Topic Course on Probabilistic Methods
(Week 1)
Linearity of Expectation (1)

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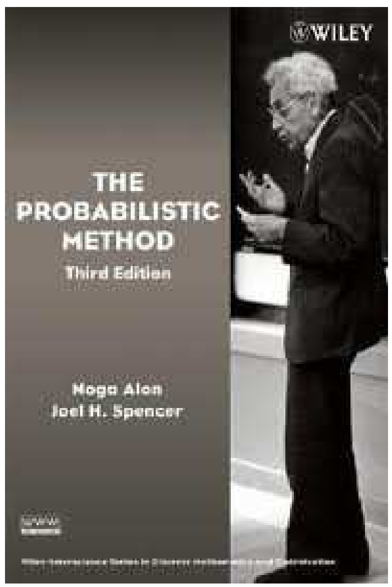
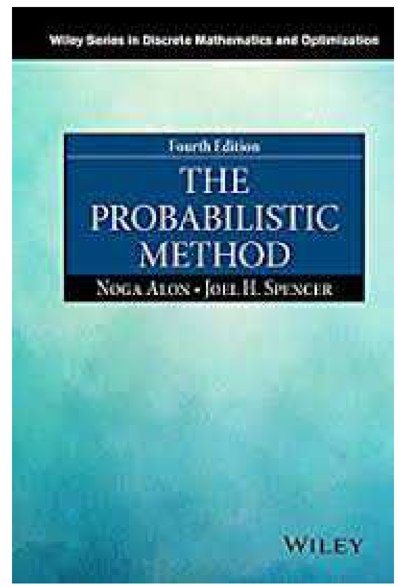
Introduction

The topic course is mostly based the textbook "The probabilistic Method" by Noga Alon and Joel Spencer (third edition 2008, John Wiley & Sons, Inc. ISBN 9780470170205 or fourth edition ISBN-13: 978-1119061953.)





Selected topics

- Linearity of Expectation (2 weeks)
- Alterations (1 week)
- The second moment method (1 week)
- The Local Lemma (1-2 weeks)
- Correlation Inequalities (1 week)
- Large deviations (1-2 weeks)
- Poisson Paradigm (1 week)
- Random graphs (2 weeks)
- Discrepancy (1 week)
- Entropy (1 week)

Subtopics

Linearity of Expectation

- Ramsey numbers
- Tournament
- Dominating set
- Property B problem
- A (k, l)-system
- Sum-free sets
- Erdős-Ko-Rado Theorem

History

Paul Erdős: 1913–1996
1525 papers
511 coauthors

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Main contributions:

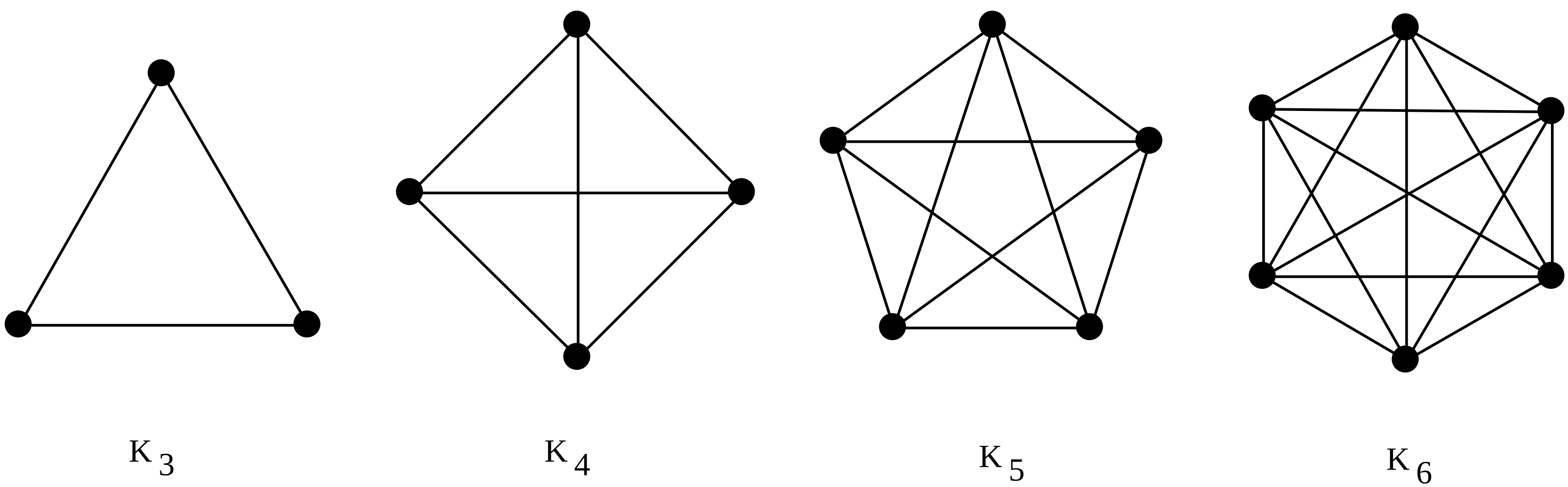
- Ramsey theory
- Probabilistic method
- Extremal combinatorics
- Additive number theory

Notation

A graph G consists of two sets V and E.

- V is the set of vertices (or nodes).
- E is the set of edges, where each edge is a pair of vertices.

Complete graphs K_n:



Ramsey number $R(k, k)$

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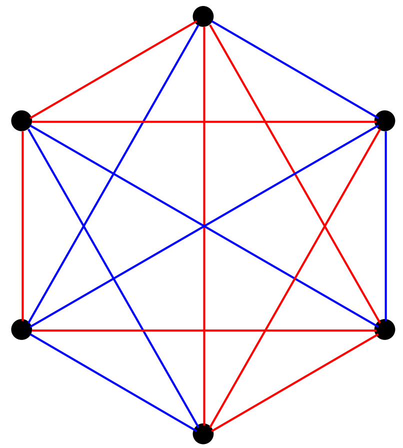
Major question: How large is $R(k,k)$?

Proposition (by Erdős): If $\binom{n}{2}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. Thus

$$R(k,k) > \frac{k}{e\sqrt{2}}2^{k/2}.$$

Ramsey number $R(3,3) = 6$

- If edges of K_6 are 2-colored then there exists a monochromatic triangle.



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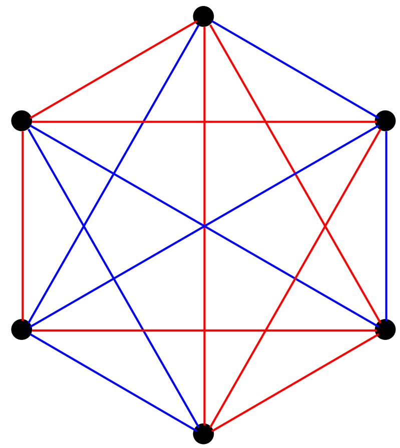
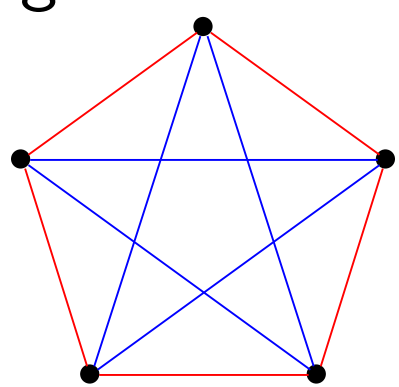


- There exists a 2-coloring of edges of K_5 with no monochromatic triangle.



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To prove $R(k,k) > n$, we need construct a 2-coloring of K_n so that it contains no red K_k or blue K_k.

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Make the set of all 2-colorings of K_n into a probability space, then show the event " no red K_k or blue K_k" with positive probability.

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- $\Omega := \{s_1, s_2, \ldots, s_n\}$: a set of n elements.

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- Two events A and B are independent if

$$\Pr(AB) = \Pr(A)\Pr(B).$$

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Hence $\Pr(\wedge_R \bar{A}_R) = 1 - \Pr(\vee_R A_R) > 0.$

Estimation of n

Since $\binom{n}{k} \leq \frac{1}{e}\left(\frac{en}{k}\right)^k$ for all $n \geq k \geq 1$, we have

$$\begin{aligned}
\binom{n}{k} 2^{1-\binom{k}{2}} &\leq \frac{1}{e}\left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} \\
&\leq \frac{2}{e}\left(\frac{en}{k2^{(k-1)/2}}\right)^k \\
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Hence,

$$R(k,k) > \frac{k}{e\sqrt{2}} 2^{k/2}. \qquad \square$$

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Spencer [1975] (using Lovasz Local Lemma)

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Upper bound of $R(k,k)$

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Conlon [2009]:

$$R(k,k) \leq k^{-C\frac{\log k}{\log \log k}} \binom{2k-2}{k-1}.$$

Diagonal Ramsey Problem

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If $\lim_{k\to\infty} R(k,k)^{1/k}$ exists, then it is between $\sqrt{2}$ to 4.

Tournament

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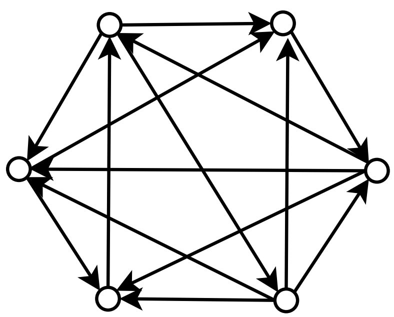
- (x, y) means player x beats y.
- **Tournament on V:** an orientation $T = (V, E)$ of complete graphs on V. For each pair of plays x and y, either (x, y) or (y, x) is in E.



We say T has **property** S_k if for every set of k players there is one beats all.

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$$\Pr(A_K) = (1-2^{-k})^{n-k}.$$

Proof continues

$$\Pr\left(\vee_{K\in\binom{V}{k}} A_K\right) \leq \sum_{K\in\binom{V}{k}} \Pr(A_K)$$
$$= \binom{n}{k}(1-2^{-k})^{n-k} < 1.$$

Therefore, with positive probability, no event A_K occurs; that is, there is a tournament on n vertices that has the property S_k.

Estimation of n

Let $f(k)$ denote the minimum possible number of vertices of a tournament that has the property S_k.
On one hand, since $\binom{n}{k} < (en/k)^k$ and $(1-2^{-k})^{n-k} < 2^{(n-k)/2^k}$, we have

$$f(k) \leq (1+o(1)) \ln 2 \cdot k^2 \cdot 2^k.$$

On the other hand, **Szekeres** proved

$$f(k) \geq c_1 k 2^k.$$

Random variable

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Linearity of expectation:

$$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y).$$

Dominating set

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Theorem: Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then G has a dominating set of at most $\frac{1+\ln(\delta+1)}{\delta+1}n$.

Proof

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$$\begin{aligned} \mathrm{E}(|Y|) &= \sum_v \Pr(v \in Y) \\ &\leq n(1-p)^{\delta+1}. \end{aligned}$$

continue

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$$\begin{aligned} \mathrm{E}(|U|) &= \mathrm{E}(X) + \mathrm{E}(Y) \\ &\leq np + n(1-p)^{\delta+1} \\ &\leq n(p + e^{-p(\delta+1)}). \end{aligned}$$

Choose $p = \frac{\ln(\delta+1)}{\delta+1}$ to minimize the upper bound. There is a dominating set of size at most

$$\frac{1 + \ln(\delta+1)}{\delta+1} n.$$

Hypergraphs

$H = (V, E)$ is an r-uniform hypergraph (r-graph, for short).

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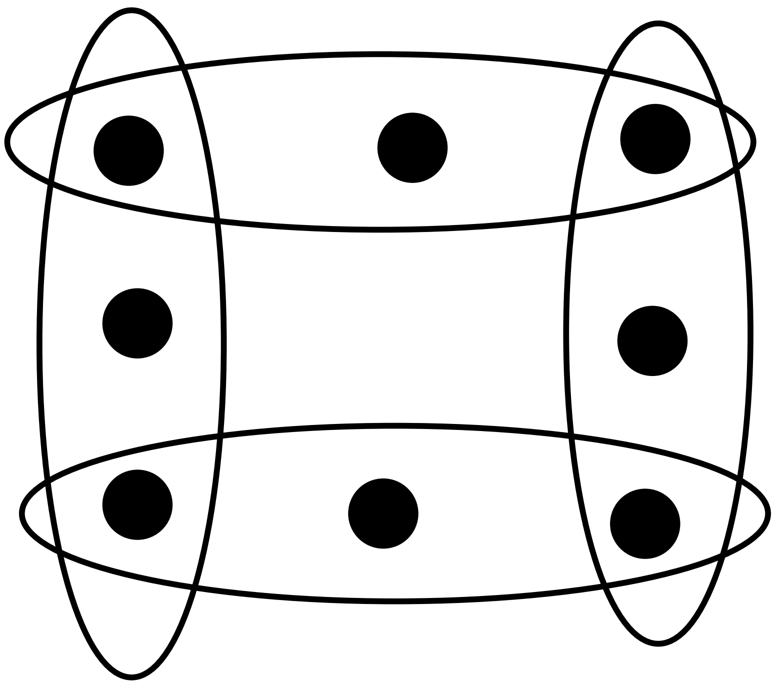
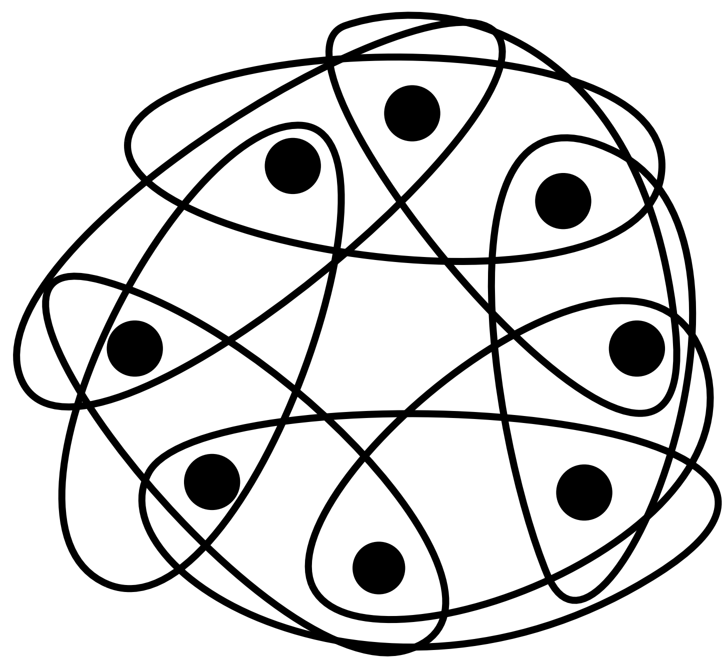


A 3-uniform loose cycle



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Let $m(r)$ denote the minimum possible number of edges of an r-uniform hypergraph that does not have property B.

Proposition [Erdős (1963)] Every r-uniform hypergraph with less than 2^{r-1} edges has property B. Therefore $m(r) \geq 2^{r-1}$.

Proof

Let H be an r-uniform hypergraph with less than 2^{r-1} edges. Color V randomly by two colors. For each edge $e \in E$, let A_e be the event that e is monochromatic.

$$\Pr(A_e) = 2^{1-r}.$$

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Therefore,

$$\Pr\left(\vee_{e \in E} A_e\right) \leq \sum_{e \in E} \Pr(A_e) < 1.$$

There is a two-coloring without monochromatic edges. $\square$

Upper bound

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$$\Pr(S \text{ is monochromatic under } \chi) = \frac{\binom{a}{r} + \binom{b}{r}}{\binom{n}{r}}.$$

Assume $n = 2k$ is even. Then $\binom{a}{r} + \binom{b}{r}$ reaches the minimum when $a = b = k$. Thus

$$\Pr(S \text{ is monochromatic under } \chi) \geq \frac{2\binom{k}{r}}{\binom{n}{r}}.$$

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$$\Pr(\vee_\chi A_\chi) \leq \sum_\chi \Pr(A_\chi) \leq 2^n (1-p)^m.$$

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$$m(r) \leq \lceil \frac{n \ln 2}{p} \rceil.$$

$$\begin{aligned} p &= \frac{2\binom{k}{r}}{\binom{n}{r}} \\ &= 2^{1-r} \prod_{i=0}^{r-1} \frac{n-2i}{n-i} \\ &\approx 2^{1-r} e^{-r^2/2n}. \end{aligned}$$

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Hence $m(r) < (1 + o(1)) \frac{e \ln 2}{4} r^2 2^r$. $\square$

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Radhakrishnan-Srinivasan [2000]: (best lower bound)

$$m(r) \geq \Omega\left(\left(\frac{r}{\ln r} \right)^{1/2} 2^r \right).$$

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$m(2) = 3$, $m(3) = 7$, $20 \leq m(4) \leq 23$.

A (k, l)-system

A family of pairs of sets $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ is called a (k, l)-system if

- for $1 \leq i \leq h$, $|A_i| = k$, $|B_i| = l$, $A_i \cap B_i = \emptyset$.
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Theorem [Bollobás 1965]: If $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ is a (k, l)-system, then $h \leq \binom{k+l}{k}$.

Proof

Let $V = \cup_{i=1}^{h}(A_i \cup B_i)$ and consider a random order π of V.

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Observe that all X_i's are disjoint events. We have

$$1 \geq \Pr(\vee_{i=1}^{h} X_i) = \sum_{i=1}^{h} \Pr(X_i) = \frac{h}{\binom{k+l}{k}}.$$

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Randomly pick an integer x in $[1,p-1]$. Define

$$A=\{b_i\colon xb_i(\bmod p)\in C\}.$$

continue

Claim: A is a sum-free set.

Let X_i be the indicator random variable that $b_i \in A$.

$$\Pr(X_i) = \frac{|C|}{p-1} = \frac{k+1}{3k-1} > \frac{1}{3}.$$

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There is a subset $A \subset B$ with greater than $n/3$ elements. $\square$

Erdős-Ko-Rado Theorem

Let $\mathcal{F} \subset \binom{[n]}{k}$. A family $\mathcal{F}$ of k-sets is called **intersecting** if for any $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

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This is tight since we can take $\mathcal{F} = \{F \in \binom{[n]}{k} : 1 \in F\}$.

Katona's book proof

Katona (1974) proof: Consider a random permutation $\sigma \in S_n$ chosen randomly. List the elements of $[n]$ in the order of σ on a cycle C_σ.

- For $A \in \mathcal{F}$, X_A be the indicator variable that A forms a consecutive block on C_σ.

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- $X := \sum_{A \in \mathcal{F}} X_A$: the number of consecutive blocks in $\mathcal{F}$.

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Since $\mathcal{F}$ is intersecting, $X \le k$. We have $\frac{n|\mathcal{F}|}{\binom{n}{k}} \le k$. $\square$

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