Topic Course on Probabilistic Methods (Week 12) Random Graphs (II)

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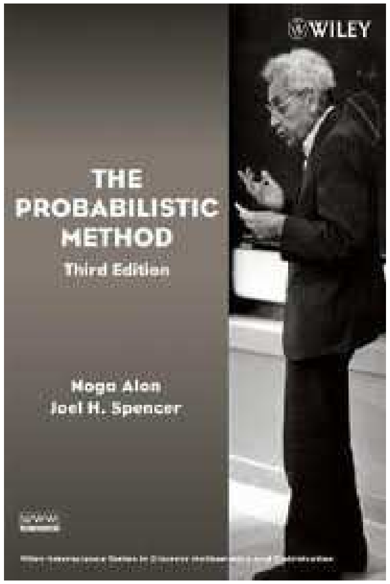
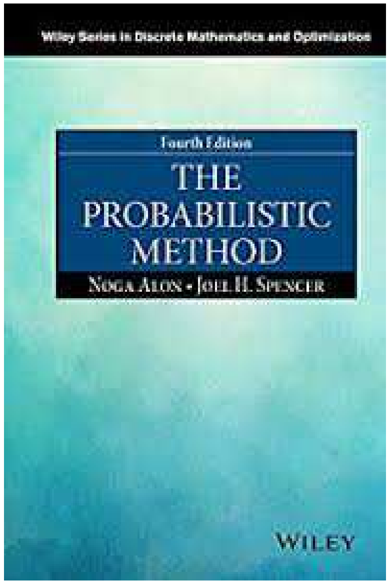
Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviation inequalities (3 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Random graphs

- Supercritical regimes
- Barely Supercritical Phase
- The critical window
- Range V
- Threshold of connectivity
- Range VI

Supercritical regimes

Now we consider $G(n,p)$ for $p = c/n$, with $c > 1$ constant.
Let $y := y(c)$ be the positive real solution of $e^{-cy} = 1 - y$.
Choose a large constant $K > 0$ and a small constant $\delta > 0$.
Let $C(v)$ be the component of $G(n,p)$ containing v.

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Claim: The probability of having any awkward component is $o(n^{-20})$.

No middle ground

Proof: We will show for any awkward t, $\Pr(|C(v)| = t) = o(n^{-22})$. Note

$$\Pr(|C(v)| = t) \leq \Pr(B(n-1, 1-(1-\frac{c}{n})^t) = t-1.$$

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If $t = o(n)$, then $1-(1-\frac{c}{n})^t \approx \frac{ct}{n}$. So the mean is about ct, which is not close to t. If $t = xn$, then $1-(1-\frac{c}{n})^t) \approx 1-e^{-cx}$. Since $1-e^{-cx} \neq x$, so the mean is not near t.

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$$\Pr\left(B(n-1, 1-(1-\frac{c}{n})^t\right) = O(e^{-Ct})$$

for some constant C. Since $t \geq K \log n$ and K large enough, this probability is $o(n^{-22})$ as required.

Escape Probability

Let $\alpha = \Pr(C(v)$ is not small $)$. Then

$$\alpha = \Pr(T_c^{po} \geq S) \approx \Pr(T_c^{po} = \infty) = y.$$

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- Each giant component has size between $(y - \delta)n$ and $(y + \delta)n$.

It remains to show the giant component is unique and of size about yn.

Sprinkling

Set $p_1 = n^{-3/2}$. Let $G_1 = G(n, p_1)$, $G = G(n, p)$, and $G^+ = G \cup G_1$. Note $G^+ \sim G(n, p^+)$ with $p^+ = p + p_1 - pp_1$.

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Since δ can be made arbitrarily small, the unique giant component has size $(1 + o(1))yn$.

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The following statements hold.

- $\Pr(\exists$ an awkward component $) = O(n^{-20})$.
- The escape probability $\alpha \approx y \approx 2\epsilon$.
- Sprinkling works with $p_1 = n^{-4/3}$.

The critical window

Now consider $G(n,p)$ with $p = \frac{1}{n} + \lambda n^{-4/3}$ for a fixed λ. This critical window has been studied by **Bollabás, Łuczak, Janson, Knuth, Pittel** and many others. It requires delicate calculations.

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For fixed $c > 0$, Let X be the number of tree components of size $k = cn^{2/3}$. Then

$$\mathrm{E}(X) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-(k-1)}.$$

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Recall

$$\ln(1+x) = x - \frac{1}{2}x^2 + O(x^3).$$

Estimation

We estimate

$$\binom{n}{k} \approx \frac{n^k}{(k/e)^k\sqrt{2\pi k}} \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right),$$

and

$$\begin{aligned}\prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right) &= e^{\sum_{i=1}^{k-1}\ln(1-i/n)} \\ &= e^{-\sum_{i=1}^{k-1}(i/n+i^2/2n^2+O(i^3/n^3))} \\ &= e^{-\frac{k^2}{2n}-\frac{k^3}{6n^2}+o(1)} \\ &= e^{-\frac{k^2}{2n}-\frac{c^3}{6}+o(1)}.\end{aligned}$$

Continue

We also estimate

$$\begin{aligned} p^{k-1} &= n^{1-k}(1+\lambda n^{-1/3})^{k-1} \\ &= n^{1-k} e^{(k-1)\ln(1+\lambda n^{-1/3})} \\ &= n^{1-k} e^{k\lambda n^{-1/3} - \frac{1}{2}c\lambda^2 + o(1)}, \end{aligned}$$

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$$\begin{aligned} (1-p)^{k(n-k)+\binom{k}{2}-(k-1)} &= e^{(kn-k^2/2+O(k))\ln(1-p)} \\ &= e^{-(kn-k^2/2+O(k))(p+p^2/2+O(p^3))} \\ &= e^{-k+\frac{k^2}{2n}-\frac{\lambda k}{n^{1/3}}+\frac{\lambda c^2}{2}+o(1)}. \end{aligned}$$

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Putting together

We get

$$E(X) \approx nk^{-5/2}(2\pi)^{-1/2}e^{A},$$

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For any fixed a, b, λ, let X be the number of tree components of size between $an^{2/3}$ and $bn^{2/3}$. Then

$$\lim_{n\to\infty} \mathrm{E}(X) = \int_a^b e^{A(c)}c^{-5/2}(2\pi)^{-1/2}dc.$$

Other components

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Let $X^{(l)}$ be the number of components on k vertices with $k-1+l$ edges. Then a similar calculation shows

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Let X^* be the total number of components of size between $an^{2/3}$ and $bn^{2/3}$. Let $g(c) = \sum_{l=0}^{\infty} c_l c^{3l/2}$. Then

$$\lim_{n\to\infty} \mathrm{E}(X^*) = \int_a^b e^{A(c)} c^{-5/2} (2\pi)^{-1/2} g(c) dc.$$

Duality

For a fixed k, consider two random graphs $G(n,p)$ and $G(n',p')$. Assume $c = np > 1$ and $c' = n'p' < 1$. We say $G(n,p)$ and $G(n',p')$ are **dual** to each other if $ce^{-c} = c'e^{-c'}$.

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Let $y = 1 - c'/c$. Then y satisfies the equation $e^{-cy} = 1 - y$. Hence the size of the giant component in $G(n,p)$ is roughly yn. We have

$$
\begin{aligned}
&\lim_{n\to\infty} \Pr(C(v) = k \text{ in } G(n,p) | C(v) \text{ is small}) \\
&\qquad = \frac{1}{1-y} \frac{e^{-ck}(ck)^{k-1}}{k!} = \frac{e^{-c'k}(c'k)^{k-1}}{k!} \\
&\qquad = \lim_{n'\to\infty} \Pr(C(v) = k \text{ in } G(n',p')).
\end{aligned}
$$

Range V

Consider $G(n, p)$ with

$$p = \frac{\log n}{kn} + \frac{(k-1)\log\log n}{kn} + \frac{t}{n} + o(\frac{1}{n}),$$

then there are only trees of size at most k except for the giant component. Let X be the number of trees of k vertices.

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$$\begin{aligned} \mathrm{E}(X) &= \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1} \\ &\approx \frac{1}{k^2 p \cdot k!} (knp)^k e^{-knp} \approx \frac{e^{-kt}}{k \cdot k!}. \end{aligned}$$

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$$\approx \frac{1}{k^2 p \cdot k!} (knp)^k e^{-knp} \approx \frac{e^{-kt}}{k \cdot k!}.$$

Further, X follows the Poisson distribution.

Threshold of connectivity

For $k = 1$ and $p = \frac{\log n}{n} + \frac{t}{n} + o(\frac{1}{n})$, $G(n,p)$ consists of a giant component with $n - O(1)$ vertices and bounded number of isolated vertices.

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- The distribution of the number of isolated vertices again has a Poisson distribution with mean value e^{-t}.
- The probability that $G(n,p)$ is connected tends to $e^{-e^{-t}}$.
- As $t \to \infty$, $G(n,p)$ is almost surely connected.

Range VI

Consider $G(n, p)$ with $p \sim \omega(n) \log n / n$ where $\omega(n) \to \infty$.

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Let $X = d_v$ be the degree of v. By Chernoff's inequality, With probability at least $1 - O(n^{-2})$, we have

$$|X - \mathrm{E}(X)| < 2\sqrt{\omega(n)} \log n.$$

Almost surely, for all v, d_v is in the interval $[d - 2\sqrt{\omega(n)} \log n, d + 2\sqrt{\omega(n)} \log n]$, where $d = np$ is the expected degree.

Subgraphs

Theorem: Let H be a strictly balanced graph with v vertices, m edges, and a automorphisms. Let $c > 0$ be arbitrary. Then with $p = cn^{-v/m}$,

$$\lim_{n\to\infty} \Pr(G(n,p) \text{ contains no } H) = e^{-c^m/a}.$$

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Proof: Let A_α, $1 \leq \alpha \leq \binom{n}{v} v!/a$, range over the edge sets of possible copies of H and B_α be the event $A_\alpha \subset G(n,p)$. We will apply Janson's Inequality.

$$\lim_{n\to\infty} \mu = \lim_{n\to\infty} \binom{n}{v} v! p^m/a = c^m/a.$$

$$\lim_{n\to\infty} M = e^{-c^m/a}.$$

Proof

Consider $\Delta = \sum_{\alpha \sim \beta} \Pr(B_\alpha \wedge B_\beta)$. We split the sum according to the number of vertices in $A_\alpha \cap A_\beta$. For $2 \le j \le v$, let f_j be the maximal number of edges in $A_\alpha \cap A_\beta$ where $\alpha \sim \beta$ and α and β intersect in j vertices. Since H is strictly balanced,

$$\frac{f_j}{j} < \frac{m}{v}.$$

There are $O(n^{2v-j})$ choices of α, β For such α, β,

$$\Pr(B_\alpha \wedge B_\beta) = p^{|A_\alpha \cup A_\beta|} \le p^{2m - f_j}.$$

Continue

$$\Delta \leq \sum_{j=2}^{v} O(n^{2v-j}) O(n^{(v/m)(2m-f_j)}).$$

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Each term is $o(1)$ and hence $\Delta = o(1)$. By Janson's inequality,

$$\lim_{n\to\infty} \Pr(\wedge \bar{B}_\alpha) = \lim_{n\to\infty} M = e^{-c^m/a}.$$

The proof is finished. □

Clique number of $G(n, \frac{1}{2})$

For the rest of slides, we assume $p = \frac{1}{2}$ and $G := G(n, 1/2)$. Let $\omega(G)$ Be the clique number. For a fixed $c > 0$, let $n, k \to \infty$ so that

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We get $n \sim \frac{k}{e\sqrt{2}} 2^{k/2}$ and $k \sim \frac{2 \ln n}{\ln 2}$.

Clique number of $G(n, \frac{1}{2})$

For the rest of slides, we assume $p = \frac{1}{2}$ and $G := G(n, 1/2)$. Let $\omega(G)$ Be the clique number. For a fixed $c > 0$, let $n, k \to \infty$ so that

$$\binom{n}{k} 2^{-\binom{k}{2}} \to c.$$

We get $n \sim \frac{k}{e\sqrt{2}} 2^{k/2}$ and $k \sim \frac{2 \ln n}{\ln 2}$.

For this k, apply Poisson paradigm to X: the number of k-cliques. We have

$$\Pr(\omega(G < k) = \Pr(X = 0) = (1 + o(1))e^{-c}.$$

Two points concentration

Let $n_0(k)$ be the minumum n for which $\binom{n}{k}2^{-\binom{k}{2}} \geq 1$. For any $\lambda \in (-\infty, +\infty)$ if $n = n_0(k)\left[1 + \frac{\lambda+o(1)}{k}\right]$, then

$$\binom{n}{k}2^{-\binom{k}{2}} = \left[1 + \frac{\lambda + o(1)}{k}\right]^k = e^{\lambda} + o(1).$$

and so

$$\Pr(\omega(G) < k) = e^{-e^{\lambda}} + o(1).$$

Note that $e^{-e^{\lambda}}$ ranges from 1 to 0 as λ ranges from $-\infty$ to $+\infty$. Let K be arbitrarily large and set

$$I_k = \left[n_0(k)(1 - K/k), n_0(k)(1 + K/k)\right].$$

continues

For $k \geq k_0(K)$, $I_{k-1} \cap I_k = \emptyset$ since $n_0(k+1) \sim \sqrt{2} n_0(k)$.

- If n lies between the intevals, $I_k < n < I_{k+1}$, then

$$\Pr(\omega(G) = k) \geq e^{-e^{-K}} - e^{-e^{K}} + o(1).$$

With probability near one, we have $\omega(G) = k$.

continues

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- If n lies between the intevals, $I_k < n < I_{k+1}$, then

$$\Pr(\omega(G) = k) \geq e^{-e^{-K}} - e^{-e^{K}} + o(1).$$

  With probability near one, we have $\omega(G) = k$.

- If n lies in the inteval I_k, then we still have $I_{k-1} < n < I_{k+1}$, then

$$\Pr(\omega(G) = k - 1 \text{ or } k) \geq e^{-e^{-K}} - e^{-e^{K}} + o(1).$$

  With probability near one, we have $\omega(G) = k - 1$ or k.

Chromatic number

Let $f(k) = \binom{n}{k} 2^{-\binom{k}{2}}$ and $k_0 = k_0(n)$ be that value for which

$$f(k_0 - 1) > 1 > f(k_0).$$

Setting $k := k_0(n) - 4$, then $f(k) > n^{3+o(1)}$.
We apply the Extended Janson Inquality to estimate $\Pr(\omega(G) < k)$. We have $\frac{\Delta}{\mu^2} = \sum_{i=2}^{k-1} g(i)$, where $g(i) = \frac{\binom{k}{i}\binom{n-k}{k-i}}{\binom{n}{k}} 2^{\binom{i}{2}}$. As $k \sim 2\log_2 n$, $g(2) \sim k^4/n^2$ dominates. Thus,

$$\Pr(\omega(G) < k) < e^{-\mu^2/2\Delta} = e^{-\Theta(n^2/\ln^4 n)}.$$

Chromatic number $\chi(G)$

Theorem Bollabás (1988): Almost surely

$$\chi(G) \sim \frac{n}{2\log_2 n}.$$

Chromatic number $\chi(G)$

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$$\chi(G) \sim \frac{n}{2\log_2 n}.$$

Proof: Note that $\alpha(G) = \omega(\bar{G})$ and $\bar{G}$ has the same distribution as $G(n, 1/2)$. We have $\alpha(G) \leq (2 + o(1)) \log_2 n$. Thus almost surely

$$\Pr(\chi(G) \geq \frac{n}{\alpha(G)}) \geq (1 + o(1)) \frac{n}{2\log_2 n}.$$

reverse direction

Let $m = \lfloor \frac{n}{\ln^2 n} \rfloor$. For any set S of m vertices the restriction $G|_S$ has the distribution $G(m, \frac{1}{2})$. Let $k := k(m)$ as before. Note

$$k \sim 2\log_2 m \sim 2\log_2 n.$$

There are at most $\binom{n}{m} < 2^n = 2^{m^{1+o(1)}}$ such set of S. Hence

$$\Pr(\exists S(\alpha(G|_S) < k)) < 2^{m^{1+o(1)}} e^{-m^{2+o(1)}} = o(1).$$

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$$\Pr(\exists S(\alpha(G|_S) < k)) < 2^{m^{1+o(1)}} e^{-m^{2+o(1)}} = o(1).$$

Almost surely every m vertices contain a k-element independent set.

continue

Now we pull out k-element independent sets and give each a distinct color until there are less than m vertices left. Then we given each point a distinct color. We have

$$\begin{aligned}\chi(G) &\leq \left\lceil \frac{n-m}{k} \right\rceil + m \\ &= (1+o(1))\frac{n}{2\log_2 n} + o\left(\frac{n}{\log_2 n}\right) \\ &= (1+o(1))\frac{n}{2\log_2 n}.\end{aligned}$$

The proof is finished. □