \neq z \text{ or } y \neq w \\
0 & \text{otherwise.}
\]

\[ U^1_{i\gamma\epsilon}(x, y, z, w) = \begin{cases} 
F_{i\gamma\epsilon}(x, y, w) & \text{if } x < z \text{ and } F_{i\gamma\epsilon}(x, y, w) \neq 0 \text{ and } y = z \\
F_{i\gamma\epsilon}(x, y, w) & \text{if } x < z \text{ and } F_{i\gamma\epsilon}(x, y, w) \neq 0 \text{ and } y \neq z \\
0 & \text{otherwise.}
\]

\[ U^2_{i\gamma\epsilon}(x, y, z, w) = \begin{cases} 
F_{i\gamma\epsilon}(y, z, w) & \text{if } x < z \text{ and } F_{i\gamma\epsilon}(y, z, w) \neq 0 \text{ and } x = y \\
F_{i\gamma\epsilon}(y, z, w) & \text{if } x < z \text{ and } F_{i\gamma\epsilon}(y, z, w) \neq 0 \text{ and } x \neq y \\
0 & \text{otherwise.}
\]

Finally, we need the \(4m + 4\) ternary operations \(F_{i\gamma\epsilon}\) introduced in Lecture 4 (but extended by the mathematicians’ construction of bars everywhere) and one further unary operation which serves to set up initial configurations:

\[ I(x) = \begin{cases} 
C^0_{1,0} & \text{if } x = 1, \\
M^0_{1} & \text{if } x = H, \\
D^0_{1,0} & \text{if } x = 2, \\
0 & \text{otherwise.}
\]

While all this is relatively intricate, the \(F\)’s and the \(I\) plainly help us emulate the computations of the Turing machine. The role of the \(S\)’s is to prevent certain kinds of elements from getting into the picture during the construction of finite subdirectly irreducible algebras. \(T\) was crucial to get a kind of unique decomposition result for \(\cdot\) in the finite subdirectly irreducible algebras. The \(U\) operations play a similar role in connection with the \(F\) operations.

**What Happens If \(T\) Does Not Halt**

Now we expand \(Q_Z\) to the similarity type appropriate to \(T\) by the simple expedient of insisting that all the additional operations are constantly 0. This sort of inessential expansion leaves its key properties intact: any locally finite variety to which (this expanded) \(Q_Z\) belongs will be inherently nonfinitely based, and \(Q_Z\) has a countably infinite subalgebra \(Q_\omega\) which is subdirectly irreducible.

**Theorem 7.** If \(T\) does not halt, then \(Q_Z\) belongs to the variety generated by \(A(T)\). In particular, if \(T\) does not halt, then \(A(T)\) is inherently nonfinitely based and the variety it generates is not residually finite.

**Proof:** We follow the pattern set in the proofs of Theorems 3 and 4. For each \(p \in \mathbb{Z}\) we take \(\alpha_p, \beta_p \in A(T)^Z\) to be the same elements we used before:
\[ \alpha_p := \ldots 1 1 1 H 2 2 2 \ldots \]
\[ \beta_p := \ldots C C C D D D D \ldots \]

where the change is taking place at the \( p \)th position. Next we let \( B_1 = \{ \alpha_p : p \in \mathbb{Z} \} \cup \{ \beta_p : p \in \mathbb{Z} \} \) and we take \( B \) to be the subalgebra of \( \mathbb{A}(T)^\mathbb{Z} \) generated by \( B_1 \). Let \( B_0 \) denote the subset of \( B \) consisting of all those \( \mathbb{Z} \)-tuples in \( B \) which contain at least one 0. The set \( \{ \alpha_p : p \in \mathbb{Z} \} \) is sequentiable and consists of all the tuples in \( B \) belonging to \( U^\mathbb{Z} \), since none of the operations of \( \mathbb{A}(T) \) ever produces an element of \( U \). Now for every \( p \in \mathbb{Z} \)

\[ I(\alpha_p) := \ldots C^0_{1,0} C^0_{1,0} C^0_{1,0} M^0_1 D^0_{1,0} D^0_{1,0} D^0_{1,0} \ldots \]

which gives the code of a configuration (the all-0 tape with the machine in state 1 reading square \( p \)). The \( F_{\gamma\varepsilon} \)'s may now be applied, step by step, to produce the codes of further configurations reached as the computation of \( T \) proceeds. Plainly, all these codes of configurations belong to \( B \). Let \( C \) denote the set of all these configuration codes. We will prove that \( C \cup B_0 \cup B_1 \) is a subuniverse of \( \mathbb{A}(T)^\mathbb{Z} \), and therefore \( B = C \cup B_0 \cup B_1 \).

Now let \( \Phi \) be the map defined from \( B \) to \( Q_\mathbb{Z} \) via

\[ \Phi(x) = \begin{cases} a_p & \text{if } x = \alpha_p \text{ for some } p \in \mathbb{Z}, \\ b_p & \text{if } x = \beta_p \text{ for some } p \in \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases} \]

We contend that \( \Phi \) is a homomorphism from \( B \) onto \( Q_\mathbb{Z} \). To verify this, as well as that \( C \cup B_0 \cup B_1 \) is a subuniverse, requires us to examine the behavior of each of our operations on \( C \cup B_0 \cup B_1 \). For each operation in turn, we show that this set in closed and that \( \Phi \) preserves the operation.

**Case 0:** Evidently \( 0 = \ldots, 0, 0, 0, 0, \ldots \in B_0 \) and so \( \Phi(0) = 0 \).

**Case \( \wedge \):** Evidently, \( u \wedge v = u \) if \( u = v \) and \( u \wedge v \in B_0 \) if \( u \neq v \), for all \( u, v \in C \cup B_0 \cup B_1 \). Hence, our set is closed under \( \wedge \) and \( \Phi(u \wedge v) = \Phi(u) \wedge \Phi(v) \).

**Case \( \cdot \):** Clearly, \( \alpha_p \cdot \beta_{p+1} = \beta_p \) for all \( p \in \mathbb{Z} \), with all other \( \cdot \)-products resulting in elements of \( B_0 \). So our set is closed under \( \cdot \) and \( \Phi \) preserves \( \cdot \).

**Case \( F_{\gamma\varepsilon} \):** According to the Key Coding Lemma, the results of applying \( F_{\gamma\varepsilon} \) to members of \( C \cup B_0 \cup B_1 \) lie in \( C \cup B_0 \). Hence, \( C \cup B_0 \cup B_1 \) is closed under this operation and \( \Phi \) preserves the operation.

**Case \( I \):** Applied to elements of \( C \cup B_0 \cup B_1 \), \( I \) produces only elements of \( C \cup B_0 \). Hence, \( C \cup B_0 \cup B_1 \) is closed with respect to \( I \), and \( \Phi \) preserves \( I \).

Observe that no barred elements occur in any of the members of \( C \cup B_1 \). It follows that

\[ \star \text{ if } u, v \in C \cup B_0 \cup B_1 \text{ with } u(p) = v(p) = 0 \text{ or } u(p) = \overline{v(p)} \in V \cup W \text{ for all } p \in \mathbb{Z}, \text{ then } u = v. \]

**Case \( J \):** Evidently, \( J(x, y, z) \in B_0 \) if \( x \in B_0 \) or \( y \in B_0 \) or \( x \neq y \). Otherwise, according to \( \star \), \( J(x, y, z) = x \). This entails that \( C \cup B_0 \cup B_1 \) is closed under \( J \) and \( \Phi \) preserves \( J \).

**Case \( J' \):** Likewise, \( J'(x, y, z) \in B_0 \) if \( x \in B_0 \) or \( y \in B_0 \) or \( x \neq y \). Otherwise, according to \( \star \), \( J'(x, y, z) = x \wedge z \). This entails that \( C \cup B_0 \cup B_1 \) is closed under \( J \) and \( \Phi \) preserves \( J \).
5. \( \mathbf{A(T)} \) AND WHAT HAPPENS IF \( T \) DOESN’T HALT

CASE \( T \): If \( xy \in B_0 \) or \( zw \in B_0 \) or \( xy \neq zw \), then we have \( T(x, y, z, w) \in B_0 \) and \( (xy) \land (zw) \in B_0 \), for any elements \( x, y, z, w \in C \cup B_0 \cup B_1 \). On the other hand, if \( xy = zw \notin B_0 \) it must be that \( x = z = \alpha_p \) and \( y = w = \beta_{p+1} \) for some \( p \in \mathbb{Z} \). In that case, \( T(x, y, z, w) = xy = zx = (xy) \land (zw) \). Thus, \( C \cup B_0 \cup B_1 \) is closed under \( T \) and \( \Phi \) preserves \( T \).

Observe that for \( u, v \in C \cup B_0 \cup B_1 \), we have \( u \prec v \) only when \( u = \alpha_p \) and \( v = \alpha_{p+1} \) for some \( p \in \mathbb{Z} \). In particular,

* With respect to \( \prec \), every element of \( C \cup B_0 \cup B_1 \) has at most one predecessor and at most one successor.

CASE \( U_1^{1\gamma\varepsilon} \): In case \( F_1^{\gamma\varepsilon}(x, y, w) \in B_0 \) or \( x \neq z \), we have \( U_1^{1\gamma\varepsilon}(x, y, z, w) \in B_0 \). In the alternative case, it follows from the definition of \( F_1^{\gamma\varepsilon} \) that \( x \prec y \). In view of \((*)\) it must be that \( y = z \). So \( U_1^{1\gamma\varepsilon}(x, y, z, w) = F_1^{\gamma\varepsilon}(x, y, w) \in C \). Therefore, the application of \( U_1^{1\gamma\varepsilon} \) always results in an element of \( C \cup B_0 \). Consequently, \( C \cup B_0 \cup B_1 \) is closed with respect to \( U_1^{1\gamma\varepsilon} \) and \( \Phi \) preserves this operation.

CASE \( U_2^{2\gamma\varepsilon} \): This case is like the one above, but it exploits the uniqueness of predecessors instead of successors.

CASE \( S_0 \): Since \( T \) does not halt, the set \( V_0^Z \) is disjoint from \( C \cup B_0 \cup B_1 \). It follows that the application of \( S_0 \) always results in an element of \( B_0 \). Thus \( S_0 \) is preserved by \( \Phi \) and \( C \cup B_0 \cup B_1 \) is closed with respect to \( S_0 \). It should be noted that this is the sole place in the argument where the fact the \( T \) does not halt comes into play.

CASE \( S_1 \): The set \( \{1, 2\}^Z \) is disjoint from \( C \cup B_0 \cup B_1 \). It follows that the application of \( S_1 \) always results in an element of \( B_0 \). Thus \( S_1 \) is preserved by \( \Phi \) and \( C \cup B_0 \cup B_1 \) is closed with respect to \( S_1 \).

CASE \( S_2 \): It follows from \(*\) that the application of \( S_2 \) always results in an element of \( B_0 \). Thus \( S_2 \) is preserved by \( \Phi \) and \( C \cup B_0 \cup B_1 \) is closed with respect to \( S_2 \).

So \( \mathbb{Q}^Z \) belongs to the variety generated by \( \mathbf{A(T)} \). \( \square \)
LECTURE 6

When $T$ Halts: Finite Subdirectly Irreducible Algebras of Sequentiable Type

Throughout this lecture we assume that $T$ is a Turing machine that eventually halts when started on the all-0 tape. We denote by $\pi(T)$ the number of squares examined by $T$ in the course of its computation. Thus $\pi(T)$ is the length of the stretch of tape which comes into use for this computation. Our ambition is to describe all the finite subdirectly irreducible algebras in the variety generated by $A(T)$, or at any rate to bound their size. From the facts developed in Lectures 1 and 2 we already have a lot of information at our disposal. Once again we take $S$ to be a finite subdirectly irreducible algebra in the variety and we fix a finite set $T$, $B$, $\theta$, and $\tilde{\theta}$ so that

- $B \subseteq A(T)^T$
- $\theta, \tilde{\theta} \in \text{Con } B$
- $\theta$ covers $\tilde{\theta}$ in $\text{Con } B$.
- $T$ is as small as possible for representing $S$ in this way.
- $|T| > 1$ (i.e. $S \notin \text{HSA}(T)$).

Among other things, we know that $(x \land y) \lor (x \land z)$ is not a polynomial of $B$ (Fact 8). We also have an element $p \in B$ so that $(p, 0)$ is critical for $\tilde{\theta}$ over $\theta$. In Lecture 2 the analysis revealed that all the elements of $S$, except 0, arose from a unique longest factorization of $p$ using the product $\cdot$. We want, loosely speaking, to do the same thing now; but the machine operations $I$ and $F_{i\gamma}$ have to be considered along with $\cdot$. We will change the definition of $B_1$. Thus, the facts that grew out of our analysis of the old version of $B_1$ must be re-examined. Also, Fact 13 was proved using an analysis by cases, with one case for each basic operation. Now we have more operations. Finally, we have modified all the old operations by extending their domains, (in the case of $J, J'$, and $S_2$, we have done this by treating the new elements in $V$ like the elements in $W$). However, in all its essential features the old analysis can be carried forward.

We take $B_0$ to be the collection of all elements of $B$ which contain at least one 0. In $B$ ranges of $S_0, S_1,$ and $S_2$ lie entirely in $B_0$. Moreover, $V_0^T$ and $\{1, 2\}^T$ are disjoint from $B$ and there are no elements $u, v \in B$ so that $u = \tilde{v} \in (V \cup W)^T$. This is just a direct consequence of Fact 8.

**FACT 20.** Every sequentiable subset of $B$ has fewer than $\pi(T)$ members.

**Proof:** By the Key Coding Lemma any large enough sequentiable set would allow us, using $I$ and the $F_{i\gamma}$’s, to emulate in $B$ the entire halting computation of $T$, producing an element of $V_0^T$ in $B$. Then, via $S_2$, $(x \land y) \lor (x \land z)$ would be a polynomial of $B$. \hfill $\Box$

Next we restate Fact 9 in our expanded setting. The only difference is the insertion of $V$ in the statement and the proof.

**FACT 21.** If $v \in B$ and $p(s) = v(s)$ or $p(s) = \overline{v(s)} \in V \cup W$ for all $s \in T$, then $p = v$. \hfill $\Box$
The next fact splits our analysis into two cases.

**FACT 22.** Either \( p \in V^T \) or \( p \in W^T \).

**Proof:** First notice that there must be a nonconstant unary polynomial \( f \) and \( u \in B \) with \( f(u) = p \) but \( u \neq p \). Otherwise, it follows from Fact 6 that \( B - \{p\} \) is a \( \theta \)-class. This means that our subdirectly irreducible algebra \( S \) has only two elements, and indeed is isomorphic to a subalgebra of \( A(T) \). This contradicts our assumption that \( T \) has at least two elements.

Let \( f \) be a nonconstant unary polynomial of least complexity so that for some \( u \in B \) with \( u \neq p \) we have \( f(u) = p \). Also fix such a \( u \). Now the rest of the argument falls into cases according to the leading operation symbol of \( f \).

**Case \( \land \):** \( f(x) = g(x) \land h(x) \). Then \( p = g(u) \land h(u) \). Since \( p \) is maximal, we conclude that \( p = g(u) = h(u) \). This leads to a violation of the minimality of \( f \).

**Case \( \lor \):** \( f(x) = g(x) h(x) \). So \( p = g(u) h(u) \). This means \( p \in W^T \).

**Case \( I \):** \( f(x) = I(g(x)) \). So \( p = I(g(u)) \in V^T \).

**Cases \( F_{i_{\gamma_0}} \):** \( f(x) = F_{i_{\gamma_0}}(g(x), h(x), k(x)) \). So \( p = F_{i_{\gamma_0}}(g(u), h(u), k(u)) \in V^T \).

**Cases \( S_i \):** \( f(x) = S_i(g(x), h(x), k(x), \ell(x)) \). Impossible: the range of each \( S_i \) is included in \( B_0 \).

**Cases \( T, U_i \):** These cases put \( p \in W^T \) (for \( T \)) or \( p \in V^T \) (for \( U_i \)).

**Case \( J \):** \( f(x) = J(g(x), h(x), k(x)) \). Then \( p = g(u) = h(u) \) by Fact 21 and the maximality of \( p \). This violates the minimality of \( f \) unless \( g \) and \( h \) are constant (and hence constantly \( p \)). But in the latter event \( f \) would be constant. Hence this case cannot happen.

**Case \( J' \):** This is like the last case, but easier. \( \square \)

\( S \) is of sequentiable type if \( p \in W^T \) and of machine type otherwise.

**FACT 23.** Finite subdirectly irreducibles of sequentiable type have fewer than \( 2\pi(T) \) members.

**Proof:** We can just follow the old analysis for \( A \), paying a modest amount of attention to the additional operations, and observing that a sequentiable set arises in a natural way.

Now \( p \in W^T \). Let \( B_1 \) be the set of all factors of \( p \) with respect to \( \cdot \). Now all our previously established facts hold, as is evident in all cases except for Fact 13. This fact asserts that, if \( u \in B \) and \( f(u) \in B_1 \) for some nonconstant unary polynomial \( f \), then \( u \in B_1 \). The proof of Fact 13 relied on a case-by-case analysis according to the leading operation symbol. To get a proof for Fact 13 in our expanded similarity type, we have to consider the operations \( I, F_{i_{\gamma_0}}, U_{i_{\gamma_0}}, U_{i_{\gamma_0}^2} \), and \( S_0 \). (Actually, there are also minor changes in the definitions of \( J, J' \), and \( S_2 \), which merit a small amount of attention not provided here.) All these cases are trivial because \( f(u) \notin B_1 \) for any \( u \) if the leading operation is any of these, since \( B_1 \subseteq U^T \cup W^T \).

As in our analysis for \( A \), we have \( B_1 = \{a_0, a_1, \ldots, a_{n-1}\} \cup \{b_0, b_1, \ldots, b_n\} \) where \( b_k = a_k b_{k+1} \) for all \( k < n \) and \( b_0 = p \). Also \( B - B_1 \) is the \( \theta \)-class of 0, \( B_1 \) splits into singletons modulo \( \theta \), and \( a_k \in U^T \) and \( b_k \in W^T \) for all \( k \). It remains to see that \( \{a_k : k < n\} \) is a sequentiable set. Since \( \pi(T) \) bounds the size of sequentiable sets, we would be finished. We need \( a_k < a_{k+1} \) for all \( k \). Let \( t \in T \), and suppose first that \( a_{k+1}(t) = 1 \). Then \( b_{k+1}(t) \in \{C, C\} \), so \( a_k(t) \in \{1, H\} \). Hence \( a_k(t) < a_{k+1}(t) \). Next, suppose that \( a_{k+1}(t) = H \). Then \( b_{k+1}(t) \in \{D, D\} \), so \( a_k(t) = 2 \). Hence, \( a_k(t) < a_{k+1}(t) \). Finally, suppose \( a_{k+1}(t) = 2 \). Then \( b_{k+1}(t) \in \{D, D\} \), so \( a_k(t) = 2 < 2 = a_{k+1}(t) \). Thus, \( a_k < a_{k+1} \) and \( \{a_k : k < n\} \) is sequentiable. \( \square \)
Lecture 7

When $T$ Halts: Finite Subdirectly Irreducible Algebras of Machine Type

We now consider the case when the finite subdirectly irreducible algebra $S$ introduced in Lecture 6 is of machine type. So we have $p \in V^T$. In this case, we let $B_1$ be the smallest subset of $B$ which includes $p$ and which is closed under the inverses of all the machine operations $I$ and $F_{i\gamma\varepsilon}$. Hence,

$$x \in B_1 \text{ whenever } I(x) \in B_1,$$

$$x, y, u \in B_1 \text{ whenever } F_{i\gamma\varepsilon}(x, y, u) \in B_1.$$

It is easy to see that since $p \in V^T$, then $B_1 \subseteq U^T \cup V^T$, because this latter set is closed under the kind of inversion displayed above.

Since we have now substantially altered the definition of $B_1$, we will need to re-examine Facts 10 and 13. Here is the new version of Fact 10.

**Fact 24.** If $u \in B_1$ and $v \in B$ so that for all $s \in T$ either $u(s) = v(s)$ or $u(s) = \overline{v(s)} \in V \cup W$, then $u = v$.

**Proof:** There are two kinds of elements in $B_1$—those in $U^T$ and those in $V^T$. Clearly, we can restrict our attention to the case when $u \in V^T$. There is a term $t(x, y_0, \ldots, y_{m-1})$ built up from $I$ and the $F_{i\gamma\varepsilon}$’s and elements $d_0, \ldots, d_{m-1} \in B_1$ with $p = t(u, d_0, \ldots, d_{m-1})$. Let $p' = t(v, d_0, \ldots, d_{m-1})$. So for all $s \in T$ either $p(s) = p'(s)$ or $p(s) = \overline{p'(s)}$. By Fact 21, it follows that $p = p'$ and, therefore, that $t(u, d_0, \ldots, d_{m-1}) = t(v, d_0, \ldots, d_{m-1})$. But the operations $I$ and $F_{i\gamma\varepsilon}$ were one-to-one except where they produced 0. Hence, $u = v$. 

Here is the new version of Fact 13. The statement has not changed, but the proof is different, accommodating the change in the definition of $B_1$.

**Fact 25.** If $u \in B$ and $f(u) \in B_1$ for some unary nonconstant polynomial $f$, then $u \in B_1$.

**Proof:** The proof is by induction on the complexity of $f$. The initial step of the induction is obvious, since the identity function is the only simplest nonconstant unary polynomial. The inductive step breaks down into cases, one for each basic operation of positive rank.

**Case $\wedge$:** $f(x) = g(x) \wedge h(x)$.

We have $f(u) \leq g(u), h(u)$. But every element of $B_1$ is maximal with respect to the semilattice order. So $f(u) = g(u) = h(u) \in B_1$. Now at least one of $g$ and $h$ must be nonconstant. Invoking the induction hypothesis, we get $u \in B_1$.

**Case $\cdot$:** $f(x) = g(x)h(x)$.

We have $g(u)h(u) = f(u) \in B_1$. This cannot happen since $-$product produces either 0 or an element of $W$. But $B_1$ is disjoint from both $B_0$ and $W^T$. 

26
CASES $F_{\gamma\xi}$: $f(x) = F_{\gamma\xi}(g(x), h(x), k(x))$
We have $f(u) = F_{\gamma\xi}(g(u), h(u), k(u)) \in B_1$. By the definition of $B_1$, we have $g(u), h(u), k(u) \in B_1$. One of $g(x), h(x), k(x)$ is not constant. So the inductive hypothesis gives $u \in B_1$.
CASE I: $f(x) = I(g(x))$
We have $f(u) = I(g(u)) \in B_1$. By the definition of $B_1$ we get $g(u) \in B_1$. Since $f(x)$ is not constant, $g(x)$ cannot be constant either. So $u \in B_1$ by the inductive hypothesis.
CASE J: $f(x) = J(g(x), h(x), k(x))$
We have $f(u) = J(g(u), h(u), k(u)) \leq g(u)$. By the maximality of $f(u)$ we get

$$f(u) = J(g(u), h(u), k(u)) = g(u) \in B_1.$$ 

Moreover, the definition of $J$ implies that $g(u)$ and $h(u)$ fulfill the hypothesis of Fact 24. So $g(u) = h(u)$. Now $g$ and $h$ cannot both be constant, since then they would have to be the same constant (namely $f(u)$), forcing $f$ to be constant by the definition of $J$. Hence we can invoke the inductive hypothesis (on $g$ or $h$) to conclude that $u \in B_1$.
CASE $J'$: $f(x) = J'(g(x), h(x), k(x))$.
This case is easier than the last one and is omitted.

CASES $S_0, S_1$ AND $S_2$: Too easy, they produce elements of $B_0$.
CASE T: $f(x) = T(g(x), h(x), k(x), \ell(x))$
We have $f(u) = T(g(u), h(u), k(u), \ell(u)) \in B_1$. This cannot happen since $T$ produces either 0 or elements in $W$.

CASES $U_{\gamma\xi}$: $f(x) = U_{\gamma\xi}(g(x), h(x), k(x), \ell(x))$.

We have $f(u) = U_{\gamma\xi}(g(u), h(u), k(u), \ell(u)) \in B_1$. The other case being similar, we suppose that $j = 1$. Evidently, $f(u)$ and $F_{\gamma\xi}(g(u), h(u), \ell(u))$ satisfy the hypotheses of Fact 24. So $f(u) = F_{\gamma\xi}(g(u), h(u), \ell(u)) = F_{\gamma\xi}(g(u), k(u), \ell(u))$ (since also $h(u) = k(u)$) follows from the definition of $U_{\gamma\xi}$. So $g(u), h(u), k(u), \ell(u) \in B_1$, by the definition of $B_1$. Now at least one of $g, h, k$, and $\ell$ is nonconstant since $f$ is not constant. So $u \in B_1$ by the induction hypothesis.

Here is the new version of Fact 14. Again, the statement is the same, but $B_1$ has a new meaning.

**FACT 26.** If $u \in B_1$ and $u \sigma v$, then $u = v$.

PROOF: As we did in the proof of Fact 24 above, pick a term $t(x, y_0, \ldots, y_{m-1})$ and elements $d_0, \ldots, d_{m-1} \in B_1$ so that $p = t(u, d_0, \ldots, d_{m-1})$. Hence $p \sigma t(v, d_0, \ldots, d_{m-1})$. But this entails $t(u, d_0, \ldots, d_{m-1}) = p = t(v, d_0, \ldots, d_{m-1})$. Consequently, $u = v$.

So at this stage we know Fact 15: $B - B_1$ is the $\theta$-class of $0$, and that $\theta$ splits $B_1$ into singleton classes. Thus to bound the cardinality of $S$ we need to bound $|B_1|$. This will be the focus of our efforts in the next lecture. However, here we can remark that in fact a complete analysis of finite subdirectly irreducible algebras of machine type, as well as those of sequentiable type, is at hand. This further analysis would describe the behavior of all the operations. We will not pursue this more detailed analysis, except to point out that all these subdirectly irreducible algebras are flat.
LECTURE 8

When $T$ Halts: Bounding the Subdirectly Irreducibles

In this lecture we will complete our analysis of the subdirectly irreducible algebras generated by $A(T)$ in the case when $T$ halts. Fact 23 already provides a bound on the size of the finite subdirectly irreducible algebras of sequentiable type. The last lecture provided a description of the finite subdirectly irreducible algebras of machine type. Our next task is to bound the size of these algebras. So we continue to consider the case when $S$ is of machine type.

We can suppose that no component of $p \in V^T$ is a barred element. (The basic reason is that the operations $F_{i\gamma\varepsilon}$ do not alter whether a symbol is barred. Hence the distribution of bars in any member of $B_1 \cap V^T$ is the same as the distribution of bars in $p$.) Now $B_1 \subseteq U^T \cup V^T$. Let $\Omega = B_1 \cap V^T$ and $\Sigma = B_1 \cap U^T$. Look first in more detail at $\Omega$.

\[
\begin{align*}
\Omega_0 &= \{p\} \\
\Omega_{n+1} &= \Omega_n \cup \{u \in B_1 : F_{i\gamma\varepsilon}(f, g, u) \in \Omega_n \text{ for some } f, g \in B \text{ and some } i, \gamma, \varepsilon\}
\end{align*}
\]

Evidently, $\Omega = \bigcup_n \Omega_n$. We will say that $f \in U^T$ matches $v \in V^T$ provided for all $t \in T$

\[
\begin{align*}
f(t) &= 1 \iff \nu(t) \text{ is a } C_i^\gamma \\
f(t) &= H \iff \nu(t) \text{ is an } M_i^\gamma \\
f(t) &= 2 \iff \nu(t) \text{ is a } D_i^\varepsilon
\end{align*}
\]

Observe that every $v \in V^T$ matches exactly one $f \in U^T$. For each natural number $n$, we let $\Sigma_n = \{f \in \Sigma : f \text{ matches } v \text{ for some } v \in \Omega_n\}$. By referring to the definition of $F_{i\gamma\varepsilon}$, we have that the elements of the two element set $\{f, g\}$ match the elements of the two element set $\{u, v\}$ whenever $F_{i\gamma\varepsilon}(f, g, u) = v \in \Omega$ (the order in which this matching occurs depends on whether the underlying Turing machine instruction is right-moving or left-moving). It follows that $\Sigma = \bigcup_n \Sigma_n$.

**Fact 27.** $\Sigma$ is a sequentiable set.

**Proof:** We argue by induction that $\Sigma_n$ is sequentiable.

**Initial Step:** Observe that $\Sigma_0$ has only one element. ($\Sigma_0$ cannot be empty, since then our subdirectly irreducible $S$ would be in $HSA(T)$.) Since $\Sigma_0 \subseteq B_1 \cap U^T$ we see that its element has to have $H$ in at least one place. Thus, $\Sigma_0$ is a sequentiable set.

**Inductive Step:** Suppose $h \in \Sigma_{n+1} - \Sigma_n$. Pick $u \in \Omega_{n+1} - \Omega_n$ so that $h$ matches $u$. Further, pick $F_{i\gamma\varepsilon}, f, g$, and $v$ so that $F_{i\gamma\varepsilon}(f, g, u) = v \in \Omega_n$. It does no harm to suppose that we have a left-moving operation. So $g$ matches $u$ and $f$ matches $v$. It follows that $h = g$, that $f \in \Sigma_n$, and that $f \preceq g$. By the inductive hypothesis, we have that $\Sigma_n$ is sequentiable. Let us display $\Sigma_n$ as
$f_a < f_{a+1} < \ldots < f_b$

In the event that $f = f_b$ we have $\Omega_n \cup \{h\}$ sequentiable as desired. On the other hand, if $f = f_c$ for some $c < b$, then, in view of Fact 24, we know $U^1_{i\gamma\varepsilon}(f, h, f_{c+1}, u) = F_{i\gamma\varepsilon}(f, h, u)$. So we would be able to conclude that $h = f_{c+1} \in \Sigma_n$, contrary to our choice of $h$. Reasoning in the same way, we see that it is not possible that $\Sigma_{n+1}$ extends $\Sigma_n$ on the right in any more elaborate way. Indeed, suppose $h' \in \Sigma_{n+1} - \Sigma_n$ and that $F_{i'\gamma'\varepsilon'}(f_b, g', u') = v' \in \Omega_n$, where $h'$ matches $u'$. We take this operation to be left-moving. Then from $U^1_{i'\gamma'\varepsilon'}(f_b, h', h, u') = F_{i'\gamma'\varepsilon'}(f_b, h', u')$ we are able to conclude that $h = h'$.

Right-moving operations are handled in a way similar to what we just did for left-moving operations, but using $U^2_{i\gamma\varepsilon}$.

**Fact 28.** $\Sigma$ has fewer than $\pi(T)$ elements.

To obtain a bound on the cardinality of $\Omega$ we must recall that the sequentiable set $\Sigma$ partitions $T$ into $T_L, T_a, \ldots, T_b, T_R$ where $\Sigma = \{f_a, \ldots, f_b\}$.

**Fact 29.** $u \upharpoonright T_c$ is constant for each $u \in \Omega$ and each $c \in \{a, \ldots, b\}$.

**Proof:** The proof is accomplished in stages, each stage showing that more elements of $\Omega$ are constant on more $T_i$'s until everything is accomplished. This proof needs some preliminary observations.

Suppose that $u \in \Omega_{n+1} - \Omega_n$ with $F_{i\gamma\varepsilon}(f_c, f_{c+1}, u) = v \in \Omega_n$. In this case we will say that $u, c$ and $c+1$ become active at stage $n+1$. (We regard $p$ as the only element active at stage 0 and no member of $c \in \{a, \ldots, b\}$ as active at stage 0.) The definition of $F_{i\gamma\varepsilon}$ entails that $u \upharpoonright T_c, u \upharpoonright T_{c+1}, v \upharpoonright T_c$ and $v \upharpoonright T_{c+1}$ are all constant. Moreover, for all $d$, $u \upharpoonright T_d$ is constant if and only if $v \upharpoonright T_d$ is constant. In checking this, it helps to notice that the relevant subscripts and superscripts can all be determined from $F_{i\gamma\varepsilon}$ and the related Turing machine instruction $[i, \gamma, \delta, M, j]$. Also, if $I(f) = u \in \Omega$, then $u \upharpoonright T_d$ is constant for all $d$.

Now we argue by induction on $n$, that every member of $\Omega_n$ is constant on $T_c$ for all $c$ that have become active by stage $n$ and that, for all $d$ and all $v, v' \in \Omega_n$, $v \upharpoonright T_d$ is constant if and only if $v' \upharpoonright T_d$ is constant.

The initial step of the induction holds vacuously.

For the inductive step, suppose $u, u' \in \Omega_{n+1} - \Omega_n$ with

\[ F_{i\gamma\varepsilon}(f_c, f_{c+1}, u) = v \in \Omega_n \quad \text{and} \quad F_{i'\gamma'\varepsilon'}(f_c', f_{c+1}', u') = v' \in \Omega_n \]

Now our preliminary observations give the conclusions that $u$ and $u'$ are constant on all the $d$'s active by stage $n$ as well as for $c, c', c + 1$, and $c' + 1$, some of which may have become active for stage $n+1$. Moreover, we also conclude that, for all $d$, $u$ is constant on $T_d$ if and only if $v$ is constant on $T_d$ if and only if $v'$ is constant on $T_d$ if and only if $u'$ is constant on $T_d$. In this way, the inductive step is complete.

Now we just count things to obtain:

**Fact 30.** $\Omega$ has no more than $2^s ms$ elements where $s = |\Sigma|$ and $m$ is the number of nonhalting states of $T$. 

8. WHEN $T$ HALTS: BOUNDING THE SUBDIRECTLY IRREDUCIBLES

**Proof:** For each $u \in \Omega$ there are no more than $s$ possibilities for $c \in \{a, \ldots, b\}$ so that $u(t) = M_i^\gamma$, for some $i$ and some $\gamma$ and all $t \in T_c$. Having fixed one of these possibilities there are $m$ choices for $i$ and two choices for $\gamma$. Now for $d$ with $a \leq d < c$ we must have a $\nu$ so that $u(t) = C_i^\nu$ for all $t \in T_d$. Thus for such each $d$ there are no more than two possibilities for $\nu$. Likewise, if $c < d \leq b$, then there is some $\nu$ so that $u(t) = D_i^\nu$ for all $t \in T_d$. Again, for each such $d$ there are no more than two possibilities for $\nu$. Thus far we have bounded the number of possibilities for $u$ by $2^s m$, as desired—but we still have to examine what $u(t)$ is like when $t \in T_L \cup T_R$. Suppose $t \in T_L$. Then $f_c(t) = 1$ for all $c \in \{a, \ldots, b\}$. From the definition of the operations $F_{i\gamma}$ it follows that $u(t) = C_i^\nu$, where $\nu$ is determined by $p(t) = C_i^{\nu'}$, and $i$ and $\gamma$ are the same subscripts that occur throughout $u$. So $u$ is determined on $T_L$ by our previous choices and by the structure of $p$. Likewise, $u$ is determined on $T_R$. So the desired bound is established.

**Theorem 8.** If $T$ halts, then the cardinality of any subdirectly irreducible member of the variety generated by $A(T)$ is no greater than the maximum of $2^{(\pi - 1)m(\pi - 1)} + \pi$ and $20m + 28$, where $\pi$ is the number of tape squares used by $T$ in its halting computation and $m$ is the number of nonhalting states of $T$; moreover, every subdirectly irreducible algebra in the variety is flat.

The $20m + 28$ that occurs above is just the cardinality of $\mathbf{A}(T)$. Its inclusion eliminates the need to analyze the homomorphic images of subalgebras of $\mathbf{A}(T)$ to determine which are subdirectly irreducible.

It is clear that much more was accomplished than just establishing the bound on subdirectly irreducible algebras given above. Our analysis is very close to a complete description (given a description of the behavior of $T$) of all the subdirectly irreducible algebras, even in the case that $T$ does not halt. The only way in which the hypothesis that $T$ does not halt entered into consideration of the finite subdirectly irreducible algebras was in bounding their size. The analysis of their structure holds regardless. In the case that $T$ does not halt, McKenzie describes how to carry this description of the finite subdirectly irreducible algebras up to the infinite subdirectly irreducibles, via an argument relying on Quackenbush’s Theorem. His conclusion is that such varieties have residual character $\omega_1$: while they have countably infinite subdirectly irreducible algebras, they have none of any larger cardinality.

Finally, we have in hand all the pieces of McKenzie’s first undecidability result about finite algebras:

**Theorem 9.** The set of finite algebras of finite type which generate residually very finite varieties is not recursive. Indeed, that set is recursively inseparable from the set of finite algebras of finite type which generate varieties of residual character $\omega_1$. 

\[\square\]
When $T$ Halts: $A(T)$ is Finitely Based—The Plan

The figure above displays the essentials of the situation at hand, as well as the strategy of attack. $V$ is the variety generated by $A(T)$, $W$ is the variety of the same type which is determined by the stipulations that $\land$ is a meet-semilattice operation with least element 0, and that the remaining (finitely many) operations are all monotone. $W_0$ is that subvariety of $W$ determined by also insisting that $S_i(u,v,x,y,z) \approx 0$ for $i = 0, 1, 2$. As we are assuming that $T$ halts, we know that $V$ is residually very finite, and that if $S \in V$ is subdirectly irreducible then $S \in \text{HSA}(T)$ or $S \in W_0$. The role of $U$ is revealed in the following theorem.

**Theorem 10.** Let $K$ be class of algebras and let $V$ be variety with $K \subseteq V$. Let $W_0$ be a finitely axiomatizable elementary class. If

1. there is a finitely axiomatizable elementary class $U$ with $V \subseteq U$ such that every subdirectly irreducible algebra in $U$ lies either in $K$ or in $W_0$, and
2. $W_0 \cap V$ is a finitely axiomatizable elementary class,

then $V$ is finitely based.

**Proof:** Let $\Gamma_0$ be an elementary sentence axiomatizing $W_0$, let $\Sigma_0$ be an elementary sentence axiomatizing $V \cap W_0$, and let $\Delta$ be an elementary sentence axiomatizing $U$. Evidently the sentence $\Delta \land (\Gamma_0 \rightarrow \Sigma_0)$ is true in $V$. Since $V$ is a variety, let $\Sigma$ be a finite set of equations true in $V$ so that $\Sigma \models \Delta \land (\Gamma_0 \rightarrow \Sigma_0)$. To see that $\Sigma$ is a base for $V$ it suffices to prove that every subdirectly irreducible model of $\Sigma$ lies in $V$. So let $S$ be a subdirectly irreducible model of $\Sigma$. So $S \models \Delta$ and
\[ S \vdash \Gamma_0 \rightarrow \Sigma_0. \text{ Thus } S \in \mathcal{U} \text{ and if } S \in \mathcal{W}_0 \text{ then } S \in \mathcal{V}. \text{ Since } S \text{ is subdirectly irreducible and in } \mathcal{U}, \text{ we conclude that either } S \in \mathcal{K} \text{ or } S \in \mathcal{W}_0. \text{ Since } \mathcal{K} \subseteq \mathcal{V}, \text{ we have in either alternative that } S \in \mathcal{V}, \text{ as desired.} \]

Observe that \( \mathcal{K} \) can be eliminated from Theorem 10 by letting \( \mathcal{V} \) itself play the role of \( \mathcal{K} \). However, the strategy we will follow exploits \( \mathcal{K} \) by setting it equal to \( HSA(T) \). Aside from finite axiomatizability, we know that \( \mathcal{V} \) has all the properties attributed to \( \mathcal{U} \) in Theorem 10. To get our hands on \( \mathcal{U} \) we need to find finitely many elementary sentences true in \( \mathcal{V} \) from which the dichotomy among subdirectly irreducible algebras expressed in condition (1) of Theorem 10 can be deduced. In fact, we will end up with finitely many equations true in \( A(T) \) which serve this purpose. So \( \mathcal{U} \) will be turn out to be a variety. The dichotomy itself emerged in Fact 8, where, however, the proof depended in an essential way on the fact the \( A(T) \) is flat. While flatness can be expressed by an elementary sentence, such a sentence certainly cannot be true throughout \( \mathcal{V} \), which contains many algebras that are far from flat. Fortunately, a closer examination of the equations true in \( A(T) \) reveals how the dichotomy can be deduced. Jönsson’s Lemma and Baker’s Finite Basis Theorem play prominent roles. Whether \( T \) halts plays no role in this phase of the argument.

To establish condition 2 of Theorem 10 turns out to rely on a vintage theorem of McKenzie connecting the finite basis property with definable principal congruences as well as recent work by Kearnes. It is crucial here that \( \mathcal{V} \) is very finite, and this is the way in which the fact that \( T \) halts enters the argument. In fact, what we need for \( \mathcal{V} \cap \mathcal{W}_0 \) is that it is residually very finite, its operations are monotone and 0-absorbing, and that its subdirectly irreducible algebras are flat. All these things have already been established.

The work that remains before us breaks into two tasks: the discovery of finitely many equations true in \( \mathcal{V} \) to use as a base for the \( \mathcal{U} \) required in condition (1) of Theorem 10, and the proof of condition (2).

The remainder of this lecture is devoted to listing a number of equations true in \( A(T) \) (regardless of whether \( T \) halts). It is more or less fair to say that they are all evidently true in \( A(T) \). The following notational conventions make it easier to write down and manipulate these equations.

Let \( Q \) be an \( r \)-ary operation symbol. Let \( \bar{u} \) denote the \( r \)-tuple \((u_0, u_1, \ldots, u_{r-1})\) of variables, and let \( i < r \). Then set

\[ Q^0_{(i)}(x) = Q(u_0, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_{r-1}) \]

Also, for \( i = 0, 1, 2 \) and for variables \( u, v \) and \( x \), we use \( S^i_{(u,v)}(x) \) to denote the term \( S_i(u, v, x, x, x) \). We take \( \Delta^* \) to be the set of all the following equations.

**Group I**

This group expresses that \( \land \) and 0 provide a meet-semilattice structure with least element 0.

\[
\begin{align*}
(1) & \quad x \land (y \land z) \approx (x \land y) \land z \\
(2) & \quad x \land y \approx y \land x \\
(3) & \quad x \land x \approx x \\
(4) & \quad x \land 0 \approx 0
\end{align*}
\]
### Group II

This group expresses that every operation is monotone. Let $Q$ be any operation symbol and suppose that $u, x, y$ and $i$ have been chosen appropriately for the rank of $Q$.

\[ Q^u_{(i)}(x \wedge y) \leq Q^u_{(i)}(x) \]

### Group III

This group expresses how certain operations commute with $\wedge, J$ and $J'$. Let $Q$ be any of the operation symbols $\cdot, F_{j\gamma}$, or $I$ and suppose that $\bar{u}$ and $i$ have been chosen appropriately.

\[ Q^u_{(i)}(x \wedge y) \approx Q^u_{(i)}(x) \wedge Q^u_{(i)}(y) \]

\[ Q^u_{(i)}(J(x, y, z)) \approx J(Q^u_{(i)}(x), Q^u_{(i)}(y), Q^u_{(i)}(z)) \]

\[ Q^u_{(i)}(J'(x, y, z)) \approx J'(Q^u_{(i)}(x), Q^u_{(i)}(y), Q^u_{(i)}(z)) \]

### Group IV

This group expresses some essential properties of the operations $S_0, S_1,$ and $S_2$ as well as the related term operations $S_0^{(u,v)}, S_1^{(u,v)},$ and $S_2^{(u,v)}$. Below we take $i = 0, 1, 2$, and $Q$ to be an operation symbol with rank denoted by $r$.

\[ S_1(u, v, x, y, z) \leq x \]

\[ S_1^{(u,v)}(S_i(u, v, x, y, z)) \approx S_i(u, v, x, y, z) \]

\[ S_1^{(u,v)}(Q(x_0, \ldots, x_{r-1})) \approx Q(S_i^{(u,v)}(x_0), \ldots, S_i^{(u,v)}(x_{r-1})) \]

In this group, we also include all equations of the form below except where $j = 2$ and $Q$ is $J$ or $J'$ and where $j = 1$ and $Q$ is $S_0$ or $S_1$. (In $A(T)$ these excluded cases are where 0-absorption fails.)

\[ S_1^{(u,v)}(Q^y_{(j)}(x)) \approx Q^y_{(j)}(S_i^{(u,v)}(x)) \]

### Group V

This group expresses key facts about the operations $U_1^{1, \gamma}$, $U_2^{2, \gamma}$, and $T$.

\[ (x \wedge z)(y \wedge w) \leq T(x, y, z, w) \]

\[ F_{\gamma}(x, y \wedge z, w) \leq U_1^{1, \gamma}(x, y, z, w) \]

\[ F_{\gamma}(x \wedge y, z, w) \leq U_2^{2, \gamma}(x, y, z, w) \]

\[ J'(T(x, y, z, w), xy, (x \wedge z)(y \wedge w)) \approx T(x, y, z, w) \]

\[ J'(U_1^{1, \gamma}(x, y, z, w), F_{\gamma}(x, y, w), F_{\gamma}(x, y \wedge z, w)) \approx U_1^{1, \gamma}(x, y, z, w) \]

\[ J'(U_2^{2, \gamma}(x, y, z, w), F_{\gamma}(x, z, w), F_{\gamma}(x \wedge y, z, w)) \approx U_2^{2, \gamma}(x, y, z, w) \]
This group of equations expresses the salient properties of $J$ and $J'$.

\begin{align*}
(19) & \quad x \land y \leq J(x, y, z) \\
(20) & \quad x \land y \land z \leq J'(x, y, z) \\
(21) & \quad J(x, y, 0) \leq x \land y \\
(22) & \quad J(x, y, z) \leq J'(x, y, x) \\
(23) & \quad J'(x, y, z) \land w \leq J(x, y, S_2^{(x,y)}(w)) \\
(24) & \quad J'(x, y, z) \land w \leq J'(x, y, x \land y \land z \land w) \\
(25) & \quad J(x, y, z) \land J'(x, y, z) \leq z
\end{align*}

The set $\Delta^*$ consists of this finite but uncounted list of equations in Groups I through VI.
Lecture 10

Laying Hands on $\mathcal{U}$

We are searching for a finite set $\Delta$ of equations true in $\mathbf{A}(\mathcal{T})$ so that if $\mathbf{B}$ is a subdirectly irreducible model of $\Delta$, then either it is in $\text{HSA}(\mathcal{T})$ or it is in $\mathcal{W}_0$. Let $\mu$ denote the monolith of $\mathbf{B}$. Here is our plan:

Case I: There are elements $a, b, p, q \in B$ and $i \in \{0, 1, 2\}$ with $(p, q) \in \mu$ and $p > q$ such that $S_i(a, b, p, p, p, p) = p$.

To handle this case, we insert in our set of equations a slightly disguised finite base for an expansion of $\mathbf{A}(\mathcal{T})$. This expansion generates a congruence distributive variety, because we will be able to find Jonsson terms. The subdirectly irreducible $\mathbf{B}$ is a model of the disguised basis. The equation displayed above can be used to undisguise the basis, discovering that the expansion of $\mathbf{B}$ by naming $a$ and $b$ is a model $\Gamma_i$. Hence by Jonsson’s Lemma it is a hom-sub of the expansion of $\mathbf{A}(\mathcal{T})$. By forgetting the names of the constants, $\mathbf{B}$ is seen to be in $\text{HSA}(\mathcal{T})$.

Case II: $S_i(a, b, p, p, p, p) < p$, for all $a, b, p, q \in B$ and for all $i \in \{0, 1, 2\}$, with $(p, q) \in \mu$ and $p > q$.

In this case we will argue that $S_i(u, v, x, y, z) \approx 0$ holds in $\mathbf{B}$ for all $i \in \{0, 1, 2\}$. This means $\mathbf{B} \in \mathcal{W}_0$.

We must accomplish these things on the basis of $\Delta$. Of course, we can use all the equations listed in the last lecture, but we need some others in Case I.

We introduce two new constant symbols $c$ and $d$. For each $i \in \{0, 1, 2\}$ define

\[ P_i = \{(a, b) : a, b \in A(\mathcal{T}) \text{ and } \mathbf{A}(\mathcal{T}) \models S_i(a, b, x, y, z) \approx (x \land y) \lor (x \land z)\} \]

\[ K_i = \{\langle \mathbf{A}(\mathcal{T}), a, b \rangle : (a, b) \in P_i\} \]

$V_i$ be the variety generated by $K_i$.

Now notice that for each $i$, the three terms $d_0(x, y, z) = S_i(c, d, x, y, z), d_1(x, y, z) = x \land z$, and $d_2(x, y, z) = S_i(c, d, z, y, x)$ constitute a system of Jonsson terms. Hence each of $V_0, V_1, V_2$ is a finitely generated congruence distributive variety. Using Baker’s Theorem, let

\[ \Gamma_i = \{s_j(c, d, x) \approx t_j(c, d, x) : j < m\} \]

be a finite base for $V_i$. Finally, let

\[ \Delta_i = \{S_i^{(u, v)}(s_j(u, v, x)) \approx S_i^{(u, v)}(t_j(u, v, x)) : j < m\}. \]

It is easy to see that $\mathbf{A}(\mathcal{T}) \models \Delta_i$ for $i = 0, 1, 2$. Now we can define $\mathcal{U}$. Let $\Delta$ be the set $\Delta^*$ of equations listed in the last lecture together with $\Delta_0 \cup \Delta_1 \cup \Delta_2$. We take $\mathcal{U}$ to be the variety based on $\Delta$. Evidently, $V \subseteq \mathcal{U} \subseteq \mathcal{W}$ and $\mathcal{U}$ is finitely based. Our hope for $\mathcal{U}$ is that it also possesses the required dichotomy among its subdirectly irreducible members.
Lemma 0. Let \( i \in \{0, 1, 2\} \) and let \( B \) be a subdirectly irreducible model of \( \Gamma_i \). If there are \( a, b \in B \) such that \( B \models S_i(a, b, x, x, x) \approx x \), then \( B \in HSA(T) \).

Proof: Let \( B^+ = \langle B, a, b \rangle \). Then \( B^+ \models \Gamma_i \), so \( B^+ \in \mathcal{V}_i \). Since \( B^+ \) is subdirectly irreducible, we have that \( B^+ \in HSP_\mu \mathcal{K}_i \) by Jónsson’s Lemma. But \( \mathcal{K}_i \) is a finite set of finite algebras, so \( B^+ \in HSK_i \). But this means \( B \in HSA(T) \). \( \qed \)

For the remainder of this lecture fix \( B \) as a subdirectly irreducible model of \( \Delta \). Let \( \mu \) denote the monolith of \( B \) and pick \( (p, q) \in \mu \) so that \( p > q \).

In what follows \( e(x) \) denotes the unary polynomial \( S_i(a, b, x, x, x) \) of \( B \), and \( C \) denotes the range of \( e(x) \). Notice that \( e(p) = p \), so \( p \in C \). Let \( B_1 \) denote \( C - \{0\} \).

The following facts are useful. The first restates equation schemas (10) and (11) from the last lecture.

Fact 31. \( e \) is an idempotent endomorphism of \( B \). \( \Box \)

Fact 32. If \( x \leq y \), then \( e(y) \land x = e(x) \).

Proof: \( e(x) = e(y \land x) = e(y) \land x \) by equation schema (12). \( \Box \)

Fact 33. If \( y \in C \) and \( x \leq y \), then \( x \in C \).

Proof: Since \( e \) is idempotent (Fact 31) and \( y \in C \), we have \( e(y) = y \). Thus Fact 32 gives \( e(x) = e(y) \land x = y \land x = x \). So \( x \in C \). \( \Box \)

In particular, since \( p \in C \), we have \( 0, q \in C \).

Fact 34. \( C \) is a subuniverse of \( B \).

Proof: This is true about the image of any algebra under an idempotent endomorphism. Let \( Q \) be an operation symbol. Let \( r \) be the rank of \( Q \). Let \( x_0, \ldots, x_{r-1} \in C \). Then \( Q^B(x_0, \ldots, x_{r-1}) = Q^B(e(x_0), \ldots, e(x_{r-1})) = e(Q^B(x_0, \ldots, x_{r-1})) \in C \). \( \Box \)

Fact 35. Restriction to \( C \) is a lattice homomorphism from \( \text{Con} B \) onto \( \text{Con} C \).

Proof: Again this is true about the image of any algebra under an idempotent endomorphism. Recalling that congruences are exactly the equivalence relations on the universe which are also subalgebras of the square of the algebra eases most of this through. To see that restriction preserves joins and that it is onto, one only has to apply the idempotent endomorphism \( e \) to a sequence appropriately chosen via Maltsev’s Lemma on congruence generation. \( \Box \)

For the purposes of Case I, we suppose \( i \in \{0, 1, 2\} \) and \( a, b, p, q \in B \) with \( (p, q) \in \mu \) and \( p > q \) so that \( S_i(a, b, p, p, p) = p \).

Since \( (p, q) \in \mu \), we see that \( \mu | C \neq 0_C \). It follows that \( C \) is subdirectly irreducible.

Fact 36. \( C \in HSA(T) \); hence, \( C \) is flat, \( q = 0 \), and \( p \) covers \( 0 \) in \( B \).
Proof: To invoke Lemma 0, we have to verify $C \models S_i(e(a),e(b),x,x,x) \approx x$. To see this, let $x \in C$. Since $e$ is an idempotent endomorphism, we obtain $x = e(x) = e(e(x)) = e(S_i(a,b,x,x,x)) = S_i(\epsilon(a),\epsilon(b),\epsilon(x),\epsilon(x),\epsilon(x)) = S_i(\epsilon(a),\epsilon(b),x,x,x)$. So by Lemma 0, we know that $C \in \text{HSA}(T)$. Fact 33 helps to establish the last bit of the statement.

By a basic translation of $B$ (or any algebra), we mean a unary polynomial of the form $Q_{(j)}(x,d)$ where $Q$ is a basic operation, $j$ is a natural number less than the rank of $Q$, and $d$ is an appropriate length tuple of elements of $B$. A translation of $B$ is a composition of basic translations. The translations are the unary polynomials of $B$ which come up in Maltsev’s Lemma on Congruence Generation.

**Fact 37.** For any $u,v \in B$ with $u < v$, there is a translation $\lambda(x)$ of $B$ so that $\lambda(u) = 0$ and $\lambda(v) = p$.

**Proof:** Since $u \neq v$ we have that $(p,0) \in C_B(u,v)$ because $(p,0)$ is a critical pair. By Maltsev’s Lemma there is a sequence $\tau_0(x), \ldots, \tau_{m-1}$ of translations connecting $0$ to $p$ over $(u,v)$. Let $\lambda_j(x) = \tau_j(x) \wedge p$ for each $j < m$. Then $\lambda_0, \ldots, \lambda_{m-1}$ is also a sequence of translations connecting $0$ to $p$. Since all these translations are monotone and since $p$ covers $0$ in $B$, we obtain $\lambda(u) = 0$ and $\lambda(v) = p$ for some translation in this last sequence.

**Fact 38.** Suppose $v \in B$ and $u \in B_1$. If $u \leq v$, then $u = v$.

**Proof:** $0 < u = e(u) \leq e(v)$ by monotonicity. Since $C$ is flat, this means $e(u) = e(v)$. But suppose $u < v$. By Fact 37 pick a translation $\lambda(x)$ so that $\lambda(u) = 0$ while $\lambda(v) = p$. So $0 = e(0) = e(\lambda(u)) = \lambda'(e(u)) = e(\lambda(v)) = e(p) = p$, where $\lambda'(x)$ is obtained from $\lambda(x)$ by applying $e$ to each element occurring as a constant in $\lambda(x)$.

The fact below resembles Facts 13 and 25.

**Fact 39.** For every translation $\lambda$, if $\lambda(v) \in B_1$ and $\lambda(0) = 0$, then $v \in B_1$.

**Proof:** Suppose $v \notin B_1$. Then $e(v) < v$ according the equation schema (9). Since $e(v) \in C$, from Fact 38 we conclude that $e(v) = 0$. But now $\lambda(v) = e(\lambda(v)) = \lambda'(e(v)) = \lambda'(0) \leq \lambda(0) = 0$, where $\lambda'$ is obtained from $\lambda$ as in the proof of Fact 38. The last inequality in this chain follows by equation schema (9). This concludes the proof.

**Lemma 1.** Let $B$ be a subdirectly irreducible model of $\Delta$ and let $\mu$ be its monolith. If there are $i \in \{0,1,2\}$ and $a,b,p,q \in B$ such that $q < p, (p,q) \in \mu$, and $S_i(a,b,p,p,p) = p$, then $B \in \text{HSA}(T)$.

**Proof:** We know that $C \in \text{HSA}(T)$ and that $0, q, p \in C$. Say $0 \neq v \in B$. By Fact 37, there is a translation $\lambda$ so that $\lambda(0) = 0$ and $\lambda(v) = p$. So $v \in C$ by Fact 39. Hence $B = C \in \text{HSA}(T)$.

So we are finished with Case I.
LECTURE 11

The Dichotomy Holds for $\mathcal{U}$

In this lecture we assume that $B$ is a subdirectly irreducible model of $\Delta$ and we denote the monolith of $B$ by $\mu$. We also assume

**Basic Tenet of Case II:** $S_i(a,b,p,p,p) < p$ holds in $B$ for all $i \in \{0,1,2\}$ and all $a,b,p,q \in B$ with $(p,q) \in \mu$ and $p > q$.

Our objective is to prove that $B \in \mathcal{W}_0$. We begin by observing that a stronger version of the Basic Tenet holds.

**Fact 40.** $S_i(a,b,p,p,p) \leq q$ holds in $B$ for all $i \in \{0,1,2\}$ and all $a,b,p,q \in B$ with $(p,q) \in \mu$ and $p > q$.

**Proof:** Let $e(x) = S_i^{(a,b)}(x)$. As noted in the last lecture, $e$ is an idempotent endomorphism of $B$. Let $p' = e(p)$ and $q' = e(q)$. Clearly, $(p',q') \in \mu$ and $p' \geq q'$ by monotonicity. Now by idempotence, $p' = e(p')$. Hence, the Basic Tenet forces $p' = q'$. By equation schema (9), we have $e(q) \leq q$. So $e(p) = p' = q' = e(q) \leq q$, as desired. □

A basic translation is *innocent* provided the basic operation symbol involved is $I, \cdot$, or one of the $F_{i\gamma\varepsilon}$. An *innocent* translation is one that is the composition of innocent basic translations. The identity map is an innocent translation.

The key to establishing that $B \in \mathcal{W}_0$ is the following definition. Let $B_1 \subseteq B \times B$ be defined so that $(v,u) \in B_1$ if and only if the following conditions hold

- $u \leq v$,
- $\lambda(v) \geq p$, and
- $\lambda(u) \land p \leq q$,

for some $(p,q) \in \mu$ with $p > q$ and for some innocent translation $\lambda$. The picture below illustrates these conditions.

Notice that the critical pairs $(p,q) \in \mu$ with $p > q$ all belong to $B_1$ and that $B_1$ itself is essentially the set of inverse images of these critical pairs under translations of the form $\lambda(x) \land p$, where $\lambda$ is innocent, (apart from innocence, translations of this form are just the kind we have been using in conjunction with Maltsev’s Congruence Generation scheme). Thus, this incarnation of $B_1$, even though it is now a binary relation, still strongly resembles the sets that bore the name $B_1$ in earlier lectures. In particular, the operations that give rise to innocent translations were the same ones used in definitions of these earlier sets, and as we will see, the task before us, as in the earlier situations, is essentially to show that the other non-innocent operations could be included without substantially changing $B_1$.

The next fact gives us a useful condition for membership in $B_1$. 38
11. THE DICHOTOMY HOLDS FOR $u$

Fact 41. If $v \geq u$, $(p, q) \in \mu$ with $p > q$, and $\lambda$ is an innocent translation such that
1. $\lambda(u) \land p \leq q$ and
2. $\lambda(v) \land p \not\leq q$,
then $(v, u) \in B_1$.

Proof: Let $p' = \lambda(v) \land p$ and $q' = \lambda(v) \land q$. Then $(p', q') \in \mu$ and $p' > q'$ by (b). Evidently $\lambda(v) \geq p'$. To get the remaining condition, notice that both $\lambda(u) \land p = \lambda(u) \land \lambda(v) \land p = \lambda(u \land v) \land p = \lambda(u) \land p \leq q$ (equation schema (6) came into play), and $\lambda(u) \land p' \leq \lambda(u) \leq \lambda(v)$. Therefore $\lambda(u) \land p' \leq \lambda(v) \land q = q'$, as desired.

Here is a picture that shows how this condition could differ from the one used in the definition.

The usefulness of $B_1$ is revealed in the next fact.
FACT 42. If \((v, u) \in B_1\) and \(i \in \{0, 1, 2\}\), then \(v\) is not in the range of the operation \(S_i\).

**Proof:** Suppose that \(v\) is in the range of \(S_i\). Pick \(a, b, r, s, t \in B\) so that \(S_i(a, b, r, s, t) = v\). Let \(e(x) = S_i^{(a, b)}(x)\). According to equation schema (10), \(e(v) = v\). Also, in view of equation schema (12), \(\lambda(e(x)) = e(\lambda(x))\), for any innocent translation \(\lambda\). So \(\lambda(v)\) is in the range of \(e\), for every innocent translation \(\lambda\). Now pick an innocent translation \(\lambda\) and a critical pair \((p, q) \in \mu\) with \(p > q\), so that \(\lambda(v) \geq p\) and \(\lambda(u) \land p \leq q\). By Fact 33, we conclude that \(p\) is in the range of \(e\). Since \(e\) is idempotent, we have \(e(p) = p\). But since \(S_i(a, b, p, p, p) = e(p) = p\), we have a violation of the Basic Tenet. Thus we must abandon the supposition that \(v\) is in the range of \(S_i\).

To conclude that \(B \in W_0\) we will prove that \((v, 0) \in B_1\) for all \(v \in B - \{0\}\). This is an immediate consequence of the next two facts.

**Fact 43.** If \(0 < v \in B\), then there is a translation \(\lambda\) so that \((\lambda(v), \lambda(0)) \in B_1\).

**Proof:** Let \((p', q') \in \mu\) with \(p' > q'\). According Maltsev, there is a sequence \(\tau_0, \ldots, \tau_{n-1}\) of translations connecting \(q'\) to \(p'\) over \(\{v, 0\}\). Let \(\lambda_i(x) = \tau_i(x) \land p'\) for each \(i < n\). Then \(\lambda_0, \ldots, \lambda_{n-1}\) is also a sequence of translations connecting \(q'\) to \(p'\) over \(\{v, 0\}\). Pick \(\lambda(x)\) from this sequence so that \(\lambda(0) \leq q'\) while \(\lambda(v) \leq q'\). But observe that \(\lambda(v) \leq p'\). Set \(p = \lambda(v)(\lambda(0) \land p')\) and \(q = \lambda(v) \land q'\). Then we have \((p, q) \in \mu\) with \(p > q\). Thus \(\lambda(v) \geq p\) and \(\lambda(0) \land q = \lambda(0) \land \lambda(v) \land q' \leq \lambda(v) = p\). Therefore, \((\lambda(v), \lambda(0)) \in B_1\), since the identity map is an innocent translation.

The following fact is an analog of Facts 13, 25, and 39.

**Fact 44.** For all translations \(\lambda\), if \((\lambda(v), \lambda(u)) \in B_1\), then \((v, u) \in B_1\).

**Proof:** Since every translation is a composition of basic translations, it suffices (by an easy induction) to establish this fact under the stipulation that \(\lambda\) is a basic translation. So our proof breaks down into cases, one for each kind of basic translation.

**Innocent Cases:** \(\lambda(x)\) is an innocent basic translation.

Pick an innocent translation \(\kappa\) and a critical pair \((p, q) \in \mu\) with \(p > q\) which witnesses that \((\lambda(v), \lambda(u)) \in B_1\). So \(\kappa(\lambda(v)) \geq p\) and \(\kappa(\lambda(u)) \land p \leq q\). Since the composition \(\kappa(\lambda(x))\) is innocent, we see that \((v, u) \in B_1\).

**\& Cases:** \(\lambda(x) = x \land b\) for some \(b \in B\).

Pick an innocent translation \(\kappa\) and a critical pair \((p, q) \in \mu\) with \(p > q\) which witnesses that \((\lambda(v), \lambda(u)) \in B_1\). So \(\kappa(\lambda(v)) \geq p\) and \(\kappa(\lambda(u)) \land p \leq q\). This means \(\kappa(v \land b) \geq p\) and \(\kappa(u \land b) \land p \leq q\). So \(\kappa(v) \geq p\) by monotonicity, and

\[
\kappa(u) \land p = \kappa(u) \land (\kappa(b) \land p) = (\kappa(u) \land \kappa(b)) \land p = \kappa(u \land b) \land p \leq q.
\]

Equation schema (6) plays a role above. So \(\kappa, p,\) and \(q\) also witness that \((v, u) \in B_1\).

**J Cases:**

**Subcase J(a, b, x):** \(\lambda(x) = J(a, b, x)\) for some \(a, b \in B\).

We will argue that this subcase is vacuous. Suppose our hypotheses can be satisfied. Pick an innocent translation \(\kappa\) and a critical pair \((p, q) \in \mu\) with \(p > q\) which witnesses that \((\lambda(v), \lambda(u)) \in B_1\). So \(\kappa(\lambda(v)) \geq p\) and \(\kappa(\lambda(u)) \land p \leq q\). This means

\[
\kappa(J(a, b, v)) \geq p\text{ and }\kappa(J(a, b, u)) \land p \leq q.
\]
But now consider

\[ p = \kappa(J(a, b, v)) \land p \]
\[ = J(\kappa(a), \kappa(b), \kappa(v)) \land p \quad \text{schema (7)} \]
\[ \leq J'(\kappa(a), \kappa(b), \kappa(a)) \land p \quad \text{schema (22)} \]
\[ \leq J(\kappa(a), \kappa(b), S_2(\kappa(a), \kappa(b), p, p, p)) \quad \text{schema (23)} \]
\[ \leq J(\kappa(a), \kappa(b), q) \quad \text{by Fact ??} \]

Also

\[ p \leq J'(\kappa(a), \kappa(b), \kappa(a)) \land p \quad \text{as above} \]
\[ \leq J'(\kappa(a), \kappa(b), \kappa(a) \land \kappa(b) \land p) \quad \text{by schema (24)} \]
\[ \leq J'(\kappa(a), \kappa(b), \kappa(J(a, b, u)) \land p) \quad \text{schemas (19) and (7)} \]
\[ \leq J'(\kappa(a), \kappa(b), q) \quad \text{by monotonicity} \]
\[ \quad \text{and our supposition} \]

So

\[ p \leq J(\kappa(a), \kappa(b), q) \land J'(\kappa(a), \kappa(b), q) \]
\[ \leq q \quad \text{by schema (25)} \]

But this is impossible, since we have \( p > q \).

**Subcases \( J(x, a, b) \) and \( J(a, x, b) \):** Let \( \lambda(x) = J(x, a, b) \), putting the other subcase aside.

We will argue, using Fact ??, that \((v \land a, u \land a) \in B_1\) and then invoke the \( \land \)-case above.

Pick an innocent translation \( \kappa \) and a critical pair \((p, q) \in \mu\) with \( p > q \) which witnesses \((\lambda(v), \lambda(u)) \in B_1\). So

\[ \kappa(\lambda(v)) \geq p \text{ and } \kappa(\lambda(u)) \land p \leq q. \]

This means

\[ \kappa(J(v, a, b)) \geq p \text{ and } \kappa(J(u, a, b)) \land p \leq q \]

To use Fact ??, we need the following two claims.

**Claim A:** \( \kappa(u \land a) \land p \leq q \).

**Proof:** Since \( u \land a \leq J(u, a, b) \) by equation schema (19), we get \( \kappa(u \land a) \leq \kappa(J(u, a, b)) \) by monotonicity. So \( \kappa(u \land a) \land p \leq \kappa(J(u, a, b)) \land p \leq q \), as desired. \( \square \)

**Claim B:** \( \kappa(v \land a) \land p \not\leq q. \)

**Proof:** Suppose not. Then \( \kappa(v \land a) \land p \leq q. \)

Now
11. THE DICHOTOMY HOLDS FOR $u$

\[ p = \kappa(J(v, a, b)) \land p \]
\[ = J(\kappa(v), \kappa(a), \kappa(b)) \land p \quad \text{schema (7)} \]
\[ \leq J'(\kappa(v), \kappa(a), \kappa(b)) \land p \quad \text{schema (22)} \]
\[ \leq J(\kappa(v), \kappa(a), S_2(\kappa(v), \kappa(a), p, p)) \quad \text{schema (23)} \]
\[ \leq J(\kappa(v), \kappa(a), q) \quad \text{by Fact ??} \]

Also

\[ p \leq J'(\kappa(v), \kappa(a), \kappa(v)) \land p \quad \text{as above} \]
\[ \leq J'(\kappa(v), \kappa(a), \kappa(v) \land \kappa(a) \land p) \quad \text{by schema (24)} \]
\[ \leq J'(\kappa(v), \kappa(a), \kappa(v \land a) \land p) \quad \text{schema (6)} \]
\[ \leq J'(\kappa(a), \kappa(b), q) \quad \text{by monotonicity and our supposition} \]

So

\[ p \leq J(\kappa(a), \kappa(b), q) \land J'(\kappa(a), \kappa(b), q) \]
\[ \leq q \quad \text{by schema (25)} \]

But this is impossible, since we have $p > q$. \qed

So, by Fact ??, we have $(v \land a, u \land a) \in B_1$. By the $\land$-case above we conclude $(v, u) \in B_1$. This finishes the $J$ cases.

\textbf{$J'$ CASES: $\lambda(x) = J'(a, b, x)$, leaving the other subcases aside.}

We will argue, using Fact ??, that $(a \land b \land v, a \land b \land u) \in B_1$ and then invoke the $\land$-case above.

Pick an innocent translation $\kappa$ and a critical pair $(p, q) \in \mu$ with $p > q$ which witnesses $(\lambda(v), \lambda(u)) \in B_1$. So

\[ \kappa(\lambda(v)) \geq p \quad \text{and} \quad \kappa(\lambda(u)) \land p \leq q. \]

This means

\[ \kappa(J'(a, b, v)) \geq p \quad \text{and} \quad \kappa(J'(a, b, u)) \land p \leq q \]

To use Fact ??, we need the following two claims.

\textbf{CLAIM A:} $\kappa(a \land b \land u) \land p \leq q$.

\textbf{PROOF:} Since $a \land b \land u \leq J'(a, b, u)$ by equation schema (20), we get $\kappa(a \land b \land u) \leq \kappa(J'(a, b, u))$ by monotonicity. So $\kappa(a \land b \land u) \land p \leq \kappa(J'(a, b, u)) \land p \leq q$, as desired. \qed

\textbf{CLAIM B:} $\kappa(a \land b \land v) \land p \leq q$.

\textbf{PROOF:} Suppose not. Then $\kappa(a \land b \land v) \land p > q$.

Now
11. THE DICHOTOMY HOLDS FOR \( \mathcal{U} \)

\[ p = \kappa(J'(a, b, v)) \land p \]
\[ = J'(\kappa(a), \kappa(b), \kappa(v)) \land p \]  
\[ \leq J(\kappa(a), \kappa(b), S_2(\kappa(a), \kappa(b), p, p, p)) \]  
\[ \leq J(\kappa(a), \kappa(b), q) \]  

Also

\[ p \leq J'(\kappa(a), \kappa(b), \kappa(v)) \land p \]
\[ \leq J'(\kappa(a), \kappa(b), \kappa(a) \land \kappa(b) \land (v) \land p) \]
\[ \leq J'(\kappa(a), \kappa(b), \kappa(a \land b \land v) \land p) \]
\[ \leq J'(\kappa(a), \kappa(b), q) \]  

by Fact ??

So

\[ p \leq J(\kappa(a), \kappa(b), q) \land J'(\kappa(a), \kappa(b), q) \]
\[ \leq q \]  

by schema (25)

But this is impossible, since we have \( p > q \).

So, by Fact ??, we have \((a \land b \land v, a \land b \land u) \in B_1\). By the \( \land \)-case above we conclude \((v, u) \in B_1\).

This finishes the \( J' \) cases.

**S_i Cases:** These cases are vacuous by Fact ??.

**T and \( U_{j \in \gamma} \) Cases:**

**T Subcase:** \( \lambda(x) = T(a, b, c, x) \), leaving the other subcases for \( T \) aside.

We will argue, using Fact ??, that \(((a \land c)(b \land v), (a \land c)(b \land u)) \in B_1\) and then invoke the innocent and \( \land \) cases above.

Pick an innocent translation \( \kappa \) and a critical pair \((p, q) \in \mu\) with \( p > q \) which witnesses \((\lambda(v), \lambda(u)) \in B_1\). So

\[ \kappa(\lambda(v)) \geq p \text{ and } \kappa(\lambda(u)) \land p \leq q. \]

This means

\[ \kappa(T(a, b, c, v)) \geq p \text{ and } \kappa(T(a, b, c, u)) \land p \leq q \]

To use Fact ??, we need the following two claims.

**Claim A:** \( \kappa((a \land c)(b \land u)) \land p \leq q \).

**Proof:** Since \((a \land c)(b \land u) \leq T(a, b, c, u)\) by equation schema (13), we get \( \kappa((a \land c)(b \land u)) \leq \kappa(T(a, b, c, u)) \) by monotonicity. So \( \kappa((a \land c)(b \land u)) \land p \leq \kappa(T(a, b, c, u)) \land p \leq q \), as desired. \( \square \)

**Claim B:** \( \kappa((a \land c)(b \land v)) \land p \not\leq q \).

**Proof:** Suppose not. Then \( \kappa((a \land c)(b \land v)) \land p \leq q \).

Now
11. THE DICHOTOMY HOLDS FOR $U$

\[ p = \kappa(T(a, b, c, v)) \land p \]
\[ = \kappa(T'(a, b, c), ab, (a \land c)(b \land v)) \land p \]
\[ = J'(\kappa(T(a, b, c, v)), \kappa(ab), \kappa((a \land c)(b \land v))) \land p \]
\[ \leq J(\kappa(T(a, b, c, v)), \kappa(ab), S_2(\kappa(T(a, b, c, v), \kappa(ab), p, p, p))) \]
\[ \leq J(\kappa(T(a, b, c, v)), \kappa(ab), q) \]

by schema (16)

\[ J(\kappa(T(a, b, c, v)), \kappa(ab), q) \]

by fact ??

Also

\[ p \leq J'(\kappa(T(a, b, c, v)), \kappa(ab), \kappa((a \land c)(b \land v))) \land p \]
\[ \leq J'(\kappa(T(a, b, c, v)), \kappa(ab), \kappa(T(a, b, c, v)) \land \kappa(ab) \land \kappa((a \land c)(b \land v)) \land p) \]
\[ \leq J'(\kappa(T(a, b, c, v)), \kappa(ab), \kappa((a \land c)(b \land v)) \land p) \]
\[ \leq J'(\kappa(T(a, b, c, v)), \kappa(ab), q) \]

as above

\[ J'(\kappa(T(a, b, c, v)), \kappa(ab), q) \]

by schema (24) monotonicity

\[ J(\kappa(T(a, b, c, v)), \kappa(ab), q) \]

by monotonicity and our supposition

So

\[ p \leq J(\kappa(T(a, b, c, v)), \kappa(ab), q) \land J'(\kappa(T(a, b, c, v)), \kappa(ab), q) \]
\[ \leq q \]

by schema (25)

But this is impossible, since we have \( p > q \).

So, by fact ??, we have \( ((a \land c)(b \land v), (a \land c)(b \land u)) \in B_1 \). By the innocent and \( \land \) cases above we conclude \((v, u) \in B_1 \). This finishes the \( T \) subcases.

\( U_{\gamma \xi}^1 \) subcases: These subcases can be argued in the same fashion as the \( T \)-subcases. For example, for \( U_{\gamma \xi}^1 (a, b, c, x) \) we would use fact ?? to establish \((F_{\gamma \xi}(a, b \land c, v), F_{\gamma \xi}(a, b \land c, u)) \in B_1 \).

To accomplish this, we write out the \( T \) subcases proof, but invoke equation schema (14) in place of (13) and schema (17) in place of (16). For \( U_{\gamma \xi}^2 \) we would use schemas (15) and (18).

Thus all the cases can be managed. Fact ?? is established.

\[ \Box \]

Lemma 2. Suppose \( B \) is a subdirectly irreducible model of \( \Delta \) and \( \mu \) is the monolith of \( B \). If \( S_i^B (a, b, p, p, p) < p \) for all \( i \in \{0, 1, 2\} \) and all \( a, b, p, q \in B \) with \((p, q) \in \mu \) and \( p > q \), then \( B \in W_0 \).

Proof: From fact ?? and fact ?? we have \((v, 0) \in B_1 \) for all \( v \in B - \{0\} \). From fact ?? we see that \( v \) cannot be in the range of \( S_i \) for any \( i \in \{0, 1, 2\} \), unless \( v = 0 \). Consequently, \( B \in W_0 \).

\[ \Box \]

Theorem 11. If \( B \) is a subdirectly irreducible member of \( U \), then either \( B \in HSA(T) \) or \( B \in W_0 \).

\[ \Box \]
FACT 45. If $B \models \Delta$ and $B \models S_2(u, v, x, y, z) \approx 0$, then the following equations are true in $B$:

$$J(x, y, z) \approx x \land y \quad J'(x, y, z) \approx x \land y \land z$$

$$T(x, y, z, w) \approx (x \land z)(y \land w)$$

$$U^1_{\gamma \varepsilon}(x, y, z, w) \approx F_{\gamma \varepsilon}(x, y \land z, w) \quad U^2_{\gamma \varepsilon}(x, y, z, w) \approx F_{\gamma \varepsilon}(x \land y, z, w)$$

**Proof:**

$x \land y \leq J(x, y, z)$ \hspace{2cm} schema (19)

$$x \land y \leq J'(x, y, x) \land J'(x, y, x)$$ \hspace{2cm} schema (22)

$$x \land y \leq J(x, y, S_2(x, y, J'(x, y, x), J'(x, y, x), J'(x, x, x)))$$ \hspace{1cm} schema (23)

$$x \land y \leq J(x, y, 0)$$ \hspace{2cm} hypothesis

$$x \land y \leq x \land y$$ \hspace{2cm} schema (21)

$$x \land y \land z \leq J'(x, y, z) \land J'(x, y, z) \land J'(x, y, z)$$ \hspace{1cm} schema (20)

$$x \land y \land z \leq J(x, y, S_2(x, y, J'(x, y, z), J'(x, y, z), J'(x, y, z)) \land J'(x, y, z)$$ \hspace{1cm} schema (23)

$$x \land y \land z \leq J(x, y, 0) \land J'(x, y, z)$$ \hspace{2cm} hypothesis

$$x \land y \land z \leq J(x, y, 0) \land J(x, y, z) \land J'(x, y, z)$$ \hspace{1cm} mononicity

$$x \land y \land z \leq x \land y \land z$$ \hspace{2cm} schema (21)

This establishes the first two equations. The others are immediate from these and the equation schemas in Group V. \qed
LECTURE 12

\( \mathcal{V} \cap \mathcal{W}_0 \) is Finitely Based

The goal we have is to show that \( \mathbf{A}(\mathcal{T}) \) is finitely based when \( \mathcal{T} \) halts. The plan for achieving this was laid out in Lecture 9 and embodied in Theorem 10. That theorem presented us with two hypotheses to establish. With Theorem 9 in the last lecture we have established the first of these hypotheses. It remains to show that \( \mathcal{V} \cap \mathcal{W}_0 \) is finitely based. We will accomplish that here.

An algebra \( \mathbf{A} \) is 0-absorbing provided each of its basic operations is 0-absorbing. We say \( \mathbf{A} \) commutes with \( \wedge \) provided

\[
\mathbf{A} \models Q(x_0, \ldots, x_{r-1}) \wedge Q(y_0, \ldots, y_{r-1}) = Q(x_0 \wedge y_0, \ldots, x_{r-1} \wedge y_{r-1})
\]

for all basic operation symbols \( Q \), with \( r \) denoting the rank of \( Q \).

**Lemma 3.** If \( \mathbf{A} \in \mathcal{W} \) is flat and 0-absorbing, and \( HSP\mathbf{A} \) is residually small, then \( \mathbf{A} \) commutes with \( \wedge \).

**Proof:** On the basis that \( \mathbf{A} \in \mathcal{W} \) is flat, 0-absorbing, and does not commute with \( \wedge \), we will argue that \( HSP\mathbf{A} \) is residually large.

Pick a failure of \( \mathbf{A} \) to commute with \( \wedge \):

\[
Q(a_0, \ldots, a_{r-1}) \wedge Q(b_0, \ldots, b_{r-1}) \neq Q(a_0 \wedge b_0, \ldots, a_{r-1} \wedge b_{r-1}).
\]

By monotonicity, we have

\[
Q(a_0, \ldots, a_{r-1}) \wedge Q(b_0, \ldots, b_{r-1}) > Q(a_0 \wedge b_0, \ldots, a_{r-1} \wedge b_{r-1}),
\]

and \( a_j \neq b_j \) for some \( j < r \). By flatness, we have

\[
Q(a_0, \ldots, a_{r-1}) = Q(b_0, \ldots, b_{r-1}) > 0.
\]

By 0-absorption, we have \( a_i \neq 0 \neq b_i \) for all \( i < r \). Let \( c = Q(a_0, \ldots, a_{r-1}) = Q(b_0, \ldots, b_{r-1}) \).

Now let \( \kappa \) be any infinite cardinal. For each \( \alpha \in \kappa \) and each \( i < r \) define \( f_i^{(\alpha)} \in A^\kappa \) via

\[
f_i^{(\alpha)}(\beta) = \begin{cases} a_i & \text{if } \beta \neq \alpha \\ b_i & \text{if } \beta = \alpha \end{cases}
\]

for all \( \beta \in \kappa \).

Let \( X = \{ f_i^{(\alpha)} : \alpha \in \kappa \text{ and } i < r \} \). Notice that \( |X| = \kappa \). For each \( d \in A \), let \( \hat{d} \in A^\kappa \) denote the constant function from \( \kappa \) into \( A \) with value \( d \). We have

\[
Q A^\kappa(f_0^{(\alpha)}, \ldots, f_{r-1}^{(\alpha)}) = \hat{c}.
\]

Let \( \delta \) be the equivalence relation on \( A^\kappa \) which collapses all \( g \in A^\kappa \) such that 0 is in the range of \( g \), but which isolates all other members of \( A^\kappa \). Since \( \mathbf{A} \) is 0-absorbing, \( \delta \in \text{Con } \mathbf{A} \), and \((\hat{0}, \hat{c}) \notin \delta \).
Let $\theta$ be a congruence of $A$ which is a maximal extension of $\delta$ separating $\hat{0}$ from $\hat{c}$. So $A^\kappa/\theta$ is subdirectly irreducible. The following claim entails that $|A^\kappa/\theta| \geq \kappa$, making $\text{HSP} A$ residually large, since $\kappa$ was an arbitrary infinite cardinal. So the proof of the claim below concludes the proof of Lemma ??.

**Claim:** If $f, g \in X$ with $f \theta g$, then $f = g$.

Say $f = f_1^{(a)}$. Let $\lambda(x) = Q(f_0^{(a)}, \ldots, f_{i-1}^{(a)}, x, f_{i+1}^{(a)}, \ldots, f_{r-1}^{(a)})$. Then $\lambda(f) = \hat{c}$, while $\lambda(\hat{0}) = \hat{0}$. Since $(\hat{0}, \hat{c}) \notin \theta$, we have $(\hat{0}, f) \notin \theta$. Since $f \theta g$, we have $f \theta (f \land g)$. Therefore, $(\hat{0}, f \land g) \notin \theta$. Since $\delta \subseteq \theta$, we get $(\hat{0}, f \land g) \notin \delta$. In view of the definition of $\delta$, we have that $f \land g$ has no component which is 0. Since $\land$ is defined coordinatewise in $A^\kappa$, by the flatness of $A$, we conclude that $f = g$. \hfill $\Box$

**Lemma 4.** Let $K$ be a finite set of finite, flat, $0$-absorbing algebras in $\mathcal{W}$, each commuting with $\land$. There is a natural number $n$ such that for every term $t(x, y)$ there are terms $s(x, z_0, \ldots, z_{n-1})$ and $p_0(y), \ldots, p_{n-1}(y)$ for which $K \models t(x, y) \approx s(x, p_0(y), \ldots, p_{n-1}(y))$.

**Proof:** Consider a term $t(x, y_0, \ldots, y_{m-1})$ in which all the listed variables occur. Let $A \in K$.

Suppose that

$$0 \neq t^A(a, b_0, \ldots, b_{m-1}) = t^A(a', b'_0, \ldots, b'_{m-1}).$$

Then

$$0 \neq t^A(a, b_0, \ldots, b_{m-1}) = t^A(a, b_0, \ldots, b_{m-1}) \land t^A(a', b'_0, \ldots, b'_{m-1})$$

by commutation with $\land$

$$0 \neq \{a \land a', b_0 \land b'_0, \ldots, b_{m-1} \land b'_{m-1}\}$$

by 0-absorption

$$a = a', b_0 = b'_0, \ldots, b_{m-1} = b'_{m-1}$$

by flatness

This means that given $i \in \{0, \ldots, m-1\}$, $A \in K$, and $e \in A - \{0\}$, with $e$ in the range of $t^A$, there is a unique $b \in A$ so that $t^A(a, b_0, \ldots, b_{m-1}) = e$ with $b_i = b$. This permits us to define a function. To more easily see this function, let $P = \{(A, e) : A \in K$ and $e \in A - \{0\} \}$ with $e$ in the range of $t^A$.

Then the function we have in mind is

$$f^t : \{0, \ldots, m-1\} \rightarrow \prod_{(A, e) \in P} A$$

where $f^t(A, e) = b$ specified uniquely as above.

Let $E_t$ denote the kernel of $f^t$. Thus

$(i, j) \in E_t$ if and only if $b_i = b_j$ whenever $A \in K, a, b_0, \ldots, b_{m-1} \in A$, and $t^A(a, b_0, \ldots, b_{m-1}) \neq 0$.

Then $E_t$ induces no more than $M^{M_q}$ equivalence classes, where $M$ is the maximum of $|A| : A \in K \}$ and $q = |K|$. Set $n = M^{M_q}$. This number only depends on $K$ and not on $t$.

We also have

$$K \models \forall x \overline{y} \overline{t}(x, \overline{y}) \not\approx 0 \rightarrow \land_{(i, j) \in E_t} (y_i \approx y_j) \quad (*)$$

Let $C_0, \ldots, C_{r-1}$ be a listing of the equivalence classes of $E_t$. Define $p_k(\overline{y}) = \land \{y_i : i \in C_k\}$ for each $k < r$, and let $g_k(\overline{y}) = p_k(\overline{y})$ where $k$ is the unique number such that $i \in C_k$. With this setup, (* ) implies

$$K \models \overline{t}(x, \overline{y}) \approx \overline{t}(x, g_0(\overline{y}), \ldots, g_{m-1}(\overline{y}))$$
by flatness and 0-absorption. Evidently, \( t(x,g_0(\bar{y}),\ldots,g_{m-1}(\bar{y})) \) can be rewritten in the form \( s(x,p_0(\bar{y}),\ldots,p_{r-1}(\bar{y})) \). Since \( r \leq n \), our proof is done.

A variety has \textit{definable principal congruences} provide there is an elementary formula \( \Phi(x,y,z,w) \) in the language of the variety, such that \((a,b) \in C_g^A(c,d)\) if and only if \( A \models \Phi(a,b,c,d) \), for all \( a,b,c,d \in A \) and for all \( A \) in the variety. In general, Maltsev’s description of principal congruences fails to be elementary in two ways. First, it quantifies over finite sequences: “There is a finite sequence \( \ldots \)”. Second, if quantifies over unary polynomials. If the length of the sequences needed in the characterization can be bounded by a finite number, uniformly across the variety, the first difficulty can be remedied, since the quantification can be replaced by a finite disjunction. The standard remedy for the second difficulty is to show that only unary polynomials of finitely many different forms are needed, where each different form could give rise to infinitely many distinct polynomials through the substitution of different elements of an algebra to play the role of constants. This strategy succeeds in the next theorem, due to Keith Kearnes.

**Theorem 12.** Let \( \mathcal{W}' \) be a subvariety of \( \mathcal{W} \) such that \( \mathcal{W}' \) is residually very finite and every subdirectly irreducible algebra in \( \mathcal{W}' \) is flat and 0-absorbing. Then \( \mathcal{W}' \) has definable principal congruences, and is therefore finitely based.

**Proof:** Let \( \mathcal{K} \) be a finite set of finite, flat, 0-absorbing, subdirectly irreducible algebras which generates \( \mathcal{W}' \). According to Lemmas 9 and 10, fix a number \( n \) so that for every term \( t(x,\bar{y}) \) there are terms \( s(x,z_0,\ldots,z_{n-1}) \) and \( p_0(\bar{y}),\ldots,p_{n-1}(\bar{y}) \) so that

\[
\mathcal{W}' \models t(x,\bar{y}) \approx s(x,p_0(\bar{y}),\ldots,p_{n-1}(\bar{y}))
\]

Now \( \mathcal{W}' \) is locally finite, so there is a finite set \( S \) of terms in \( x, z_0, \ldots, z_{n-1} \) such that every term in these variables is equivalent modulo \( \mathcal{W}' \) to a term in \( S \). In consequence, for all \( A \in \mathcal{W}' \) we have

\[
\text{Pol}_1 A = \{ s^A(x,\bar{b}) : s(x,\bar{z}) \in S \text{ and } \bar{b} \in A^n \}
\]

Therefore, across \( \mathcal{W}' \), the unary polynomials have forms drawn from the finite set \( S \).

To obtain a bound on the length of Maltsev sequences, we consider first monotone sequences. A monotone (increasing) Maltsev sequence over \( \{c,d\} \) consists of \( \langle a_0,\lambda_0, a_1, \lambda_1, \ldots, \lambda_{m-1}, a_m \rangle \) such that \( a_0 < a_1 < \cdots < a_m \) and the \( \lambda_i \)'s are unary polynomials such that \( \{ a_i, a_{i+1} \} = \{ \lambda_i(c), \lambda_i(d) \} \) for all \( i < m \). The length of this sequence is \( m \).

**Claim:** If \( i < j < m \), then the terms in \( S \) underlying \( \lambda_i \) and \( \lambda_j \) are distinct.

**Proof of the Claim:** Suppose not. Take \( i < j, s(x,\bar{z}) \in S \) and \( \bar{b}, \bar{b}' \in A^n \) so that \( \lambda_i(x) = s(x,\bar{b}) \) and \( \lambda_j(x) = s(x,\bar{b}') \). Then

\[
\begin{align*}
\lambda_i(c) & = \lambda_i(c) \land \lambda_j(d) \\
& = s(c,\bar{b}) \land s(d,\bar{b}') \\
& = s(c \land d, b_0 \land b'_0, \ldots, b_{n-1} \land b'_{n-1}) \\
& = s(d,\bar{b}) \land s(c,\bar{b}') \\
& = \lambda_i(d) \land \lambda_j(c) \\
& = \lambda_i(d)
\end{align*}
\]

So \( \{ \lambda_i(c), \lambda_i(d) \} \) is a one-element set, whereas \( \{ a_i, a_{i+1} \} \) is a two-element set, a contradiction.
Thus every monotone Maltsev sequence has length no more than $|S|$.

Now let $(a_0, \lambda_0, a_1, \lambda_1, \ldots, \lambda_{m-1}, a_m)$ be a Maltsev sequence over \{c, d\}. Then polynomials can be found to make the sequence displayed below a Maltsev sequence over \{c, d\}.

\[
\begin{array}{cccc}
a_0 & & & a_m \\
a_0 \land a_1 & & & a_{m-1} \land a_m \\
\vdots & & & \vdots \\
a_0 \land \ldots \land a_{m-1} & & & a_1 \land \ldots \land a_m \\
& & & a_0 \land \ldots \land a_m
\end{array}
\]

The polynomials we need are the $\mu_j(x)$ defined below.

For $j < m$:

$$\mu_j(x) = a_0 \land a_1 \land \ldots \land a_j \land \lambda_j(x)$$

For $m \leq j < 2m + 1$:

$$\mu_j(x) = \lambda_{j-m}(x) \land a_{j-m} \land \ldots \land a_m$$

The sequence we just built is a Maltsev sequence from $a_0$ to $a_m$ over \{c, d\}. It is a monotone decreasing sequence followed by a monotone increasing sequence. So, its length is no more than $2|S|$. Given this restriction on the length of Maltsev sequences, and that a finite set $S$ of terms provides a complete set of schemas for all unary polynomials on all algebras in the variety, Maltsev's characterization of principal congruences becomes elementary, and $\mathcal{W}_0$ has definable principal congruences. Since $\mathcal{W}_0$ is also a residually very finite variety of finite type, by a vintage theorem of Ralph McKenzie, $\mathcal{W}_0$ is finitely based. $\square$

**Theorem 13.** $\mathcal{V} \cap \mathcal{W}_0$ is finitely based.

**Proof:** By Fact ??, $\mathcal{V} \cap \mathcal{W}_0$ is term equivalent to a 0-absorbing variety. By Theorem 8, $\mathcal{V} \cap \mathcal{W}_0$, is residually very finite and all its subdirectly irreducible algebras are flat. So, $\mathcal{V} \cap \mathcal{W}_0$ is finitely based. $\square$

**Theorem 14.** If $T$ halts, then $A(T)$ is finitely based.

**Proof:** By Theorem 10 via Theorem ?? and Theorem ??.

**Theorem 15.** There is no algorithm which determines whether a finite algebra is finitely based.

**Proof:** By Theorem 7 and Theorem ??.

Theorem ?? is due to Ralph McKenzie. The part of its proof described in Lectures 9, 10, 11, and 12 was discovered by Ross Willard. McKenzie’s proof relies on encoding a Turing machine $T$ into a different algebra $F(T)$. McKenzie originated the use of Jónsson’s Theorem and Baker’s Theorem found at the beginning of Lecture 10, but otherwise McKenzie’s arguments focus on terms and equations, while Willard’s analyze the subdirectly irreducible algebras.