Topic Course on Probabilistic Methods

(Week 4)

Second Moment Method

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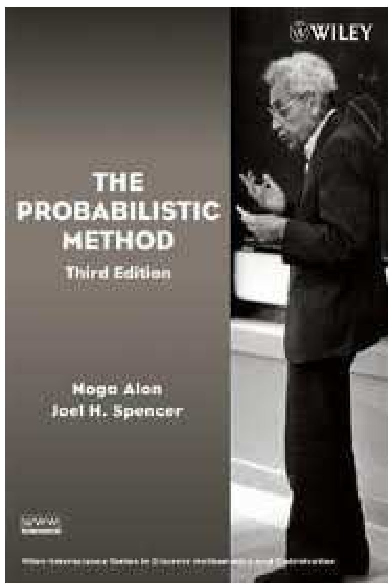
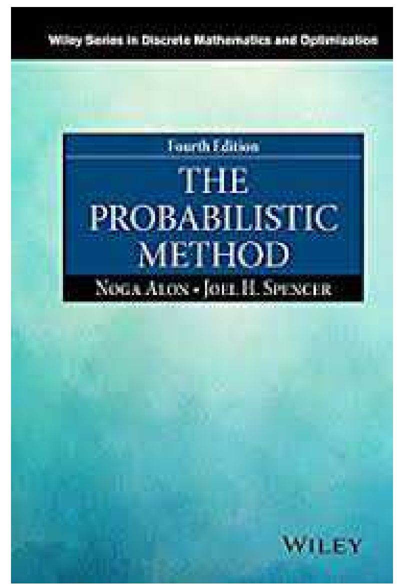
Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

The second moment method

- Variance
- Chebyshev's inequality
- Number of prime factors
- Counting K_4 in $G(n, p)$
- Counting balanced graphs
- Clique number of $G(n, \frac{1}{2})$
- Distinct sum

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If $X = \sum_{i=1}^{n} X_i$, then

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If $X_1, \ldots, X_n$ are mutually independent, then
$\mathrm{Var}(X) = \sum_{i=1}^n \mathrm{Var}(X_i).$

Chebyshev's Inequality

- $\mathrm{E}(X) = \mu$,
- $\mathrm{Var}(X) = \sigma^2$.

Theorem [Chebyshev's Inequality]: For any positive λ,

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Proof:

$$\begin{aligned} \sigma^2 &= \mathrm{Var}(X) \\ &= \mathrm{E}((X-\mu)^2) \\ &\geq \lambda^2\sigma^2 \Pr(|X - \mu| \geq \lambda\sigma). \end{aligned}$$

Number theory

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Theorem [Turán (1934)]: Let $\omega(n) \to \infty$ arbitrarily slowly. Then the number of x in $[n] := \{1, 2, \ldots, n\}$ such that

$$|\nu(x) - \ln \ln n| > \omega(n)\sqrt{\ln \ln n}.$$

is $o(n)$.

Proof

Let x be randomly chosen from $[n]$. For p prime set

$$X_p = \begin{cases} 1 & \text{if } p \mid x, \\ 0 & \text{otherwise.} \end{cases}$$

Set $M = n^{1/10}$ and $X = \sum_{p \le M} X_p$. Then

$$\nu(x) - 10 \le X(x) \le \nu(x).$$

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&= \frac{\lfloor n/pq \rfloor}{n} - \frac{\lfloor n/p \rfloor}{n}\frac{\lfloor n/q \rfloor}{n} \\
&\leq \frac{1}{pq} - \left(\frac{1}{p} - \frac{1}{n}\right)\left(\frac{1}{q} - \frac{1}{n}\right) \\
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$$\mathrm{Var}(X) = \sum_{p \le M} \mathrm{Var}(X_p) + \sum_{p \ne q} \mathrm{Cov}(X_p, X_q) = \ln\ln n + O(1).$$

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By Chebyshev's inequality, we have

$$\Pr(|X - \ln\ln n| > \lambda\sqrt{\ln\ln n}) < \lambda^{-2} + o(1).$$

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□

Theorem [Erdős-Kac (1940):] For any fixed λ, we have

$$\lim_{n\to\infty} \frac{1}{n}\left|\{x\colon 1 \le x \le n, \nu(x) \ge \ln\ln n + \lambda\sqrt{\ln\ln n}\}\right|$$
$$= \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

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- Write $X_i \sim X_j$ if $i \neq j$, and the events X_i, X_j are not independent. Let $\Delta = \sum_{i \sim j} \mathrm{Pr}(A_i \wedge A_j)$. If $E(X) \to \infty$ and $\Delta = o(E(X)^2)$, then $X > 0$ almost always.

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- Let $\Delta^* = \max_i \sum_{j \sim i} \Pr(A_j | A_i)$. If $E(X) \to \infty$ and $\Delta^* = o(E(X))$, then $X > 0$ almost always.

Erdős-Rényi model $G(n, p)$

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A function $r(n)$ is called a threshold function for some property P if

- If $p \ll r(n)$, then $G(n,p)$ does not satisfy P almost always.
- If $p \gg r(n)$, then $G(n,p)$ satisfy P almost always.

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If $p \ll n^{-2/3}$ then $\mathrm{E}(X) = o(1)$ and so $X = 0$ almost surely.

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$S \sim T$ if $|S \cap T| \geq 2$. Thus,

$$\Delta^* = O(n^2p^5) + O(np^3) = o(n^4p^6) = o(\mathrm{E}(X)).$$

Hence $X > 0$ almost surely. □

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Theorem: *Let H be a balanced graph with v vertices and e edges. Let $A(G)$ be the event that H is a subgraph (not necessarily induced) of G. Then $p = n^{-v/e}$ is the threshold function for A.*

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If $p \ll n^{-v/e}$, then $\mathrm{E}(X) = o(1)$; $X = 0$ almost surely.

If $p \gg n^{-v/e}$, then $\mathrm{E}(X) \to \infty$. We have

$$\Delta^* = O(\sum_{i=2}^{v} n^{v-i} p^{e-(ie/v)}) = o(\mathrm{E}(X)).$$

Two other results

Theorem: *Let H be a strictly balanced graph with v vertices and e edges and a automorphisms. Let X be the copies of H in $G(n,p)$. Assume $p \gg n^{-v/s}$. Then almost always*

$$X \sim \frac{n^v p^e}{a}.$$

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Theorem: *Let H be any fixed graph. For every subgraph H' of H (including H itself) let $X_{H'}$ denote the number of copies of H' in $G(n,p)$. Assume p is such that $E(X_{H'}) \to \infty$ for every H'. Then almost surely*

$$X_H \sim \mathrm{E}(X_H).$$

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Proof: For each k-set S, let X_S be the indicator random variable that S is a clique and $X = \sum_{|S|=k} X_S$.

$$\mathrm{E}(X) = \binom{n}{k} 2^{-\binom{k}{2}} = f(k).$$

Continue

$$\Delta^* = \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{\binom{i}{2} - \binom{k}{2}}.$$

$$\frac{\Delta^*}{E(|X|)} = \sum_{i=2}^{k-1} g(i),$$

where $g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}$.

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where $g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}$. Then

$$g(i) \leq \max\{g(2), g(k-1)\} = o(n^{-1}).$$

Thus, $\Delta^* = o(\mathrm{E}(X))$. $\square$

Remark

$$\frac{f(k+1)}{f(k)} = \frac{n-k}{k+1}2^{-k}.$$

For $k \sim 2\log_2 n$, then

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Remark

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Let k_0 be the value with $f(k_0) \geq 1 > f(k_0+1)$. For most of n, $f(k)$ will jump from very large to very small. With high probability, $\omega(G) = k_0$.

Distinct sums

- A set $x_1, \ldots, x_k$ of positive integers is said to have **distinct sums** if all sums

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Erdős offered $300 for a proof or disproof that

$$f(n) \leq \log_2 n + O(1).$$

An easy lower bound

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Proof: Let $k = \lfloor \log_2 n \rfloor + 1$. For $i = 1, 2, \ldots, k$, set $x_i = 2^{i-1}$. We have

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For $\epsilon_1, \ldots, \epsilon_k \in \{0, 1\}$, the binary number

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has the value $\sum_{i=1}^{k} \epsilon_i x_i$.

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Since every number has a unique base-2 representation, the set $\{x_1, x_2, \ldots, x_k\}$ has distinct sums. □

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The function $g(k) := \frac{2^k}{k}$ is an increasing function. At $k = \log_2 n + \log_2 \log_2 n + 2$, we have

$$\frac{2^k}{k} = \frac{4n \log_2 n}{\log_2 n + \log_2 \log_2 n + 2} > n.$$

Thus, $k \leq \log_2 n + \log_2 \log_2 n + 2$. □

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Proof: Fix $x_1, x_2, \ldots, x_k$ with distinct sums. Let $\epsilon_1, \ldots, \epsilon_k$ be uniform independent $\{0,1\}$-random variables. Let $X = \sum_{i=1}^{k} \epsilon_i x_i$. We have

A nontrivial result

Theorem: $f(n) < \log_2 n + \frac{1}{2}\log_2\log_2 n + O(1)$.

Proof: Fix $x_1, x_2, \ldots, x_k$ with distinct sums. Let $\epsilon_1, \ldots, \epsilon_k$ be uniform independent $\{0, 1\}$-random variables. Let $X = \sum_{i=1}^{k} \epsilon_i x_i$. We have

- $\mathrm{E}(X) = \frac{1}{2}\sum_{i=1}^{k} x_i$.

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- $\mathrm{E}(X) = \frac{1}{2} \sum_{i=1}^k x_i$.
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By Chebyshev's inequality, for any $\lambda > 1$,

$$\Pr\left[|X-\mu| \geq \lambda n\sqrt{k}/2\right] \leq \frac{1}{\lambda^2}.$$

continue

By the property of distinct sums, $\Pr(X = x)$ is either 0 or 2^{-k} for any integer x. Thus,

$$\Pr\left[|X-\mu| < \lambda n\sqrt{k}/2\right] \leq \frac{\lambda n\sqrt{k}+1}{2^k}.$$

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$$\Pr\left[|X - \mu| < \lambda n\sqrt{k}/2\right] \leq \frac{\lambda n\sqrt{k} + 1}{2^k}.$$

Combining with previous Chebyshev's inequality, we get

$$1 \leq \frac{1}{\lambda^2} + \frac{\lambda n\sqrt{k} + 1}{2^k}.$$

Choose $\lambda = \sqrt{3}$ and $k = \log_2 n + \frac{1}{2}\log_2\log_2 n + C$ with sufficiently large constant C. The above inequality is not satisfied. Contradiction! □