Ph.D. Comprehensive Examination
In Algebras, Lattices, Varieties
August 2007

Problem 0.

a. Let $\mathcal{K}$ be a class of algebras of the same signature. Prove that $\boldsymbol{P}\boldsymbol{S}\mathcal{K} \subseteq \boldsymbol{S}\boldsymbol{P}\mathcal{K}$.

b. Provide an example of an algebra $\mathbf{A}$ such that $\boldsymbol{P}\boldsymbol{S}\mathbf{A} \neq \boldsymbol{S}\boldsymbol{P}\mathbf{A}$ and prove that your example works.

Problem 1.

Let $\mathbf{L}$ be a complete lattice. Prove that every element of $\mathbf{L}$ is compact if and only if $\mathbf{L}$ has is ascending chain condition.

Problem 2.

Let $\mathbf{A}$ be an algebra which belongs to a congruence modular variety. Suppose that the lattice $\mathbf{M}_3$ is isomorphic to a sublattice of $\operatorname{Con}\mathbf{A}$ so that the top element of $\mathbf{M}_3$ is mapped to 1_A and the bottom element is mapped to 0_A. Prove that $\mathbf{A}$ is Abelian.

Problem 3.

Prove that the join irreducible elements of a complemented modular lattice are exactly the atoms of the lattice.

Problem 4.

Let $\mathbf{L}$ be a lattice. Prove that $\mathbf{L}$ is distributive if and only if $(a \vee b) \wedge c \leq a \vee (b \wedge c)$ for all $a, b, c \in L$.

Problem 5.

a. Prove that if $\mathbf{A}$ is a congruence modular algebra and θ is a congruence of $\mathbf{A}$, then $\mathbf{A}/\theta$ is also congruence modular.

b. Prove that if $\mathbf{A}$ is a congruence permutable algebra and θ is a congruence of $\mathbf{A}$, then $\mathbf{A}/\theta$ is also congruence permutable.

Problem 6.

Prove that if $\mathcal{V}$ is a congruence permutable variety such that every subdirectly irreducible algebra in $\mathcal{V}$ is simple, then every finite directly indecomposable algebra in $\mathcal{V}$ is also simple.

Problem 7.

Let $\mathbf{A}$ be an algebra which belongs to a congruence modular variety.

a. Prove that if $\theta, \varphi \in \operatorname{Con}\mathbf{A}$ are nilpotent congruences then so is $\theta \vee \varphi$.

b. Prove that if $\operatorname{Con}\mathbf{A}$ satisfies the ascending chain condition, then $\mathbf{A}$ has a unique largest nilpotent congruence.