Homework Assignment 0

Problem 0
Let $A$ be a nonempty set and let $Q$ be a finitary operation on $A$. Prove that the rank of $Q$ is unique.

Problem 1
Construct a semigroup that cannot be expanded to a monoid.

Problem 2
Construct a semigroup that is not the multiplicative semigroup of any ring.

Problem 3
Let $A$ be a set and denote by $\text{Equiv} A$ the set of all equivalence realtions on $A$. For $R, S \in \text{Equiv} A$ define

$$R \land S = R \cap S$$
$$R \lor S = R \cup R \circ S \cup R \circ S \circ R \cup R \circ S \circ R \circ S \cup \ldots$$

where $\circ$ stands for the relational product (that is $a(R \circ S)b$ means that there is some $c \in A$ such that both $aRc$ and $cSb$). Prove that $\langle \text{Equiv} A, \land, \lor \rangle$ is a lattice.

Homework Assignment 1

Problem 4
Let $A$ and $B$ be algebras. Prove

$$\text{hom}(A, B) = (\text{Sub} A \times B) \cap \{h \mid h \text{ is a function from } A \text{ into } B\}$$

Problem 5
Let $A = \langle A_i \mid i \in I \rangle$ be a system of similar algebras. Prove that each projection function on $\prod A$ is a homomorphism.

Problem 6
Let $A = \langle A_i \mid i \in I \rangle$ be a system of similar algebras. Further, assume $B$ is an algebra of the same signature and that $B = \prod A$. Prove that if each projection function on $B$ is a homomorphism, then $B = \prod A$.

Problem 7
Let $A = \langle A_i \mid i \in I \rangle$ be a system of similar algebras. Let $B$ be an algebra of the same signature and let $h_i$ be a homomorphism from $B$ into $A_i$, for each $i \in I$. Prove that there is a homomorphism $g$ from $B$ into $\prod A$ such that $h_i = p_i \circ g$ for all $i \in I$. (Here $p_i$ denotes the $i$th projection function.

Homework Assignment 2

Problem 8
Let $A$ be an algebra. Prove

$$\text{Con} A = (\text{Sub} A \times A) \cap \{\theta \mid \theta \text{ is an equivalence relation on } A\}.$$

Problem 9
Let $A$ be an algebra and let $h$ be an endomorphism of $A$. Prove that $h \circ h^{-1}$ is a congruence of $A$. Observe that $h^{-1} = \{(b, a) \mid h(a) = b \text{ and } a \in A\}$.

Problem 10
Let $A$ be an algebra and let $\theta$ be a congruence of $A$. Prove that $\theta = \bigcup \{Cg^A(a, a') \mid a \theta a'\}$. 

Problem 11
Let $A$ be an algebra and let $X \subseteq A$ such that $Sg^A X = A$. Suppose that $B$ is an algebra with the same signature and let $h$ and $g$ be homomorphisms from $A$ into $B$ such that $h(x) = g(x)$ for all $x \in X$. Prove that $h = g$.

Homework Assignment 3

Problem 12
Prove that every finite algebra is isomorphic to a direct product of directly indecomposable algebras.

Problem 13
Find two algebras $A$ and $B$ so that neither $A$ nor $B$ can be embedded into $A \times B$.

Problem 14
Prove that $A$ has factorable congruences if and only if $\beta = (\beta \lor \varphi) \land (\beta \lor \varphi^*)$ for every pair $\varphi, \varphi^*$ of complementary factor congruences of $A$ and every $\beta \in Con A$.

Problem 15
Prove that if $Con A$ is a distributive lattice, then $A$ has factorable congruences.

Homework Assignment 4

Problem 16
Let $V$ be a congruence modular variety. Let $A \in V$ and $\alpha, \beta \in Con A$. Prove the following are equivalent.

(i) $\alpha \lor \beta = \alpha \circ \beta$.
(ii) $[\alpha]^m \lor [\beta]^n = [\alpha]^m \circ [\beta]^n$ for all $m, n$.
(iii) $[\alpha]^m \lor [\beta]^n = [\alpha]^m \circ [\beta]^n$ for some $m, n$. 