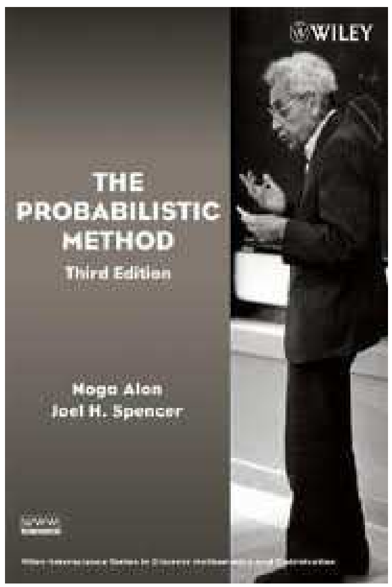
Topic Course on Probabilistic Methods
(Week 9)
Large deviation inequalities (III)

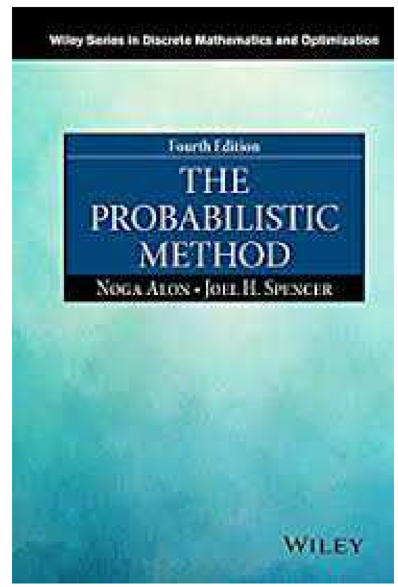
Linyuan Lu

University of South Carolina

Introduction

The topic course is mostly based the textbook “The probabilistic Method” by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviation inequalities (3 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Large deviation inequality

- Talagrand's inequality
- Kim-Vu's inequality
- Rödl's nibble method

Talagrand's inequality

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- $\rho(A, \vec{x})$: Talagrand's distance from $\vec{x} \in \Omega$ to $A \subset \Omega$:

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Theorem [Talagrand's inequality]:

$$\Pr(A)(1 - \Pr(A_t)) \le e^{-t^2/4}.$$

The distance $\rho(A, \vec{x})$

- $U(A, \vec{x}) = \{\vec{s} \in \{0,1\}^n \colon \exists \vec{y} \in A, x_i \neq y_i \Rightarrow s_i = 1\}$.

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Lemma: $\rho(A, \vec{x}) = \min_{\vec{v} \in V(A,\vec{x})} \|\vec{v}\|$.

Proof: Let $\vec{v} \in V(A, \vec{x})$ achieve this minimum. For any $\vec{s} \in V(A, \vec{x})$, we have $\vec{s} \cdot \vec{v} \geq \vec{v} \cdot \vec{v}$. Let $\vec{\alpha} = \vec{v}/\|\vec{v}\|$. We have

$$\rho(A, \vec{x}) \geq \inf_{\vec{y} \in A} \sum_{i \colon x_i \neq y_i} \alpha_i \geq \inf_{\vec{s} \in V(A,\vec{x})} \vec{s} \cdot \vec{\alpha} \geq \|\vec{v}\|.$$

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Conversely, take any unit vector $\vec{\alpha}$. Write $\vec{v} = \sum_i \lambda_i \vec{s}_i$ for some $\vec{s}_i \in U(A, \vec{x})$, $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$. Since $\|\vec{v}\| \geq \sum_i \lambda_i (\vec{\alpha} \cdot \vec{s}_i)$, we have $\alpha \cdot \vec{s}_i \leq \|\vec{v}\|$ for some i. $\square$

A general theorem

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For fixed A, consider $X = \rho(A, \vec{x})$.

$$\begin{aligned}
\Pr(\overline{A_t}) = \Pr(X > t) &\leq \Pr(X \geq t) \\
&= \Pr(e^{X^2/4} \geq e^{t^2/4}) \\
&\leq \mathrm{E}(e^{X^2/4}) e^{-t^2/4} \\
&\leq \frac{1}{\Pr(A)} e^{-t^2/4}.
\end{aligned}$$

Hence, $\Pr(A)\Pr(\overline{A_t}) \leq e^{-t^2/4}$. $\square$

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Now prove $\int_\Omega e^{\frac{1}{4}\rho^2(A,\vec{x})} d\vec{x} \le \frac{1}{\Pr(A)}$ by induction on n.

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When $n = 1$, $\rho(A, \vec{x}) = 1$ if $\vec{x} \notin A$; and 0 if $\vec{x} \in A$.

$$\int_\Omega e^{\frac{1}{4}\rho^2(A,\vec{x})} d\vec{x} = \Pr(A) + (1 - \Pr(A))e^{1/4} \leq \frac{1}{\Pr(A)}.$$

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Assume it holds for n. For any $z \in \Omega$, write $z = (x, \omega)$ with $x \in \prod_{i=1}^n \Omega_i$ and $\omega \in \Omega_{n+1}$. Let

$$B = \{x \in \prod_{i=1}^n \Omega_i \colon (x, \omega) \in A \text{ for some } \omega \in \Omega_{n+1}.\}$$

$$A_\omega = \{x \in \prod_{i=1}^n \Omega_i \colon (x, \omega) \in A\}, \qquad \text{for } \omega \in \Omega_{n+1}.$$

Continue

Two ways to move $z = (x, \omega) \in \Omega$ to A:

- By changing ω, it reduces the problem to moving from x to B. $\vec{s} \in U(B, x) \Rightarrow (\vec{s}, 1) \in U(A, (x, \omega))$.

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Taking the convex hulls, if $\vec{s} \in V(B, x)$ and $\vec{t} \in V(A_\omega, x)$, then for any $\lambda \in [0, 1]$,

$$((1 - \lambda)\vec{s} + \lambda\vec{t}, 1 - \lambda) \in V(A, (x, \omega)).$$

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$$\begin{aligned} \rho^2(A, (x, \omega)) &\leq (1-\lambda)^2 + \|(1-\lambda)\vec{s} + \lambda\vec{t}\|^2 \\ &\leq (1-\lambda)^2 + (1-\lambda)\|\vec{s}\|^2 + \lambda\|\vec{t}\|^2. \end{aligned}$$

Continue

Minimizing $\|\vec{s}\|$ and $\|\vec{t}\|$, we get

$$\rho^2(A,(x,\omega)) \leq (1-\lambda)^2 + \lambda\rho^2(A_\omega, x) + (1-\lambda)\rho^2(B, x).$$

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$$\begin{aligned}
&\int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))}dx \\
&\leq e^{\frac{(1-\lambda)^2}{4}} \int_x \left(e^{\frac{1}{4}\rho^2(A_\omega,x)}\right)^\lambda \left(e^{\frac{1}{4}\rho^2(B,x)}\right)^{1-\lambda} dx \\
&\leq e^{\frac{(1-\lambda)^2}{4}} \left(\int_x e^{\frac{1}{4}\rho^2(A_\omega,x)}dx\right)^\lambda \left(\int_x e^{\frac{1}{4}\rho^2(B,x)}dx\right)^{1-\lambda} \\
&\leq e^{\frac{(1-\lambda)^2}{4}} \left(\frac{1}{\Pr(A_\omega)}\right)^\lambda \left(\frac{1}{\Pr(B)}\right)^{1-\lambda}
\end{aligned}$$

Continue

Let $r = \frac{\Pr(A_\omega)}{\Pr(B)} \leq 1$ and $f(\lambda, r) = e^{(1-\lambda)^2/4} r^{-\lambda}$. Then

$$\int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))} dx \leq \frac{1}{\Pr(B)} f(\lambda, r).$$

Choose $\lambda = 1 + 2\ln r$ for $e^{-1/2} \leq r \leq 1$ and $\lambda = 0$ otherwise. One can show $f(\lambda, r) \leq 2 - r$. Thus,

$$\int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))} dx \leq \frac{1}{\Pr(B)} \left(2 - \frac{\Pr(A_\omega)}{\Pr(B)}\right).$$

$$\int_w \int_x e^{\frac{1}{4}\rho^2(A,(x,\omega))} dx d\omega \leq \frac{1}{\Pr(B)} \left(2 - \frac{\Pr(A)}{\Pr(B)}\right) \leq \frac{1}{\Pr(A)}. \quad \square$$

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Theorem [Alon, Krivelevich, Vu, (2002)]: For every positive integer $1 \le s \le n$, the probability that λ_s deviates from its median by more than t is at most $4e^{-\frac{t^2}{32s^2}}$. The same estimate holds for the probability that λ_{n+1-s} deviates from its median by more than t.

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It suffices to show $\mathcal{B}_{t'} \cap \mathcal{A} = \emptyset$. I.e., for any $B \in \mathcal{B}$ find an vector $\alpha = (\alpha_{ij})$, for any $A \in \mathcal{A}$, show

$$\sum_{(i,j):a_{ij} \neq b_{ij}} \alpha_{ij} \geq t' \left(\sum_{1 \leq i \leq j \leq n} \alpha_{ij} \right)^{1/2}.$$

Continue

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- For $1 \le i \le n$, let

$$\alpha_{ii} = \sum_{p=1}^{s} (v_i^{(p)})^2.$$

For $1 \le i < j \le n$, let

$$\alpha_{ij} = 2\sqrt{\sum_{p=1}^{s} (v_i^{(p)})^2}\sqrt{\sum_{p=1}^{s} (v_j^{(p)})^2}.$$

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$$
\begin{aligned}
\sum_{1\le i\le j\le n} \alpha_{ij}^2 &= \sum_{i=1}^{n} \left(\sum_{p=1}^{s} (v_i^{(p)})^2 \right)^2 \\
&\quad + 4 \sum_{1\le i<j\le n} \left(\sum_{p=1}^{s} (v_i^{(p)})^2 \right) \left(\sum_{p=1}^{s} (v_j^{(p)})^2 \right) \\
&\le 2 \left(\sum_{i=1}^{n} \sum_{p=1}^{s} (v_i^{(p)})^2 \right)^2 \\
&= 2s^2.
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Fix $A \in \mathcal{A}$. Let $u = \sum_{p=1}^{s} c_p v^{(p)}$ be a unit vector in the span of the vectors $v^{(p)}$ which is orthogonal to the eigenvectors of the largest $s-1$ eigenvalues of A. Then $\sum_{p=1}^{s} c_p^2 = 1$, $u'Au \leq \lambda_s(A) \leq M$, and $u'Bu \geq \lambda_s(B) \geq M + t$.

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$$
\begin{aligned}
t &\leq u'(B-A)u \\
&= \sum_{a_{ij}\neq b_{ij}} (b_{ij} - a_{ij}) \sum_{p=1}^{s} c_p v_i^{(p)} \sum_{p=1}^{s} c_p v_j^{(p)} \\
&\leq 2 \sum_{a_{ij}\neq b_{ij}} \left| \sum_{p=1}^{s} c_p v_i^{(p)} \sum_{p=1}^{s} c_p v_j^{(p)} \right| \leq 2 \sum_{a_{ij}\neq b_{ij}} \alpha_{ij}^2 .
\end{aligned}
$$

Putting together

$$\sum_{(i,j):a_{ij}\neq b_{ij}} \alpha_{ij} \geq \frac{t}{2\sqrt{2}s}\left(\sum_{1\leq i\leq j\leq n} \alpha_{ij}\right)^{1/2}.$$

The Talagrand distance between $\mathcal{A}$ and $\mathcal{B}$ is at least $\frac{t}{2\sqrt{2}s}$.

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The Talagrand distance between $\mathcal{A}$ and $\mathcal{B}$ is at least $\frac{t}{2\sqrt{2}s}$. Applying Talagrand's inequality, we get

$$\Pr(\mathcal{A})\Pr(\mathcal{B}) \leq e^{-t^2/32s^2}.$$

Hence, $\Pr(\lambda_s \geq m+t) \leq 2e^{-t^2/32s^2}$. Similar we get $\Pr(\lambda_s \leq m-t) \leq 2e^{-t^2/32s^2}$. Hence

$$\Pr(|\lambda_s - m| \geq t) \leq 4e^{-t^2/32s^2}. \qquad \square$$

More on eigenvalues

Let $A = (a_{ij})$ be a random symmetric $(n \times n)$-matrix with independent entry a_{ij} $(1 \leq i \leq j \leq n)$ satisfying $|a_{ij}| \leq K$ and $E(a_{ij}) = 0$.

- **Vu [2007]:** If $\mathrm{Var}(a_{ij}) \leq \sigma^2$, then

$$\|A\| \leq 2\sigma\sqrt{n} + C(K\sigma)^{1/2} n^{1/4} \ln n.$$

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- **Lu-Peng [2012+]:** If $\mathrm{Var}(a_{ij}) \le \sigma_{ij}^2$, then

$$\|A\| \le 2\sqrt{\Delta} + C\sqrt{K}\Delta^{1/4} \ln n,$$

where $\Delta = \max_{1 \le i \le n} \sum_{j=1}^{n} \sigma_{ij}^2$.

General applications

- $\Omega := \prod_{i=1}^{n} \Omega_i$.
- $h\colon \Omega \to \mathbb{R}$: a Lipschitz function.
- Given $f\colon N \to N$, h is f-certifiable if whenever $h(x) \geq s$ there exists $I \subset [n]$ with $|I| \leq f(s)$ so that all $y \in \Omega$ that agree with x on the coordinates I have $h(y) \geq s$.

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Example: Let $\Omega = G(n, p)$ and $h(G)$ be the number of triangles in G. Then h is f-certifiable with $f(s) = 3s$.

Theorem

Theorem: Suppose $X = h(\cdot)$ is f-certifiable. For any positive b and t, we have

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Let I be a set of indices of size at most $f(b)$ that certifies $h(y) \geq b$. Define $\alpha_i = |I|^{-1/2}$ if $i \in I$, and 0 otherwise. For any $x \in A$, $\sum_{x_i \neq y_i} \alpha_i \geq t\sqrt{f(b)}|I|^{-1/2} \geq t$. By Talagrand's inequality,

$$\Pr(X \leq b - t\sqrt{f(b)})\Pr(X \geq b) \leq e^{-t^2/4}. \quad \square$$

Kim-Vu's inequality

- $H = (V, E)$: a hypergraph
- $Y := \sum_{F \in E(H)} w_F \prod_{i \in F} t_i$, a polynomial of degree k with non-negative coefficients w_F.
- For each $A \subset V(H)$, let $Y_A = \sum_{F \in E(H), A \subset F} w_F \prod_{i \in F - A} t_i$, the partial derivative of Y with respect to the t_i, $i \in A$.
- $E_i := \max\{\mathrm{E}(Y_A) \colon |A| = i\}$.
- $E' := \max\{E_i : 1 \leq i \leq k\}$.
- $E := \max\{\mathrm{E}(Y), E'\}$.

Theorem [Kim-Vu 2000]:

$$\Pr(|Y - \mathrm{E}(Y)| > a_k (EE')^{1/2} \lambda^k) < d_k e^{-\lambda} n^{k-1},$$

where $a_k = 8^k \sqrt{k!}$, $d_k = 2e^2$.

Examples

Counting triangles in $G(n,p)$:

- Let $p = n^{-\alpha}$ with $0 < \alpha < \frac{2}{3}$.
- For any vertex v, let $Y := Y(v)$ be the number of triangles containing v. $Y = \sum_{i,j \neq v} t_{vi} t_{vj} t_{ij}$.

Now $\mu := \mathrm{E}(Y) = \binom{n-1}{2} p^3 \sim \frac{1}{2} n^{2-3\alpha}$ and $E' \sim \max\{np^2, 1\} = c\mu n^{-\epsilon}$ for some ϵ depending on α. Applying Kim-Vu's inequality, we have

$$\Pr(|Y - \mu| > \delta\mu) \leq Cn^2 e^{-C' n^{\epsilon/6}}.$$

Almost surely every vertex v is in $\sim \mu$ triangles.

Steiner System

A Steiner system with parameters t, k, n, written $S(t, k, n)$, is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

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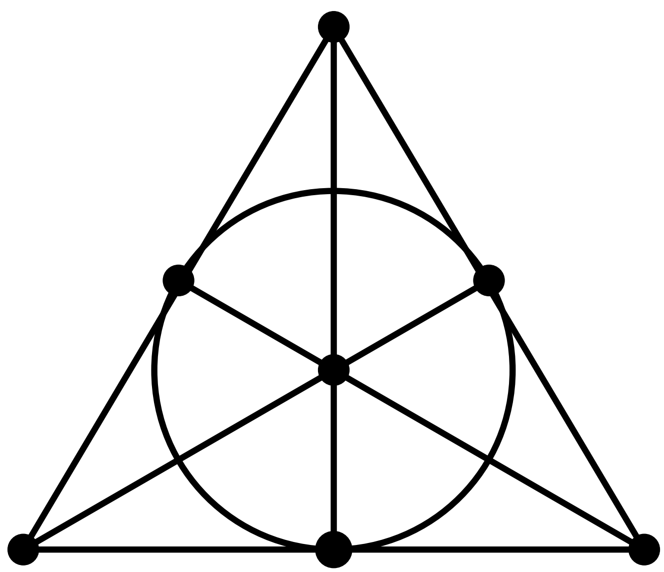
Steiner system $S(2, 3, 7)$:

1	2	4
2	3	5
3	4	6
4	5	7
5	6	1
6	7	2
7	1	3



Fano plane

Covering/Packing number

- Covering number $M(n, k, t)$: the minimal size of a family $\mathcal{K} \subset \binom{[n]}{k}$ having the property that every t-set is contained in at least one $A \in \mathcal{K}$.

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$$m(n,k,t) \leq \frac{\binom{n}{t}}{\binom{k}{t}} \leq M(n,k,t).$$

Equalities hold if and only if there exists a Steiner system $S(t,k,n)$.

Rödl's nibble method

Erdős-Hanami's conjecture: $\lim_{n\to\infty} \frac{M(n,k,t)}{\binom{n}{t}/\binom{k}{t}} = 1.$

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Theorem (Pippenger): For an integer $r \geq 2$ and a real $\varepsilon > 0$ there exists a real $\gamma = \gamma(r, \varepsilon)$ so that the following holds: If the r-uniform hypergraph H on n vertices satisfies:

1. For each vertex x, degree $d(x) \in [(1-\gamma)D, (1+\gamma)D]$.
2. For each pair of vertices x, y, codegree $d(x, y) < \gamma D$.

then $\exists$ a matching that covers all but at most ϵn vertices.

Rödl's nibble

Before nibble: A hypergraph $H = (V, E)$ with

1. For all but at most δn of vertex x, $d(x) = (1 \pm \delta)D$.
2. For any two x, y, $d(x, y) < \delta D$.

Select a random family $\mathcal{F}$ of edges with probability $p = \epsilon/D$ independently. Then delete the vertices covered by $\mathcal{F}$.

After nibble:

1. $|\mathcal{F}| \approx \frac{\epsilon n}{r}(1 \pm \delta')$.
2. The remaining set of vertices V' has size $ne^{-\epsilon}(1 \pm \delta')$.
3. For all but at most $\delta'|V'|$ of vertex x in the induced hypergraph on V', $d'(x) = De^{-\epsilon(r-1)}(1 \pm \delta')$.

Iteration

Choose $\epsilon > 0$ and $\delta > 0$ such that $\frac{\epsilon}{1-e^{-\epsilon}} + r\epsilon < 1 + \varepsilon$. Let $t = \lfloor \frac{-\ln \epsilon}{\epsilon} \rfloor$. Repeat Rödl nibbles t times.

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- $\delta_t > \delta_{t-1} > \cdots > t_1 > t_0 = \gamma$ satisfying

$$\delta_i \leq \delta_{i+1} e^{-\epsilon(r-1)}, \quad \prod_{i=0}^{t} (1 + \delta_i) < 1 + 2\delta.$$

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- $H = H_0 \supsetneq H_1 \supsetneq \cdots \supsetneq H_t$ satisfying

$$|V_i| = |V_{i-1}| e^{-\epsilon} (1 \pm \delta_i)$$

$$|E_i| = \frac{\epsilon |V_{i-1}|}{r} (1 \pm \delta_i)$$

$$D_i = D_{i-1} e^{-\epsilon(r-1)}.$$

Putting together

Note that $\mathcal{F} := \cup_{i=1}^{t} \mathcal{F}_i$ covers all vertices except V_t. The vertices in V_t need at most $|V_t|$ additional edges to cover. The edges in the final cover is at most

$$\sum_{i=0}^{t-1} \frac{\epsilon |V_i|}{r}(1+2\delta_i) + |V_t|$$

$$\leq (1+4\delta)\frac{\epsilon n}{r}\frac{1}{1-e^{-\epsilon}} + (1+2\delta)ne^{-\epsilon t}$$

$$< (1+\varepsilon)\frac{n}{r}.$$

This complete the proof. □

Improvement

Suppose that H is a r-uniform, D-regular hypergraph on n vertices with $codeg(H) = C$. Let $\mathcal{U}(H)$ be the error term of a nearly perfect matching. There is a number of improvements on the error term.

- **Grable [1996]:** If $C = o(D/\ln n)$, then
$$\mathcal{U}(H) = O\left(n\left(\frac{C\log n}{D}\right)^{1/(2r-1)+o(1)}\right).$$
- **Alon-Kim-Spencer [1997]:** If $C = 1$, then
$\mathcal{U}(H) = O\left(n\left(\frac{C}{D}\right)^{1/(r-1)+o(1))}\right)$ for $r \geq 4$ and
$\mathcal{U}(H) = O\left(n\left(\frac{C}{D}\right)^{1/2}\log^{3/2} D\right)$ for $r = 3$.
- **Vu [2000]:** For all $r \geq 3$, $\exists c > 0$, such that
$$\mathcal{U}(H) = O\left(n\left(\frac{C\log n}{D}\right)^{1/(r-1)}\log^c D\right).$$