Homework Assignment 0

Problem 0
Let A be a nonempty set and let Q be a finitary operation on A. Prove that the rank of Q is unique.

Problem 1
Construct a semigroup that cannot be expanded to a monoid.

Problem 2
Construct a semigroup that is not the multiplicative semigroup of any ring.

Problem 3
Let A be a set and denote by $\mathbf{Eqv}\, A$ the set of all equivalence realtions on A. For $R, S \in \mathbf{Eqv}\, A$ define

$$R \wedge S = R \cap S$$
$$R \vee S = R \cup R \circ S \cup R \circ S \circ R \cup R \circ S \circ R \circ S \cup \dots$$

where $\circ$ stands for the relational product (that is $a(R \circ S)b$ means that there is some $c \in A$ sucn that both aRc and cSb). Prove that $\langle \mathbf{Eqv}\, \rangle A, \wedge, \vee$ is a lattice.

Homework Assignment 1

Problem 4
Let $\mathbf{A}$ and $\mathbf{B}$ be algebras. Prove

$$\hom(\mathbf{A}, \mathbf{B}) = (\mathrm{Sub}\, \mathbf{A} \times \mathbf{B}) \bigcap \{h \mid h \text{ is a function from } A \text{ into } B\}$$

Problem 5
Let $\mathbf{A} = \langle \mathbf{A}_i \mid i \in I \rangle$ be a system of similar algebras. Prove that each projection function on $\prod \mathbf{A}$ is a homomorphism.

Problem 6
Let $\mathbf{A} = \langle \mathbf{A}_i \mid i \in I \rangle$ be a system of similar algebras. Further, assume $\mathbf{B}$ is an algebra of the same signature and that $B = \prod A$. Prove that if each projection function oo B is a homomorphism, then $\mathbf{B} = \prod \mathbf{A}$.

Problem 7
Let $\mathbf{A} = \langle \mathbf{A}_i \mid i \in I \rangle$ be a system of similar algebras. Let $\mathbf{B}$ be an algebra of the same signature and let h_i be a homomorphism from $\mathbf{B}$ into $\mathbf{A}_i$, for each $i \in I$. PRove that there is a homomorphism g from $\mathbf{B}$ into $\prod \mathbf{A}$ such that $h_i = p_i \circ g$ for all $i \in I$. (Here p_i denotes the i^{th} projection function.

Homework Assignment 2

Problem 8
Let $\mathbf{A}$ be an algebra. Prove

$$\mathrm{Con}\ \mathbf{A} = (\mathrm{Sub}\, \mathbf{A} \times \mathbf{A}) \cap \{\theta \mid \theta \text{ is an equivalence relation on } A\}.$$

Problem 9
Let $\mathbf{A}$ be an algebra and let h be an endomorphism of $\mathbf{A}$. Prove that $h \circ h^{-1}$ is a congruence of $\mathbf{A}$. Observe that $h^{-1} = \{(b, a) \mid h(a) = b \text{ and } a \in A\}$.

Problem 10
Let $\mathbf{A}$ be an algebra and let θ be a congruence of $\mathbf{A}$. Prove that $\theta = \bigcup \{\mathrm{Cg}^{\mathbf{A}}(a, a') \mid a \theta a'\}$.

PROBLEM 11

Let $\mathbf{A}$ be an algebra and let $X \subseteq A$ such that $\operatorname{Sg}^{\mathbf{A}} X = A$. Suppose that $\mathbf{B}$ is an algebra with the same signature and let h and g be homomorphisms from A into B such that $h(x) = g(x)$ for all $x \in X$. Prove that $h = g$.

Homework Assignment 3

PROBLEM 12

Prove that every finite algebra is isomorphic to a direct product of directly indecomposable algebras.

PROBLEM 13

Find two algebras $\mathbf{A}$ and $\mathbf{B}$ so that neither $\mathbf{A}$ nor $\mathbf{B}$ can be embedded into $\mathbf{A} \times \mathbf{B}$.

PROBLEM 14

Prove that $\mathbf{A}$ has factorable congruences if and only if $\beta = (\beta \vee \varphi) \wedge (\beta \vee \varphi^*)$ for every pair φ, φ^* of complementary factor congruences of $\mathbf{A}$ and every $\beta \in \operatorname{Con} \mathbf{A}$.

PROBLEM 15

Prove that if $\mathbf{Con\, A}$ is a distributive lattice, then $\mathbf{A}$ has factorable congruences.

Homework Assignment 4

PROBLEM 16

Let $\mathcal{V}$ be a congruence modular variety. Let $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta \in \mathbf{Con}\, A$. Prove the following are equivalent.

(i) $\alpha \vee \beta = \alpha \circ \beta$.
(ii) $[\alpha]^m \vee [\beta]^n = [\alpha]^m \circ [\beta]^n$ for all m, n.
(iii) $[\alpha]^m \vee [\beta]^n = [\alpha]^m \circ [\beta]^n$ for some m, n.