Topic Course on Probabilistic Methods

(Week 13)

Discrepancy

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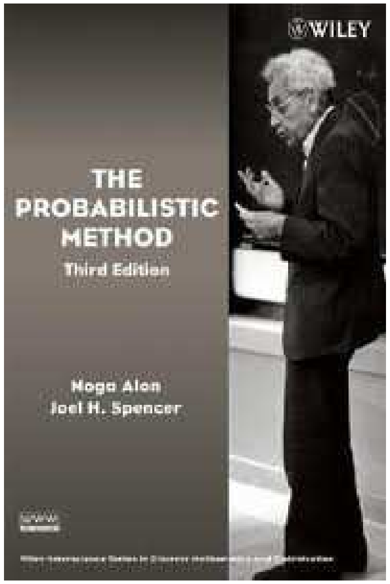
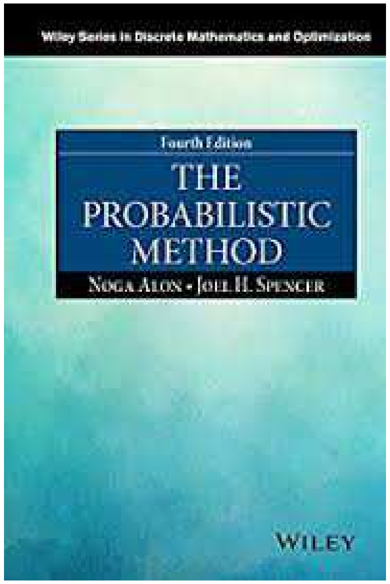
Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviation inequalities (3 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Random graphs

- Discrepancy
- Linear discrepancy
- Hereditary discrepancy
- Lower bound
- The Beck-Fiala Theorem

Discrepancy

- Ω : a finite set.
- $\chi : \Omega \to \{-1, 1\}$.
- For any $A \subset \Omega$, $\chi(A) = \sum_{a \in A} \chi(a)$.
- For $\mathcal{A} \subset 2^{\Omega}$,

$$\mathrm{disc}(\mathcal{A}, \chi) = \max_{A \in \mathcal{A}} |\chi(A)|;$$

$$\mathrm{disc}(\mathcal{A}) = \min_{\chi} \mathrm{disc}(\mathcal{A}, \chi).$$

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Geometric meaning: Assume $|\Omega| = m$, $|\mathcal{A}| = n$, and $B = (b_{ij})$ be the $m \times n$ incidence matrix. Let $v_1, v_2, \ldots, v_n$ be the column vector of B. Then

$$\mathrm{disc}(\mathcal{A}) = \min | \pm v_1 \pm v_2 \pm \cdots \pm v_n|_{\infty}.$$

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Proof: Let $\chi\colon \Omega \to \{-1, 1\}$ be random. Let $\lambda = \sqrt{2m\ln(2n)}$. By Azuma's inequality, we have

$$\Pr(|\chi(A)| > \lambda) < 2e^{-\lambda^2/(2|A|)} \leq \frac{1}{n}.$$

With positive probability, we have $|\chi(A)| \leq \lambda$ holds for every $A \in \mathcal{A}$. Therefore $\mathrm{disc}(A) \leq \lambda$. □

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- In his paper, $K = 6$ is proved; here we will prove a weaker version with $K = 11$.
- If $\mathcal{A}$ consists on n sets on m points and $m \leq n$. Then

$$\mathrm{disc}(\mathcal{A}, \chi) < K\sqrt{m}\sqrt{\ln(n/m)}.$$

Basic entropy

Let X be a random variable taking values in some range S. The **binary entropy** of X, denoted by $H(X)$ is defined by

$$H(X) = -\sum_{x \in S} \Pr(X = x) \log_2 \Pr(X = x).$$

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Sub-additive property:

$$H(X, Y) \leq H(X) + H(Y).$$

Here (X, Y) is the random variable taking values in $S \times T$ (where T is the range of Y.)

Proof of entropy inequality

Proof:

$$H(X) + H(Y) - H(X,Y)$$

$$= \sum_{i \in S} \sum_{j \in T} \Pr(X = i, Y = j) \log_2 \frac{\Pr(X = i, Y = j)}{\Pr(X = i)\Pr(Y = j)}$$

$$= \sum_{i \in S} \sum_{j \in T} \Pr(X = i)\Pr(Y = j) f(z_{ij}),$$

where $f(z) = z \log_2 z$ and $z_{ij} = \frac{\Pr(X=i,Y=j)}{\Pr(X=i)\Pr(Y=j)}$. By the convexity inequality of $f(z)$, we have

$$H(X) + H(Y) - H(X,Y) \geq f(1) = 0. \qquad \square$$

A lemma

A map $\chi \colon \Omega \to \{-1, 0, 1\}$ is called a **partial coloring**. When $\chi(a) = 0$ we say a is uncolored.

Lemma 13.2.2: *Let $\mathcal{A}$ be a family of n subsets of an n-set Ω. Then there is a partial coloring χ with at most $10^{-9}n$ points uncolored such that $|\chi(\mathcal{A})| \leq 10\sqrt{n}$ for all $A \in \mathcal{A}$.*

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Proof: Let $\mathcal{A} := \{A_1, A_2, \ldots, A_n\}$. Consider a random coloring

$$\chi\colon \Omega \to \{-1, 1\}.$$

For $1 \leq i \leq n$ define

$$b_i = \text{ nearest integer to } \frac{\chi(A_i)}{20\sqrt{n}}.$$

continue

By Chernoff's inequality, we have

$$
\begin{aligned}
\Pr(b_i = 0) &> 1 - 2e^{-50}, \\
\Pr(b_i = 1) = \Pr(b_i = -1) &< 2^{-50}, \\
\Pr(b_i = 2) = \Pr(b_i = -2) &< 2^{-450}, \\
&\vdots \\
\Pr(b_i = s) = \Pr(b_i = -s) &< 2^{-50(2s-1)^2}.
\end{aligned}
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continue

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\end{aligned}$$

Recall the entropy $H(b_i)$ is defined as

$$H(b_i) = \sum_{s=-\infty}^{s=\infty} -\Pr(b_i = s) \log_2 \Pr(b_i = s).$$

continue

$$H(b_i) \leq (1-2e^{-50})[-\log_2(1-2e^{-50})] + 2e^{-50}[-\log_2 e^{-50}] + 2e^{-550}[-\log_2 e^{-450}] + \cdots < \epsilon = 3 \times 10^{-20}.$$

By the subadditive property, we have

$$H(b_1, b_2, \ldots, b_n) \leq \sum_{i=1}^{n} H(b_i) \leq \epsilon n.$$

If a random variable Z assumes no value with probability greater than 2^{-t}, then $H(Z) \geq t$. This implies there is a particular n-tuple $(s_1, s_2, \ldots, s_n)$ so that

$$\Pr((b_1, \ldots, b_n) = (s_1, \ldots, s_n)) \geq 2^{-\epsilon n}.$$

continue

Since every coloring has equal probability 2^{-n}, there is a set $\mathcal{C}$ consisting of at least $2^{(1-\epsilon)n}$ colorings $\chi\colon \Omega \to \{-1, 1\}$, all having the same value $(b_1, b_2, \ldots, b_n)$.

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Kleitman (1966) proved that if $|\mathcal{C}| \geq \sum_{i \leq r} \binom{n}{i}$ with $r \leq n/2$ then $\mathcal{C}$ has diameter (of Hamming distance) at least $2r$.

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Let $r = \alpha n$ and $2^{H(\alpha)} \leq 2^{1-\epsilon}$. Taylor series expansion gives

$$H(\frac{1}{2} - x) \sim 1 - \frac{2}{\ln 2} x^2.$$

Thus $\mathcal{C}$ has diameter at least $n(1 - 10^{-9})$. Choose $\chi_1, \chi_2 \in \mathcal{C}$ be at the maximal distance. Let $\chi = \frac{\chi_1 - \chi_1}{2}$. Then the partial coloring χ satisfying all requirements. $\square$

Iteration

We will iterate the procedure to color the remaining uncolored points.

Lemma 13.2.3: Let $\mathcal{A}$ be a family of n subsets of an m-set Ω with at most $10^{-40}m$ points uncolored so that

$$\chi(A) < 10\sqrt{m}\sqrt{\ln(n/m)}$$

for all $A \in \mathcal{A}$.

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The proof is similar by define

$$b_i = \text{ nearest integer to } \frac{\chi(A_i)}{20\sqrt{m\ln(n/m)}}.$$

The detail is omitted. □

Proof of Theorem

Proof: Apply Lemma 13.2.2 to find a partial coloring χ^1 and then apply Lemma 13.2.3 repeatedly on the remaining uncolored points giving $\chi^2, \chi^3, \ldots$ until all points have been colored. Let $\chi = \sum_{i \geq 1} \chi^i$.

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$$\begin{aligned}
|\chi(A)| &\leq 10\sqrt{n} + 10\sqrt{10^{-9}n}\sqrt{\ln 10^9} \\
&\quad + 10\sqrt{10^{-49}n}\sqrt{\ln 10^{49}} + 10\sqrt{10^{-89}n}\sqrt{\ln 10^{89}} \\
&\leq 11\sqrt{n}.
\end{aligned}$$

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The statement of case $r < n$ can be proved similarly.

More points than sets

Suppose $m > n$, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ and $\Omega = [n]$. The **linear discrepancy** $\mathrm{lindisc}(\mathcal{A})$ is defined by

$$\mathrm{lindisc}(\mathcal{A}) = \max_{p_1,\ldots,p_m \in [0,1]} \min_{\epsilon_1,\ldots,\epsilon_m \in \{0,1\}} \max_{A \in \mathcal{A}} \left| \sum_{i \in A} (\epsilon_i - p_i) \right|.$$

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Setting all $\epsilon_i = \frac{1}{2}$ and scaling $[0, 1]$ to $[-1, 1]$, we have

$$\begin{aligned} \mathrm{disc}(A) &= \min_{\epsilon'_1,\ldots,\epsilon'_m \in \{-1,1\}} \max_{A \in \mathcal{A}} |\sum_{i \in A} \epsilon'_i| \\ &= 2 \min_{\epsilon_1,\ldots,\epsilon_m \in \{0,1\}} \max_{A \in \mathcal{A}} |\sum_{i \in A} \epsilon_i - \frac{1}{2}| \\ &\leq 2 \cdot \mathrm{lindisc}(A). \end{aligned}$$

A theorem

Theorem 13.3.1 Let $\mathcal{A}$ be a family of n sets on m points with $m \geq n$. Suppose that $\text{lindisc}(\mathcal{A}|_X) \leq K$ for every subset X of at most n points. Then $\text{lindisc}(\mathcal{A}) \leq K$.

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Proof: For $p_1, \ldots, p_m \in [0, 1]$, call index j fixed if $p_i = 0$ or 1 otherwise call it floating, and let F denote the set of floating indices.

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Our goal is to reduce $p_1, p_2, \ldots, p_m$ so that $|F| < n$.

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Our goal is to reduce $p_1, p_2, \ldots, p_m$ so that $|F| < n$.

Suppose $|F| \geq n$. Let $y_1, \ldots, y_m$ be a nonzero solution to the homogeneous system

$$\sum_{j \in A \cap F} y_j = 0, \quad A \in \mathcal{A}.$$

continue

Consider a line

$$p'_j = \begin{cases} p_j + \lambda y_j, & j \in F, \\ p_j, & j \notin F. \end{cases}$$

The line will hit the the boundary of the hypercube Q^m and the intersection point gives a set of $p'_1, \ldots, p'_m$ with the smaller floating indices. Critically, for all $A \in \mathcal{A}$.

$$\sum_{j \in A} p'_j = \sum_{j \in A} p_j + \lambda \sum_{j \in A \cap F} y_j = \sum_{j \in S} p_j.$$

Iterate this process, we get some $p^*_1, \ldots, p^*_m$ with the set X of floating indices satisfying $|X| < n$.

continue

Since $\operatorname{lindisc}(\mathcal{A}|_X) \leq K$, there exists $\epsilon_j, j \in X$ so that

$$\left| \sum_{j \in A \cap X} p_j^* - \epsilon_j \right| \leq K, \quad A \in \mathcal{A}.$$

Extend ϵ_j to $j \in \bar{X}$ by letting $\epsilon_j = p_j^*$. For any $A \in \mathcal{A}$,

$$\begin{aligned} \left| \sum_{j \in A} (p_j - \epsilon_j) \right| &= \left| \sum_{j \in A} (p_j^* - \epsilon_j) \right| \\ &= \left| \sum_{j \in A \cap X} (p_j^* - \epsilon_j) \right| \leq K. \end{aligned}$$

Thus, $\operatorname{lindisc}(\mathcal{A}) \leq K$. □

Hereditary discrepancy

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Theorem 13.3.2: $\mathrm{lindisc}(\mathcal{A}) \leq \mathrm{herdisc}(\mathcal{A})$.

Proof: Set $K = \mathrm{herdisc}(\mathcal{A})$. Let $p_1, \ldots, p_m \in [0, 1]$ be given. Firs assume all p_i have finite expansions in base 2. Let T be the minimal integer so that all $p_i 2^T \in \mathbb{Z}$. Let J be the set of i for which $p_i 2^T$ is odd. As $\mathrm{disc}(\mathcal{A}|_J) \leq K$, there exists $\epsilon_j \in \{-1, 1\}$, so that

$$\left| \sum_{j \in J \cap A} \epsilon_j \right| \leq K, \quad A \in \mathcal{A}.$$

continue

For i from T to 0, let $p_j = p_j^{(T)}$ and $p_j^{(i-1)}$ be the "roundoffs" of p_j^i. For any $A \in \mathcal{A}$,

$$\left|\sum_{j\in A}(p_j^{(i-1)} - p_j^{(i)})\right| = \left|\sum_{j\in J^{(i)}\cap A} 2^{-i}\epsilon_j^{(i)}\right| \leq 2^{-i}K.$$

Thus, for any $A \in \mathcal{A}$,

$$\left|\sum_{j\in A} p_j^{(0)} - p_j^{(T)}\right| \leq \sum_{i=1}^{T}\left|\sum_{j\in A}(p_j^{(i-1)} - p_j^{(i)})\right| \leq \sum_{i=1}^{T} 2^{-i}K \leq K.$$

continue

For $p_1, p_2, \ldots, p_m \in [0,1]$, consider the function

$$f(p_1, \ldots, p_m) = \min_{\epsilon_1, \ldots, \epsilon_m \in \{0,1\}} \max_{A \in \mathcal{A}} \left| \sum_{i \in A} (\epsilon_i - p_i) \right|.$$

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Note that $f(p_1, p_2, \ldots, p_m)$ is continuous. We just proved that

$$f(p_1, p_2, \ldots, p_m) \leq K$$

for a dense set of $[0, 1]^m$. Thus it holds for any $(p_1, \ldots, p_m) \in [0, 1]^m$. This implies

$$\mathrm{lindisc}(\mathcal{A}) \leq K.$$

Corollary

Corollary: 13.3.3: Let $\mathcal{A}$ be a family of n sets on m points. Suppose $\mathrm{disc}(\mathcal{A}|_X) \leq K$ for every subset X with at most n points. Then $\mathrm{disc}(\mathcal{A}) \leq 2K$.

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Proof: By Theorem 13.3.2, $\mathrm{lindisc}(\mathcal{A}|_X) \leq K$ for every $X \subset \Omega$ with $|X| \leq n$. By Theorem 13.3.1, $\mathrm{lindisc}(\mathcal{A}) \leq K$. Thus,

$$\mathrm{disc}(\mathcal{A}) \leq 2 \cdot \mathrm{lindisc}(\mathcal{A}) \leq 2K.$$

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$$\mathrm{disc}(\mathcal{A}) \leq 2 \cdot \mathrm{lindisc}(\mathcal{A}) \leq 2K.$$

Corollary 13.3.4: For any family $\mathcal{A}$ of n sets of arbitrary size

$$\mathrm{disc}(\mathcal{A}) \leq 12\sqrt{n}.$$

Lower bounds

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Two methods:

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Two methods:

- Using Hadamard matrices.
- Using probabilistic method.

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- $HH' = nI$.
- If A is an $n \times n$ $(\pm)$-matrix, then $|\det(A)| \leq n^{n/2}$. The equality holds if and only if A is an Hadamard matrix.
- If H_1 and H_2 are Hadamard matrices, then so is $H_1 \otimes H_2$.
- If $\exists n \times n$ Hadamard matrix, then $n = 1, 2$ or $4|n$.

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Hall (1986) For all $\epsilon > 0$ and sufficiently large n, there is a Hadamard matrix of order between $n(1-\epsilon)$ and n.

Construction I

Let H be a Hadamard matrix of order n (even) with first row and first column all ones. (Any Hadamard matrix can be so "normalized" by multiplying appropriate rows and columns by -1.) Let J be all ones square matrix of order n. Let $v = (v_1, \ldots, v_n)'$ be the column vector with each $v_i \in \{-1, 1\}$. Then

$$\langle (H+J)v, (H+J)v \rangle = n^2 + 2n(\sum_{i=1}^{n} v_i)v_1 + n(\sum_{i=1}^{n} v_i)^2 \geq n^2.$$

Setting $H^* = (H+J)/2$, then,

$$\|H^* v\|_\infty \geq \sqrt{\|H^* v\|^2/n} \geq \frac{\sqrt{n}}{2}.$$

Let $\mathcal{A}$ be the family of subsets with incidence matrix H^*.

Construction II

- M: a random $0, 1$ matrix of order n.
- d_i: i-th row sum of M, $d_i = (1 + o(1))n/2$.
- $v := (v_1, \ldots, v_n)'$, $v_i = \pm 1$, set $Mv = (L_1, L_2, \ldots, L_n)$.

$$L_i \sim B(d_i, 1/2) - B(d_i, 1/2) \sim N(0, \sqrt{n}/2).$$

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$$L_i \sim B(d_i, 1/2) - B(d_i, 1/2) \sim N(0, \sqrt{n}/2).$$

Pick λ so that

$$\int_{-\lambda}^{\lambda} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt < \frac{1}{2}.$$

Then $\Pr(|L_i| < \lambda\sqrt{n}/2) < \frac{1}{2}$. The expected number of v for which $|Mv|_\infty < \lambda\sqrt{n}/2$ is less than 1. $\exists M$ such that $|Mv|_\infty \geq \lambda\sqrt{n}/2$ for every v.

Construction II

- M: a random $0, 1$ matrix of order n.
- d_i: i-th row sum of M, $d_i = (1 + o(1))n/2$.
- $v := (v_1, \ldots, v_n)'$, $v_i = \pm 1$, set $Mv = (L_1, L_2, \ldots, L_n)$.

$$L_i \sim B(d_i, 1/2) - B(d_i, 1/2) \sim N(0, \sqrt{n}/2).$$

Pick λ so that

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Then $\Pr(|L_i| < \lambda\sqrt{n}/2) < \frac{1}{2}$. The expected number of v for which $|Mv|_\infty < \lambda\sqrt{n}/2$ is less than 1. $\exists M$ such that $|Mv|_\infty \geq \lambda\sqrt{n}/2$ for every v.
Let $\mathcal{A}$ be the family of sets with incident matrix M. Then

$$\mathrm{disc}(\mathcal{A}) \geq \lambda\lambda\sqrt{n}/2.$$

Beck-Fiala Theorem

For any $\mathcal{A}$, let $\deg(\mathcal{A})$ denote the maximal number of sets containing any particular points.

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Theorem [Beck-Fiala 1981] Let $\mathcal{A}$ be a finite family of finite sets. If $\deg(\mathcal{A}) \leq t$, then

$$\operatorname{disc}(\mathcal{A}) \leq 2t - 1.$$

Proof: Assume $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ where all $A_i \subset [n]$. Let $x = (x_1, \ldots, x_n) \in [-1, 1]^n$. A set S_i has value $\sum_{j \in S_i} x_j$. We say an index j is **fixed** if $x_j = \pm 1$; otherwise we say j is **floating**. A set S_i is **safe** if it has at most t floating points; otherwise it is **active**.

Fact: There are fewer active sets than floating points.

continue

Initially all j are floating; i.e. x is the zero vector. We will change x to x' with fewer floating points while keep the values of all sets to 0.

Iteration: For each active set, move the fixed points to the right hand side. We get a system of linear equations where the unknown variables are floating points. Since there are fewer active sets than floating points. This is an underdeterminded system. The solution contains a line, parametrized

$$x'_j = x_j + \lambda y_j, \quad j \text{ floating},$$

on which the active sets retain value zero. Choose the smallest λ on the absolute value so that one of $x'_j = 1$.

continue

After many iterations, we get a vector x so that every set is safe and has value 0. For each floating point j, setting $x_j = \pm 1$ arbitrarily. For each set, the value may change less than $2t$ and, as it is an integer, it is at most $2t - 1$. $\square$

continue

After many iterations, we get a vector x so that every set is safe and has value 0. For each floating point j, setting $x_j = \pm 1$ arbitrarily. For each set, the value may change less than $2t$ and, as it is an integer, it is at most $2t - 1$. □

Conjecture: If $\deg(\mathcal{A}) \leq t$, then $\mathrm{disc}(\mathcal{A}) \leq K\sqrt{t}$, for some absolute constant.