Topic Course on Probabilistic Methods
(Week 5)
Lovász Local Lemma

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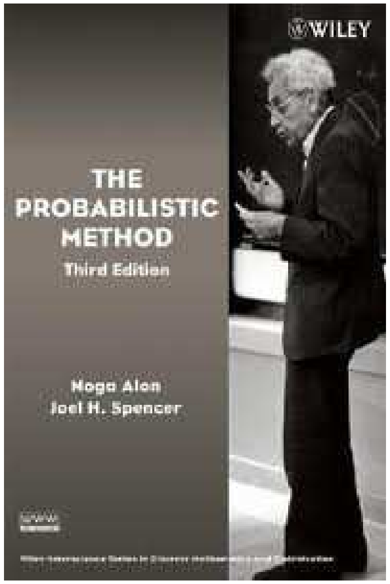
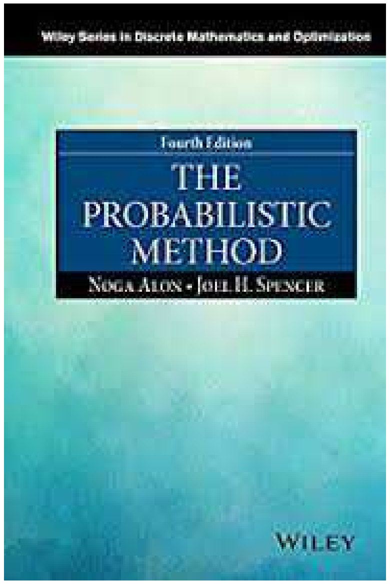
Introduction

The topic course is mostly based the textbook “The probabilistic Method” by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

The second moment method

- Lovász Local Lemma
- Property B
- k-coloring of $\mathbb{R}$
- Ramsey numbers $R(k, k)$
- Ramsey numbers $R(3, k)$
- Directed cycles
- Linear Arboricity

Lovász Local Lemma

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- $A_1, A_2, \ldots, A_n$: n events in an arbitrary probability spaces.
- A dependency digraph $D = (V, E)$: if for each A_i, A_i is mutually independent to all the events $\{A_j : A_iA_j \notin E\}$.

Lovász Local Lemma, general case: If there are real number $x_1, \ldots, x_n$ such that $0 \leq x_i < 1$ and $\Pr(A_i) \leq x_i \prod_{(i,j)\in E}(1 - x_j)$ for all $1 \leq i \leq n$. Then

$$\Pr\left(\wedge_{i=1}^{n} \bar{A}_i\right) \geq \prod_{i=1}^{n}(1 - x_i) > 0.$$

Proof

Proof: Inductively prove that for any $S \subset [n]$, $|S| = s < n$, $i \notin S$,

$$\Pr\left[A_i \mid \wedge_{j \in S} \bar{A}_j\right] \leq x_i.$$

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$$\Pr\left[A_i \mid \wedge_{j \in S} \bar{A}_j\right] = \frac{\Pr\left[A_i \wedge (\wedge_{j \in S_1} \bar{A}_j) \mid \wedge_{j \in S_2} \bar{A}_j\right]}{\Pr\left[\wedge_{j \in S_1} \bar{A}_j \mid \wedge_{j \in S_2} \bar{A}_j\right]}$$

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$$\begin{aligned}\Pr\left[A_i \wedge (\wedge_{j \in S_1} \bar{A}_j) \mid \wedge_{j \in S_2} \bar{A}_j\right] &\leq \Pr\left[A_i \mid \wedge_{j \in S_2} \bar{A}_j\right] \\ &= \Pr[A_i] \leq x_i \prod_{(i,j) \in E(G)} (1 - x_j).\end{aligned}$$

Continue

Write $S_1 = \{j_1, j_2, \ldots, j_r\}$.

$$\begin{aligned}
&\Pr\left[\wedge_{j\in S_1}\bar{A}_j \mid \wedge_{j\in S_2}\bar{A}_j\right] \\
&= \prod_{l=1}^{r}\left(1 - \Pr\left[A_{j_l} \mid \bar{A}_{j_{l+1}} \wedge \cdots \wedge A_{j_r} \wedge_{j\in S_2} \bar{A}_j\right]\right) \\
&\geq \prod_{l=1}^{r}(1 - x_{j_l}) \\
&\geq \prod_{(i,j)\in E(G)} (1 - x_j).
\end{aligned}$$

Thus,

$$\Pr\left[A_i \mid \wedge_{j\in S}\bar{A}_j\right] \leq x_i.$$

Continue

$$\Pr\left[\wedge_{i=1}^{n}\bar{A}_i\right] = (1-\Pr[A_1])(1-\Pr[A_2|\bar{A}_1])\cdots$$
$$\cdots\left(1-\Pr\left[A_n|\wedge_{i=1}^{n-1}\bar{A}_i\right]\right)$$
$$\geq \prod_{i=1}^{n}(1-x_i).$$

The proof is finished. □

Symmetric Case

Lovász Local Lemma, symmetric case: Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other event A_j but at most d, and that $\Pr(A_i) \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) < 1$, then $\Pr(\wedge_{i=1}^{n} \bar{A}_i) > 0$.

Property B

Theorem: Let $H = (V, E)$ be a hypergraph in which every edge has at least k elements, and suppose that each edge of H intersects at most d other edges. If $e(d+1) \leq 2^{k-1}$, then H has property B.

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Proof: Color each vertex in two colors randomly and independently. For each edge $f \in E$, let A_f be the event that f is monochromatic. Then

$$\Pr(A_f) = 2^{1-|f|} \leq 2^{1-k}.$$

A_f is independent to all event but at most d. Aplly LLL. $\square$

k-coloring of $\mathbb{R}$

Let $c\colon \mathbb{R} \to \{1, 2, \ldots, k\}$ be a k-coloring of $\mathbb{R}$. A set $T \subset \mathbb{R}$ is **multicolored** if $c(T) = \{1, 2, \ldots, k\}$.

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Theorem: Let m and k be two positive intergers satisfying

$$e(m(m-1)+1)k(1-\frac{1}{k})^m \leq 1.$$

Then, for any set S of m real numbers there is a k-coloring so that each translantion $x + S$ (for $x \in \mathbb{R}$) is multicolored.

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Then, for any set S of m real numbers there is a k-coloring so that each translantion $x+S$ (for $x \in \mathbb{R}$) is multicolored.

The condition is satisfied if $m > (3+o(1))k \log k$.

Proof

First we use LLL to prove "For any finite set $X \subset \mathbb{R}$, there is a k-coloring so that $x + S$ (for all $x \in X$) is multi-colored."

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Let $Y = \cup_{x \in X}(x + S)$. Color numbers in Y in k-colors randomly and independently. Let A_x be the event that $x + S$ is not multi-colored.

$$\Pr(A_x) \leq k(1 - \frac{1}{k})^m.$$

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A_x depends on A_y if $(x + S) \cap (y + S) \neq \emptyset$. Equivalently, $y - x \in S - S$. There are at most $m(m - 1)$ such events.

$$d \leq m(m - 1).$$

continue

Apllying LLL, we get

$$\Pr(\wedge_{x \in X} \bar{A}_x) > 0.$$

Then by Tikhonov's theorem, $[k]^{\mathbb{R}}$ is compact. For any $x \in \mathbb{R}$, let

$$C_x = \{c \in [k]^{\mathbb{R}} \colon x + S \text{ is multi-colored}\}.$$

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$$C_x = \{c \in [k]^{\mathbb{R}} \colon x + S \text{ is multi-colored}\}.$$

Now C_x is a closed set and $\cap_{x \in X} C_x \neq \emptyset$ for any finite X. Then $\cap_{x \in \mathbb{R}} C_x \neq \emptyset$. □

Ramsey numbers

Theorem (Spencer, 1975)

$$R(k,k) \geq (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}.$$

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Best bounds for $R(r,k)$ (for fixed r and k large),

$$c\left(\frac{k}{\log k}\right)^{(r+1)/2} < R(r,k) < (1+o(1))\frac{k^{r-1}}{\log^{r-2} k}.$$

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- For $T \in \binom{[n]}{k}$, let B_T be the event that T is an independent set of G; $\Pr(B_t) = (1-p)^{\binom{k}{2}}$.
- Dependence graph: $d_{SS} \leq 3n$, $d_{ST} \leq 3\binom{n}{k-2}$, $d_{TS} \leq \binom{k}{2}n$, and $d_{TT} \leq \binom{k}{2}\binom{n}{k-2}$.

Proof

By LLL, we only require

$$\begin{aligned} p^3 &\leq x(1-x)^{3n}(1-y)^{3\binom{n}{k-2}} \\ (1-p)^{\binom{k}{2}} &\leq y(1-x)^{\binom{k}{2}n}(1-y)^{\binom{k}{2}\binom{n}{k-2}}. \end{aligned}$$

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We can choose $p = c_1 n^{-1/2}$, $k = c_2 n^{1/2}\log n$, $x = c_3 n^{-3/2}$, and $y = c_4/\binom{n}{k}$.

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This gives $R(3,k) > c_5 k^2/\log^2 k$. $\square$

$R(4, k)$

Best bounds for $R(r, k)$ (for fixed r and k large),

$$c\left(\frac{k}{\log k}\right)^{(r+1)/2} < R(r, k) < (1 + o(1))\frac{k^{r-1}}{\log^{r-2} k}.$$

Erdős conjecture $250: Prove

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for some constants $c', c > 0$.

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The best lower bound is using LLL; $R(4,k) > c'\frac{k^{2.5}}{\log^{2.5} k}$.

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Proof: First we can assume every out-degree is δ by deleting some edges if necessary. Consider $f: V \to \mathbb{Z}_k$. Bad event A_v: no $u \in \Gamma^{+}(v)$ with $f(u) = f(v) + 1$.

$$\Pr(A_v) = (1 - 1/k)^{\delta}.$$

Each event depends on at most $\delta\Delta$ others. Apply LLL. □

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If the conjecture is true, then it is tight.

$$\mathrm{la}(G) \geq \frac{nd}{2(n-1)} > \frac{d}{2}.$$

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- G is d-regular if $d^+(v) = d^-(v) = d$ for any vertex v.
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DLA conjecture for d implies LA conjecture for $2d$.

A proposition

Proposition: *Let* $H = (V, E)$ *be a graph with maximum degree* d*, and let* $V = V_1 \cup V_2 \cup \cdots \cup V_r$ *be a partition of* V*. If* $|V_i| \geq 2ed$*, then there is an independent set of vertices* W *that contains a vertex from each* V_i*.*

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Proof: WLOG, we assume

$$|V_1| = |V_2| = \cdots = |V_r| = \lceil 2ed \rceil = g.$$

Pick from each V_i a vertex randomly and independently. Let W be the random set of the vertices picked. For each edge f, let A_f be the event that both ends in W. The maximum degree in the dependence graph is at most $2gd - 1$. We have $e \cdot 2gd \cdot \frac{1}{g^2} = \frac{2ed}{g} < 1$. Apply LLL. □

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Theorem Let $G = (U, F)$ be a d-regular digraph with directed girth $g \geq 8ed$. Then

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Proof: Using Hall's matching theorem, we can partition F into d pairwise disjoint 1-regular spanning subgraphs $F_1, \ldots, F_d$ of G.

Continue

Each F_i is a union of vertex disjoint directed cycles. Let $V_1, \ldots, V_r$ are the sets of edges of all cycles. Then

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Apply the proposition to the line-graph H of G. Note H is $4d - 2$-regular.

There exists an independent set M_1 of H. Now $M_1, F_1 \setminus M_1, \ldots, F_d \setminus M_1$ forms $d + 1$ linear directed forests. $\square$

General d-regular graphs

Theorem [Alon 1988] There is an absolute constant $c > 0$ such that for every d-regular directed graph G

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The error terms can be improved to $cd^{2/3}\log^{1/3} d$.

Proof

Pick a prime p. Color each vertex randomly and uniformly into p colors. I.e., consider a random map

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Define for $i \in \mathbb{Z}_p$,

$$E_i = \{(u, v) \in E : f(v) = f(u) + i\}.$$

Let $G_i = (V, E_i)$ and

- Δ_i^+: the maximum out-degree of G_i.
- Δ_i^-: the maximum in-degree of G_i.
- Δ_i: the maximum of Δ_i^+ and Δ_i^-.

Continue

There exists a f satisfying

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- G_i has large girth $\geq p$ for $i \neq 0$.
- All G_i can be completed to a Δ_i-regular directed graph without deceasing the girth.

$$\mathrm{dla}(G) \leq 2\Delta_0 + \sum_{i=1}^{p-1}(\Delta_i + 1) \leq d + d/p + p + C\sqrt{dp\log d}.$$

Now choose $p \sim d^{1/2}$.