Theory of Computable Functions
Problem Set on Ackermann's Function
20 September 2017

Wilhelm Ackermann, who attained his Ph.D. under the supervision of David Hilbert and later became Hilbert's assistant and secretary, invented, in 1928, a 3-place function on the natural numbers which is computable but not primitive recursive. Later, this function was simplified, first by Rózsa Péter and later by Raphael Robinson. The resulting function of two arguments is still called Ackermann's function. Here is the definition:

$$\begin{aligned} A(0,n) &= n+1 && \text{for all natural numbers } n \\ A(m+1,0) &= A(m,1) && \text{for all natural numbers } m \\ A(m+1,n+1) &= A(m, A(m+1,n)) && \text{for all natural numbers } m \text{ and } n \end{aligned}$$

Let $f(n) = A(n,n)+1$. The point of this Problem Set is to demonstrate that $f(n)$ is a recursive function which is not primitive recursive. You will see that $f(n)$ is not primitive recursive because it grows faster than (and so eventually dominates) every primitive recursive function. This should impress you since most of the functions you encounter in practise, like n^7, 7^n,$n!$, and 2^{2^n} are all primitive recursive.

Problem 0.
Prove that $A(m,n)$ is a recursive function. You can use minimization.

Problem 1.
(a) Prove $A(m,n) > n$ for all natural numbers n.
(b) Prove $A(m,n+1) > A(m,n)$ for all natural numbers m and n.
(c) Prove $A(m,n) > A(m,p)$ for all natural numbers m, n, and p such that $n > p$.

Problem 2.
(a) Prove $A(m+1,n) \geq A(m,n+1)$ for all natural numbers m and n.
(b) Prove $A(m,n) > m$ for all natural numbers m.
(c) Prove $A(m,n) > A(p,n)$ for all natural numbers m, n, and p such that $m > p$.

Problem 3.
Prove $A(m+2,n) > A(m,2n)$ for all natural numbers m and n.

Suppose that $g : \mathbb{N}^k \to \mathbb{N}$. We say that g is **within level** r provided

$$f(n_0, \ldots, n_{k-1}) \leq A(r, \max\{n_0, \ldots, n_{k-1}\})$$

for all natural numbers $n_0, \ldots, n_{k-1}$.

Problem 4.
Prove that every primitive recursive function is within level r for some natural number r.

Problem 5.
Prove that f (and hence A) is not primitive recursive.