Theory of Computable Functions
Problem Set 2
27 September 2017

Problem 0.
Let $a_0, a_1, \dots, a_n$ be a sequence of integers of $n+1$ integers. Prove that there are integers u and v so that

$$R(u, 1 + v(i+1)) = a_i \text{ for all } i = 0, \dots, n.$$

Here $R(x, y)$ is the remainder of x divided by y, if $y \neq= 0$ and is x if $y = 0$. If $a_0, \dots, a_n$ are natural numbers what can you say about u and v? [Hint: Use the Chinese Remainder Theorem.]

The next sequence of problems is intended to replicate Chapter 2 in the book of Martin Davis. However, our Turing machines are not quite the same as his, so things have to be adjusted.

Problem 1.
First give a definition of what it means for a Turing machine to be n-regular. Then prove that for every Turing machine Z there is a Turing machine Z' that is n-regular for every n and which has the same input-output behavior as Z, except that the initial state of Z' is $q_{\theta(Z)}$.

Problem 2.
Given any n-regular Turing machine Z and any $p > 0$, there is an $n+p$-regular Turing machine Z_p so that when Z_p is started on $k_0, \dots, k_{p-1}, m_0, \dots, m_{n-1}$, the computation results in $k_0, \dots, k_{p-1}, H$, where the string H is the result of Z started on $m_0, \dots, m_{n-1}$. So Z_p halts on its input if and only if Z halts on its input.

Problem 3.
For each positive integer n and each natural number p, there is am $n+p$-regular Turing machine C_p such that the output of C_p on input $k_0, \dots, k_{p-1}, m_0, \dots, m_{n-1}$ is $m_0, \dots, m_{n-1}, k_0, \dots, k_{p-1}, m_0, \dots, m_{n-1}$. The machine C_p is called a copy machine.

Problem 4.
For each n-regular Turing machine Z, there is an n-regular Turing machine Z' so that Z halts on an input if and only if Z' halts on that input, and If $r_0, \dots, r_{s-1}$ is the output of Z started on $m_0, \dots, m_{n-1}$, then $r_0, \dots, r_{s-1}, m_0, \dots, m_{n-1}$ is the output of Z' started on $m_0, \dots, m_{n-1}$.

Problem 5.
Let $Z_0, \dots, Z_p$ be Turing machines. There is an n-regular Turing machine Z so that Z halts on input $m_0, \dots, m_{n-1}$ if and only if each Z_i halts on that input, and the output of Z started on $m_0, \dots, m_{n-1}$ is $r_0, \dots, r_{p-1}$, where r_j is the output of Z_j started on $m_0, \dots, m_{n-1}$.

Problem 6.
Prove that the class of Turing computable functions is closed under composition.

Problem 7.
Prove that the class of Turing computable functions is closed under minimization.