Topic Course on Probabilistic Methods

(Week 2)

Linearity of Expectation (2)

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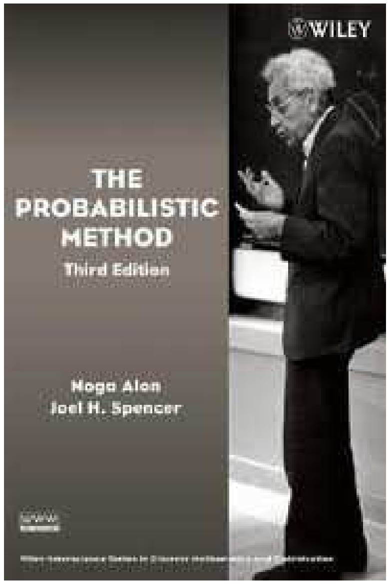
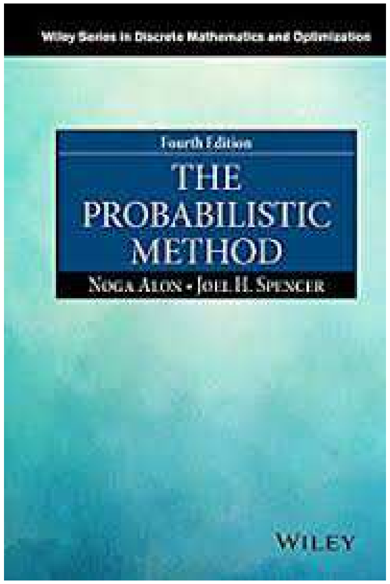
Introduction

The topic course is mostly based the textbook “The probabilistic Method” by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Linearity of Expectation (2)

- Disjoint pairs
- k-sets
- Balancing vectors
- Unbalancing lights
- Brégman's Theorem
- Hamliton paths
- Independence number
- Turán Theorem

Disjoint pairs

- $\mathcal{F} \subset 2^{[n]}$.
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Theorem [Alon-Frankl, 1985]: If $|\mathcal{F}| = 2^{(1/2+\delta)n}$, then

$$d(\mathcal{F}) < |\mathcal{F}|^{2-\delta^2/2}.$$

Proof

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$$
\begin{aligned}
& \Pr(|\cup_{i=1}^{t} A_i| \leq \frac{n}{2}) \\
\leq & \sum_{|S|=\frac{n}{2}} \Pr(\wedge_{i=1}^{t}(A_i \subset S)) \\
\leq & \; 2^n \left(\frac{2^{n/2}}{2^{(1/2+\delta)n}} \right)^t \\
= & \; 2^{n(1-\delta t)}.
\end{aligned}
$$

continue

Let $v(B) = |\{A \in \mathcal{F} \colon B \cap A = \emptyset\}|$. Then

$$\sum_B v(B) = 2d(\mathcal{F}) \geq 2m^{2-\delta^2/2}.$$

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Let Y be a random variable whose value is the number of members $B \in \mathcal{F}$ that is disjoint to all A_i $1 \leq i \leq t$. Then

$$\begin{aligned} E(|Y|) &= \sum_{B \in \mathcal{F}} \left(\frac{v(B)}{m} \right)^t \\ &\geq \frac{1}{m^{t-1}} \left(\frac{\sum_B v(B)}{m} \right)^t \\ &\geq 2m^{1-t\delta^2/2}. \end{aligned}$$

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Since $Y \leq m$, we get

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Thus, with positive probability, $|\cup_{i=1}^t A_i| > \frac{n}{2}$ and $\cup_{i=1}^t A_i$ is disjoint to more than $2^{n/2}$ members of $\mathcal{F}$. Contradiction. $\square$

Linearity of expectation

Let $X_1, X_2, \ldots, X_n$ be random variables and $X = \sum_{i=1}^{n} c_i X_i$. Then

$$\mathrm{E}(X) = \sum_{i=1}^{n} c_i \mathrm{E}(X_i).$$

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Philosophy: There is a point in the probability space for which $X \geq \mathrm{E}(X)$ and a point for $X \leq \mathrm{E}(X)$.

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- For $S \subset V$, let $h(S) = \sum_{F \subset S} h(F)$.
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Theorem: Suppose $h(F) = +1$ for all crossing k-sets F. Then there is an $S \subset V$ for which

$$|h(S)| \geq c_k n^k.$$

Here $c_k > 0$.

A Lemma

Lemma: *Let* P_k *be the set of all homogeneous polynomials* $f(p_1, \ldots, p_k)$ *of degree* k *with all coefficients have absolute value at most one and* $p_1 p_2 \cdots p_k$ *having coefficient one. Then for all* $f \in P_k$ *there exists* $p_1, \ldots, p_k \in [0, 1]$ *with*

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Proof: Let $M(f) = \max_{p_1,\ldots,p_k} |f(p_1, \ldots, p_k)|$. Note P_k is compact and M is continuous. M reaches its minimum value c_k at some point f_0. We have

$$c_k = M(f_0) > 0. \qquad \square$$

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$$\mathrm{E}(X_F) = h(F) p_1^{a_1} \cdots p_k^{a_k}.$$

continue

$$\mathrm{E}(X) = \sum_{\sum_{i=1}^{k} a_i = k} p_1^{a_1} \cdots p_k^{a_k} \sum_{F \text{ of type } (a_1,\ldots,a_k)} h(F).$$

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Now select $p_1, \ldots, p_k \in [0, 1]$ with $|f(p_1, \ldots, p_k)| \geq c_k$.
Then $\mathrm{E}(|X|) \geq |\mathrm{E}(X)| \geq c_k n^k$.

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There exists a S such that $|h(S)| \geq c_k n^k$. $\square$

Balancing vectors

Theorem: Let $v_1, \ldots, v_n$ are n unit vector in $\mathbb{R}^n$. Then there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \leq \sqrt{n},$$

and also there exist $\epsilon_1, \ldots, \epsilon_n = \pm 1$ so that

$$\|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\| \geq \sqrt{n}.$$

Proof

Let $\epsilon_1, \ldots, \epsilon_n$ be selected uniformly and independently from $\{+1, -1\}$. Let $X = \|\epsilon_1 v_1 + \cdots + \epsilon_n v_n\|^2$.

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$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}(\sum_{i,j=1}^{n} \epsilon_i \epsilon_j v_i \cdot v_j) \\
&= \sum_{i,j=1}^{n} \mathrm{E}(\epsilon_i \epsilon_j) v_i \cdot v_j \\
&= \sum_{i,j=1}^{n} \delta_i^j v_i \cdot v_j \\
&= \sum_{i=1}^{n} \|v_i\|^2 = n.
\end{aligned}$$

An extension

Theorem: Let $v_1, \ldots, v_n \in \mathbb{R}^n$, all $\|v_i\| \leq 1$. Let $p_1, p_2, \ldots, p_n \in [0,1]$ be arbitrary and set $w = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n$. Then there exist $\epsilon_1, \ldots, \epsilon_n \in \{0,1\}$ so that setting $v = \epsilon_1 v_1 + \cdots + \epsilon_n v_n$,

$$\|w - v\| \leq \frac{\sqrt{n}}{2}.$$

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Hint: Pick ϵ_i independently with

$$\Pr(\epsilon_i = 1) = p_i, \quad \Pr(\epsilon_i = 0) = 1 - p_i.$$

The proof is similar.

Unbalancing lights

Theorem: Let $a_{ij} = \pm 1$ for $1 \leq i, j \leq n$. Then there exist $x_i, y_j = \pm 1$, $1 \leq i, j \leq n$ so that

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq \left(\sqrt{\frac{2}{\pi}} + o(1) \right) n^{3/2}.$$

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Proof: Choose $y_j = 1$ or -1 randomly and independently. Let $R_i = \sum_{i=1}^{n} a_{ij} y_j$. Let x_i be the sign of R_i. Then

$$\sum_{i,j=1}^{n} a_{ij} x_i y_j = \sum_{i=1}^{n} |R_i|.$$

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Each R_i has the distribution $S_n = \sum_{i=1}^{n} X_i$, where X_i's are independent uniform $\{-1, 1\}$ random variables.

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Hence,

$$
\sum_{i=1}^{n} \mathrm{E}(|R_i|) = \left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}.
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Brégman's Theorem

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Brégman's Theorem (1973): $\mathrm{per}(A) \leq \prod_{1 \leq i \leq n} (r_i!)^{1/r_i}$.

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Claim: $\mathrm{per}(A)) \leq G(L)$.

continue

For any fixed τ. Assume $\tau(1) = 1$. By re-ordering, assume the first row has ones in the first $r := r_1$ columns. For $1 \leq j \leq r$ let t_j be the permanent of A with the first row and j-th column removed (i.e., $\sigma(1) = j$). Let

$$t = \frac{t_1 + \cdots + t_r}{r} = \frac{\text{per}(A)}{r}.$$

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By induction,

$$G(R_2 \cdots R_n | \sigma(1) = j) \geq t_j.$$

$$G(L) \geq \prod_{j=1}^{r} (rt_j)^{t_j/per(A)} = r \prod_{j=1}^{r} (t_j)^{t_j/rt}.$$

continue

Since $\left(\prod_{j=1}^{r} t_j^{t_j}\right)^{\frac{1}{r}} \geq t^t$, we have

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Now we calculate $G[L]$ conditional on a fixed σ. By reordering, assume $\sigma(i) = i$ for all i. Note

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$$G(R) = G(\prod_{i=1}^{n} R_i) = \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

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We have

$$\mathrm{E}(X) = \sum_{\sigma \in S_n} \mathrm{E}(X_\sigma) = n!2^{1-n}.$$

Done! $\square$

Related problem

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Szele [1943] proved

$$\frac{1}{2} \leq \lim_{n\to\infty} \left(\frac{P(n)}{n!}\right)^{1/n} \leq \frac{1}{2^{3/4}}.$$

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Theorem [Alon, 1990]: $P(n) \leq cn^{3/2}\frac{n!}{2^{n-1}}$.

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$$F(T) = \mathrm{per}(A_T) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Here r_i is i-th row sum of A_T; $\sum_{i=1}^{n} r_i = \binom{n}{2}$.

A convex inequality

Lemma: For every two integers a, b satisfying $b \geq a + 2 > a \geq 1$, we have

$$(a!)^{1/a}(b!)^{1/b} < ((a+1)!)^{1/(a+1)}((b-1)!)^{1/(b-1)}.$$

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It can be proved using $x! > (\frac{x+1}{2})^x$ for $x \geq 2$. $\square$

Proof of theorem

Observe that $\prod_{i=1}^{n}(r_i!)^{1/r_i}$ achieves the maximum when all r_i's are almost equal. We get

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Construct a new tournament T' for T by adding a new vertex v, where the edges from v to T are oriented randomly and independently. Every Hamiltonian path in T can be extended to a Hamiltonian cycle in T' with probability $\frac{1}{4}$. We have

$$P(T) \leq \frac{1}{4}C(T') = O\left(n^{3/2}\frac{n!}{2^{n-1}}\right). \qquad \square$$

Independence number

$\alpha(G)$: the maximal size of an independent set of a graph G.

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Let X_v be the indicator random variable for $v \in I$.

$$\mathrm{E}(X_v) = \Pr(v \in I) = \frac{1}{d_v + 1}.$$

$$\alpha(G) \geq \mathrm{E}(|I|) = \sum_v \frac{1}{d_v + 1}.$$

Turán Theorem

Turán number $t(n, H)$: the maximum integer m such that there is a graph on n vertices containing no subgraph H.

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Turán Theorem: For $n = km + r$ $(0 \leq r < k)$,

$$t(n, K_{k+1}) = m^2 \binom{k}{2} + rm(k-1) + \binom{r}{2}.$$

The equality holds if and only if G is the complete k-partite graph with equitable partitions, denoted by $G_{n,k}$.

Dual version

For any $k \leq n$, let q, r satisfy $n = kq + r$, $0 \leq r < k$. Let $e = r\binom{q+1}{2} + (m - r)\binom{q}{2}$.

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When the equality holds, I is a constant. G can not contain an induced P_2. Therefore $G = \bar{G}_{n,k}$.

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- **Erdős-Bondy-Simonovits (1963,1974):** $t(n, C_{2k}) \leq ckn^{1+1/k}$.

Open conjectures

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- **Conjecture (\$250 for proof and \$100 for disproof:)** Suppose H is a bipartite graph. Prove or disprove that $t(n, H) = O(n^{3/2})$ if and only if H does not contain a subgraph each vertex of which has degree > 2.