8) A farmer wishes to fence off three identical adjoining rectangular pens, each with 300 square feet of area. What should be the width and length of each pen so that the least amount of fence is required?

We want to minimize perimeter. We'll let

\[ P = (6x + 4y) \] (sum of all sides). We need to reduce the equation to one variable. We know that \( xy = 300 \) (area).

So, \( xy = 300 \)

\[ y = \frac{300}{x} \]

Therefore, \( P = 6x + \frac{4 \cdot 300}{x} = 6x + \frac{1200}{x} \)

\[ P = \frac{6x^2 + 1200}{x} \]

Find critical points.

\[ \frac{dP}{dx} = x(12x) - \left(6x^2 + 1200x\right) = \frac{6x^3 - 1200x}{x^2} \]

The domain of \( x \) is \((0, \infty)\).

\[ 6x^3 - 1200x = 0 \]
\[ 6x^3 = 1200 \]
\[ x^3 = 200 \]
\[ x = \sqrt[3]{200} \]

we choose the positive root because it is in our domain.

Increasing Interval: \((\sqrt[3]{200}, \infty)\)
Decreasing Interval: \((0, \sqrt{200})\)

The perimeter has a minimum when \(x = 10\sqrt{2}\) and \(y = 15\sqrt{2}\).

Why?
\[
x y = 300 \\
10\sqrt{2} \cdot y = 300 \\
y = \frac{30}{\sqrt{2}} = \frac{30}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = 15\sqrt{2}
\]

20. Show that the rectangle with maximum perimeter that can be inscribed in a circle is a square.

Let \(r\) be the radius of the circle, \(x\) the length of the rectangle, and \(y\) be the width of the rectangle. We need to show that \(x = y\). Maximize perimeter.

\[P = 2x + 2y; \quad r^2 = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2}{4} + \frac{y^2}{4}\]

Pythagorean Theorem

Reduce perimeter to one variable.

\[
\left(\frac{4}{4}\right) \left(r^2 - \frac{x^2}{4}\right) = \frac{y^2}{4} \quad (4)
\]

\[
\frac{4r^2 - x^2}{4} = y^2
\]

\[P = 2x + 2\sqrt{4r^2 - x^2}\]
\[
\frac{dP}{dx} = 2 - \frac{2x}{\sqrt{4r^2-x^2}} \quad ("r" \text{ is constant})
\]

Find critical points.

\[
2 - \frac{2x}{\sqrt{4r^2-x^2}} = 0
\]

\[
\frac{2x}{\sqrt{4r^2-x^2}} = 2
\]

\[
\frac{dx}{\frac{2}{2}} = \frac{2\sqrt{4r^2-x^2}}{2}
\]

\[
x = \sqrt{4r^2-x^2}
\]

\[
x^2 = 4r^2 - x^2
\]

\[
2x^2 = 4r^2
\]

\[
x^2 = 2r^2
\]

\[
x = \pm \sqrt{2r}
\]

\[
x = \sqrt{2r} \quad \text{(Side length)}
\]

We need a second derivative test to tell whether it is concave upward or downward.

\[
\frac{d^2P}{dx^2} = -\frac{8r^2}{(4r^2-x^2)^{3/2}} < 0 \text{ when } x = \sqrt{2r}
\]

\[
y = \frac{\sqrt{4r^2-x^2}}{\sqrt{4r^2-2r^2}} = \frac{\sqrt{2r^2}}{\sqrt{2r^2}} = \sqrt{2} \cdot \frac{\sqrt{r^2}}{\sqrt{r^2}} = \sqrt{2} \cdot r = \sqrt{2} \cdot r = x
\]

So \(y = x\). Therefore all sides are equal \(\Rightarrow\) the rectangle is a square.
24) A closed box in the form of a rectangular parallelepiped with a square base is to have a given volume. If the material used in the bottom costs 20% more per square inch than the material in the sides, and the material in the top costs 50% more per square inch than that of the sides, find the economic proportions for the box.

Let $x$ be the length of the sides of the base, $y$ be the height of the box, and $K$ be the cost per square inch of the material in the sides of the box.

$V = x^2 y$

The cost is $C = 1.2Kx^2 + 1.5Kx^2 + 4Kxy$

Reduce the equation to one variable.

$C = 1.2Kx^2 + 1.5Kx^2 + 4K \cdot \frac{V}{x^2}$

Combine

$C = 2.7Kx^2 + \frac{4KV}{x^2}$

$\frac{dC}{dx} = \frac{5.4Kx - 4K\frac{V}{x^2}}{x^2}$

(Remember $K$ is a constant).

$5.4Kx - 4K\frac{V}{x^2} = 0$

$5.4Kx = 4K\frac{V}{x^2}$

$5.4Kx^3 = 4KV$

$5.4x^3 = 4V$
\[ x^3 = \frac{4V}{5.4} \]
\[ \sqrt[3]{x^3} = \sqrt[3]{\frac{4V}{5.4}} \]
\[ x \approx 0.905 \cdot \sqrt[3]{V} \]
\[ y \approx \frac{V}{(0.905 \cdot \sqrt[3]{V})^2} \approx 1.22 \cdot \sqrt[3]{V} \]