Math576: Combinatorial Game Theory

Lecture note IV

Linyuan Lu

University of South Carolina

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Disclaimer

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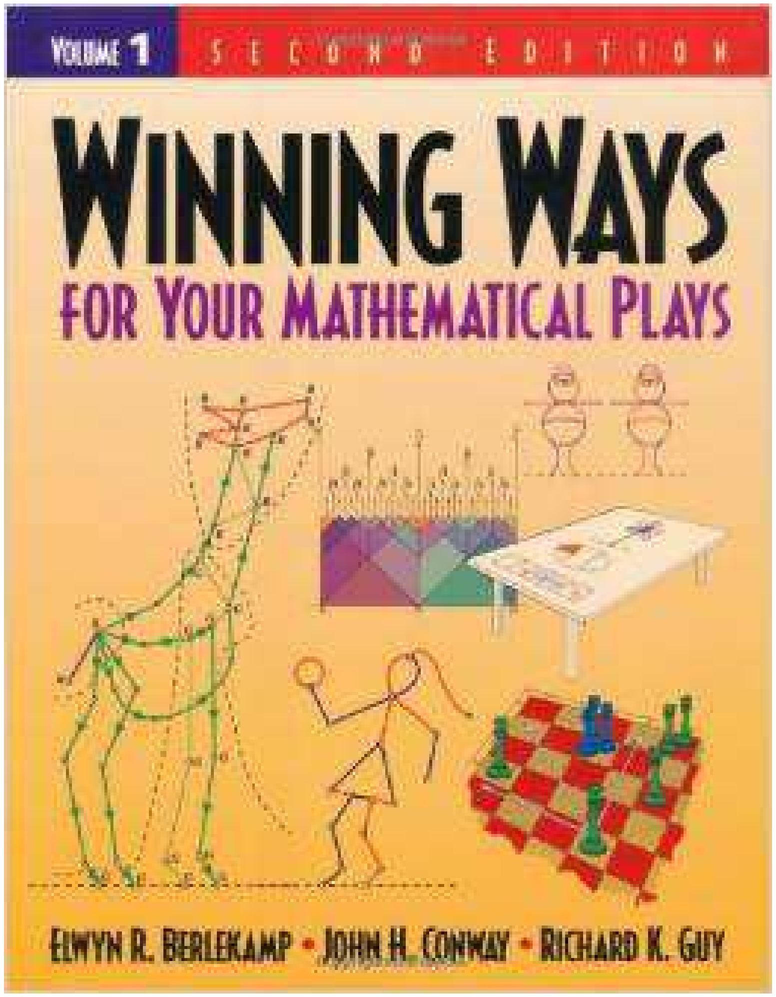
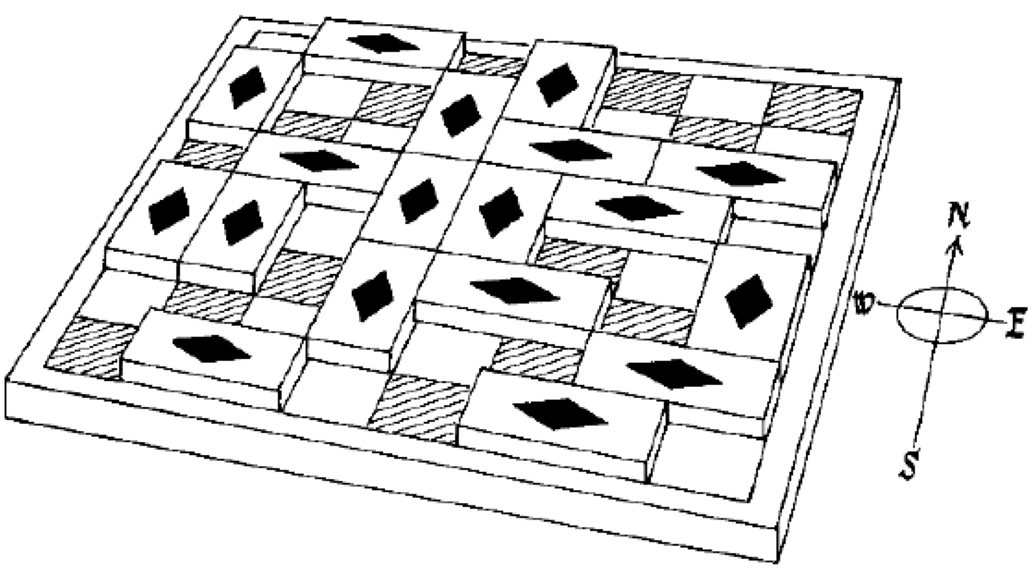
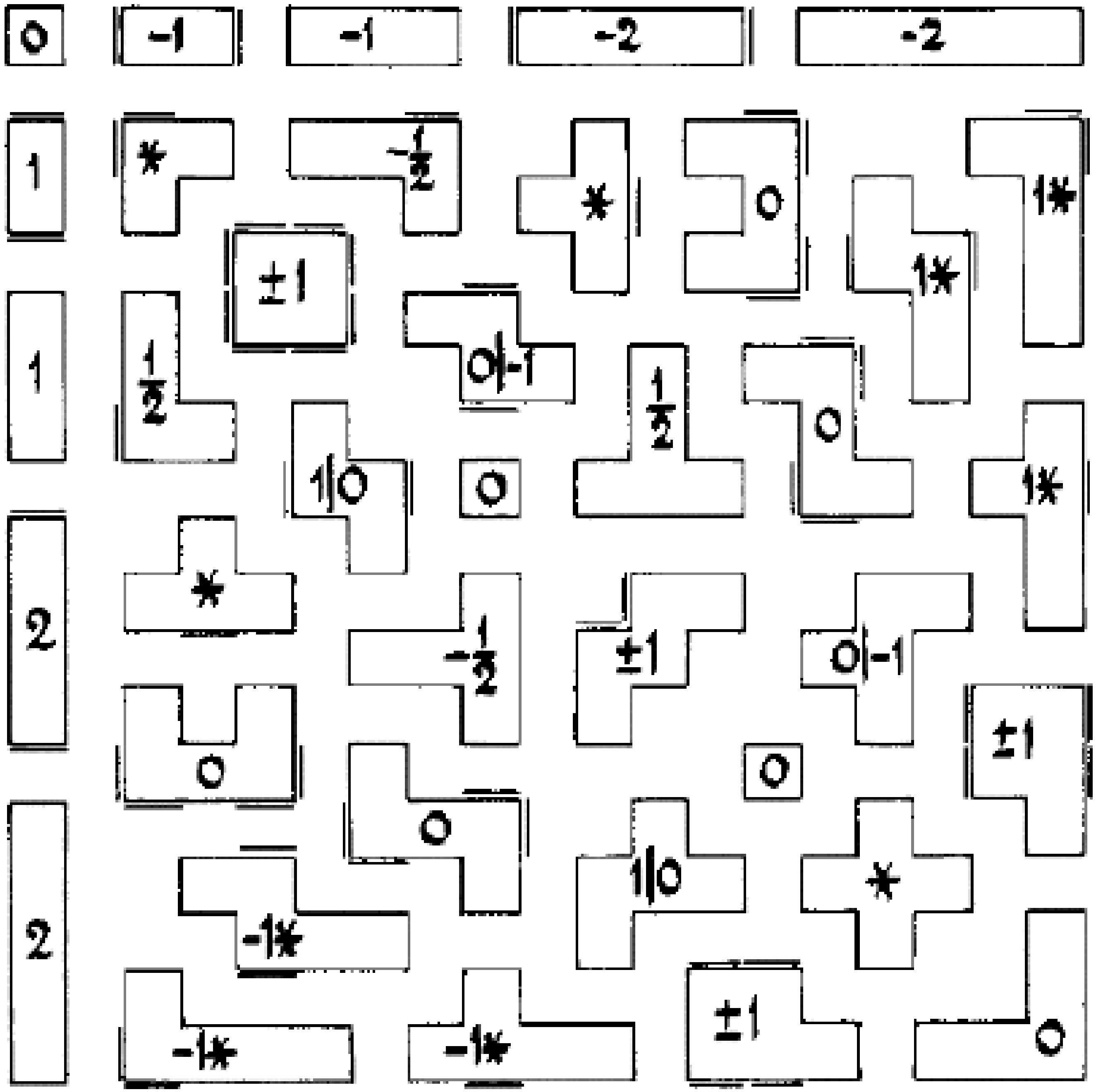


Game of Domineering

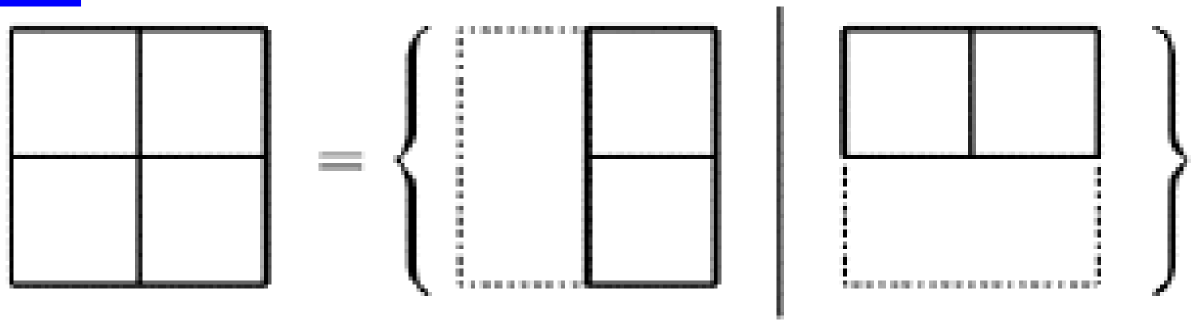
- **Two players:** "Left" and "Right".
- **Game board:** a rectangular checkerboard.
- **Rules:** Two players take turns in placing dominoes on a board. Left orients his dominoes North-South and Right East-West. Each domino must exactly cover two squares of the board and no two dominoes may overlap.
- **Ending positions:** Whoever gets stuck is the loser.



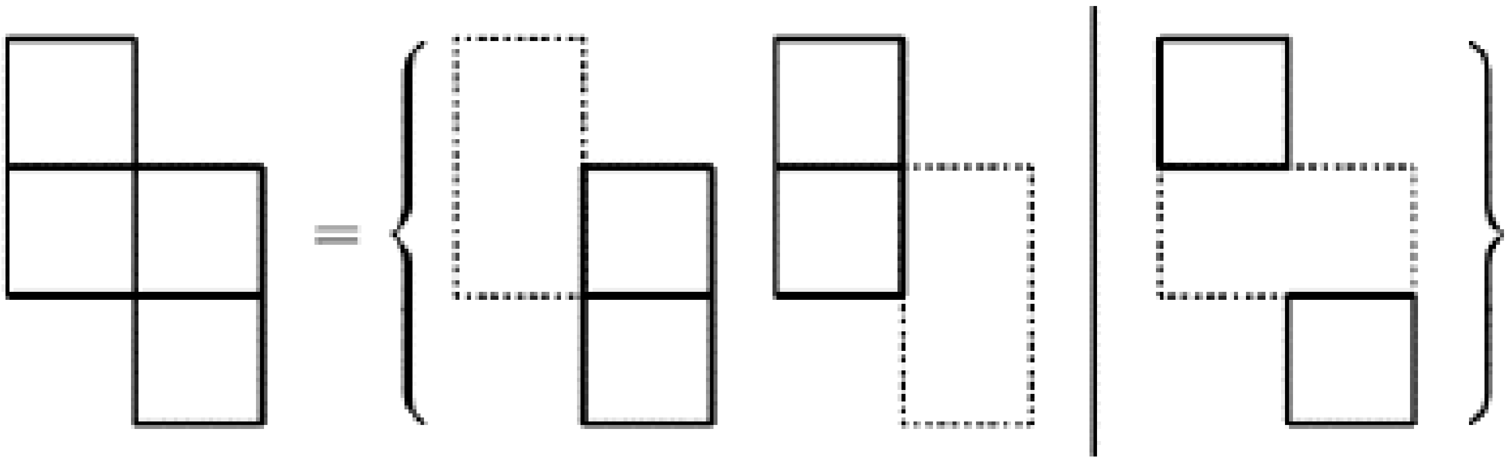
Values of Domineering



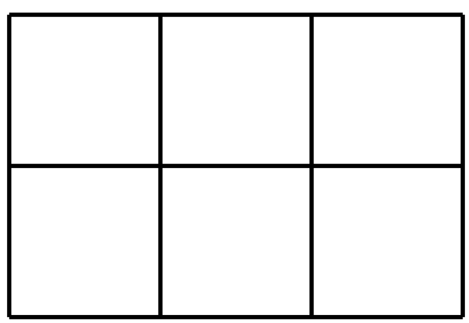
Some new values



$= \{1 \mid -1\}$



$= \{1 \mid 0\}$



$= \{2 \mid -\frac{1}{2}\}.$

Switch Games

How big is the game $\{x \mid y\}$ where x and y are numbers and $x \geq y$?

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$$z > x \text{ implies } z > \{x \mid y\}$$
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$$y \leq z \leq x \text{ implies } z \parallel \{x \mid y\}.$$

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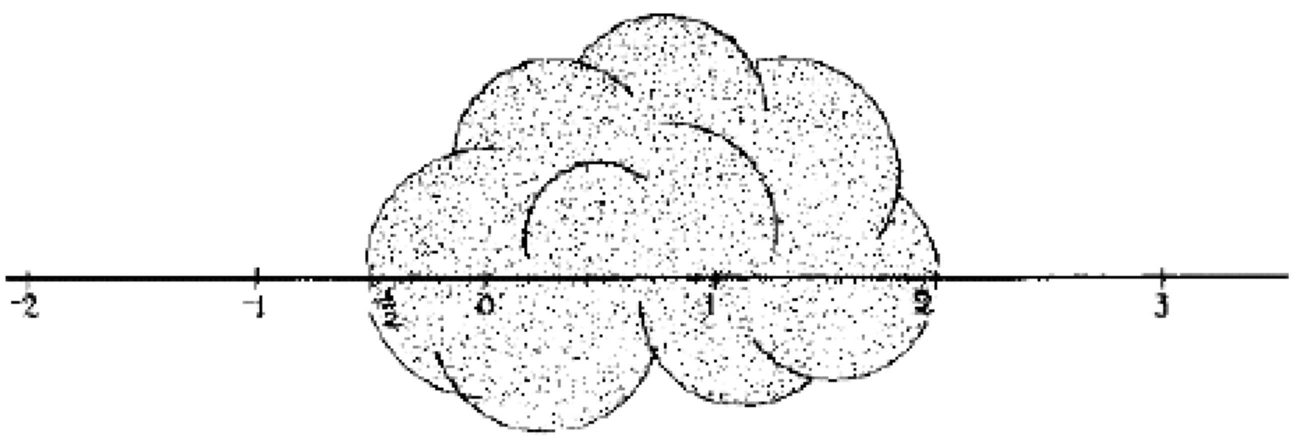

Whereabout of $\{2 \mid -\frac{1}{2}\}$.

Properties

If $x \geq y, z$ are numbers, then

$$z + \{x \mid y\} = \{z + x \mid z + y\}.$$

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Let $u = \frac{1}{2}(x + y)$, and $v = \frac{1}{2}(x - y)$, then

$$\{x \mid y\} = u + \{v \mid -v\} = u \pm v.$$

Here $\pm v$ is a short notation for $\{v \mid -v\}$, v is called temperature of $\{x \mid y\}$.

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$\{2 \mid -\frac{1}{2}\}$ is hotter than $\{4 \mid 3\}$.

Temperature policy

In any sum of switches $\{x \mid y\}$, together possibly with a number, move in any $\{x \mid y\}$ having the largest possible temperature $\frac{1}{2}(x-y)$.

Consider the game

$$z \pm a \pm b \pm c \pm \cdots \quad (a \geq b \geq c \geq \cdots \geq 0)$$

if Left starts, it soon become

$$z + a - b + c - \cdots$$

and if Right starts, it soon become

$$z - a + b - c + \cdots$$

Switch games with *

Let x and y are two numbers.

$$\{x \mid y\} + * = \{x* \mid y*\} \text{ if } x \geq y.$$
$$\{x \mid y*\} + * = \{x* \mid y\} \text{ if } x > y.$$

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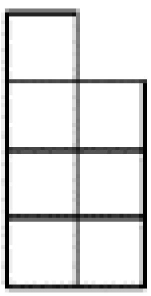

$$= \{1* \mid -1*\} = \pm(1*) = \pm 1 + * = \pm 1*.$$

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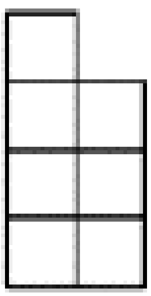


$$= \{1* \mid -1*\} = \pm(1*) = \pm 1 + * = \pm 1*.$$

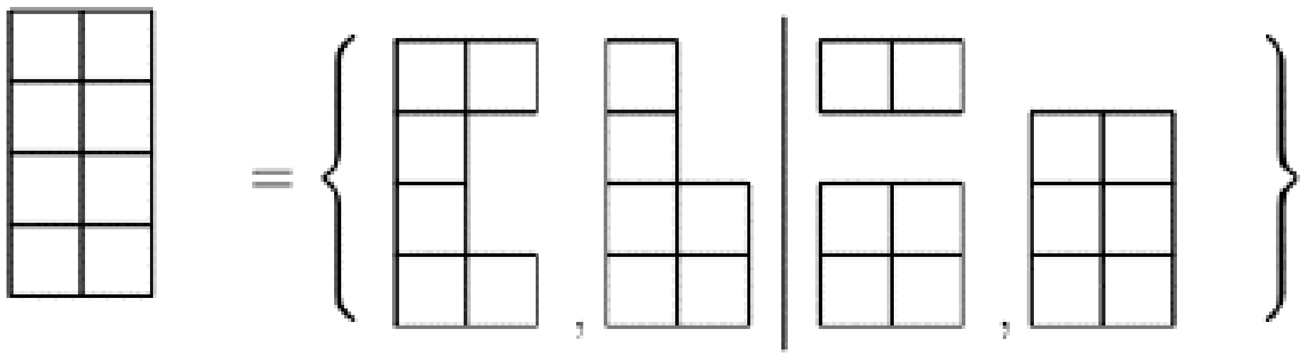
Note that the second inequality does not hold when $x = y$. For example,

$$\{0 \mid *\} + * = \uparrow *$$

$$\{* \mid 0\} = \downarrow .$$

Tiniest Games

Consider the game value of domineering game:



$$= \{0, \{2 \mid 0\} \mid \{0 \mid -2\}, \{\frac{1}{2} \mid -2\}\}$$
$$= \{0 \mid \{0 \mid -2\}\}.$$

Here we bypassed Left's reversible move and omitted the Right's dominated move. The game $\{0 \mid \{0 \mid -2\}\}$, called tiny-two and denoted by $+_2$, is a positive but much smaller than $\uparrow$.

Tiny-x and miny-x

For any value $x \geq 0$, the value $+_x = \{0 \mid \{0 \mid -x\}\}$ is called tiny-x.

For any value x, as x gets larger, $+_x$ gets smaller, rapidly. If $x > y \geq 0$, then

$$0 < +_x + +_x + \cdots + +_x < +_y.$$

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$$0 < +_x + +_x + \cdots + +_x < +_y.$$

The negation of $+_x$ is $-_x = \{\{x \mid 0\} \mid 0\}$, called miny-$x$.

$$-_y < -_x + -_x + \cdots + -_x < 0.$$

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$$-_y < -_x + -_x + \cdots + -_x < 0.$$

Note

$$+_0 = \{0 \mid \{0 \mid -0\}\} = \{0 \mid *\} = \uparrow.$$

$$-_0 = \{\{0 \mid 0\} \mid 0\} = \{* \mid 0\} = \downarrow.$$

Arithmetic operations

Two examples:

$$
\begin{aligned}
1+_2 &= 1 + \{0 \mid \{0 \mid -2\}\} \\
&= \{1 \mid 1 + \{0 \mid -2\}\} \\
&= \{1 \mid \{1 \mid -1\}\} \\
&= \{1 \mid \pm 1\}.
\end{aligned}
$$

Arithmetic operations

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&= \{1 \mid \pm 1\}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}{-}_{\frac{1}{4}} &= \frac{1}{2} + \{\{\frac{1}{4} \mid 0\} \mid 0\}\} \\
&= \{\{\frac{1}{4} \mid 0\} + \frac{1}{2} \mid \frac{1}{2}\}\} \\
&= \{\{\frac{3}{4} \mid \frac{1}{2}\} \mid \frac{1}{2}\}\}.
\end{aligned}
$$

Interpretation

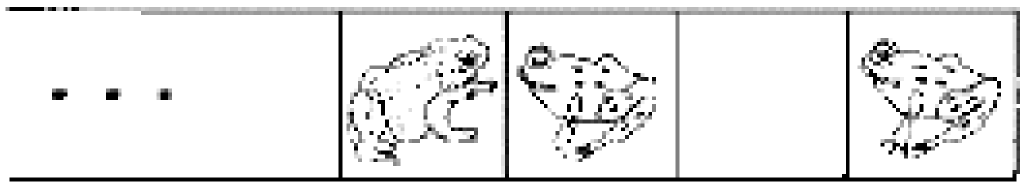
The game $+_{500} = \{0 \mid \{0 \mid -500\}\}$ can be interpreted as

> If Left has not yet filed form XYZ, then Right may issues a formal request that he do so After such a request has been issued. On any subsequent turn on which Left hast still no filed the form, Right may file a decree compelling Left to forfeit a penalty of 500 moves.

In any well-played sum of tinies and minies, the games are completed in order of increasing magnitude.

Tiny Toads-and-Frogs

the value of any position of the form



is $-_x$, where x is the value of the position obtained by making two toad moves, or is $-\frac{1}{2}$ $(= -_{-2})$ if only one toad move can be made.

Tiny Toads-and-Frogs

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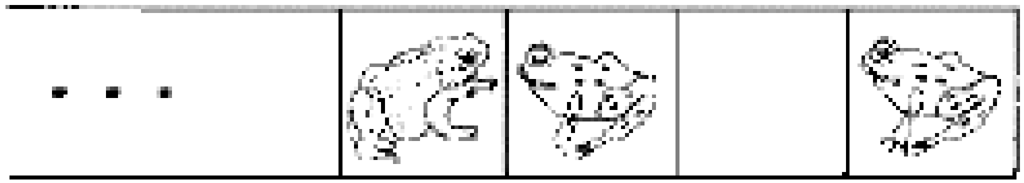
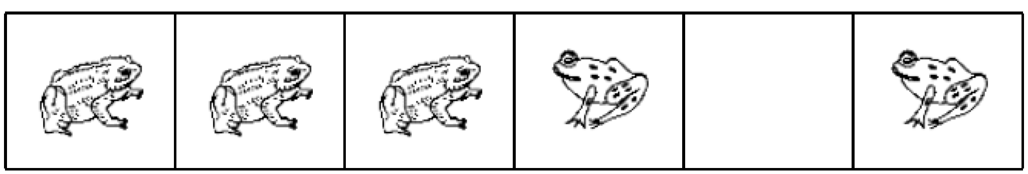


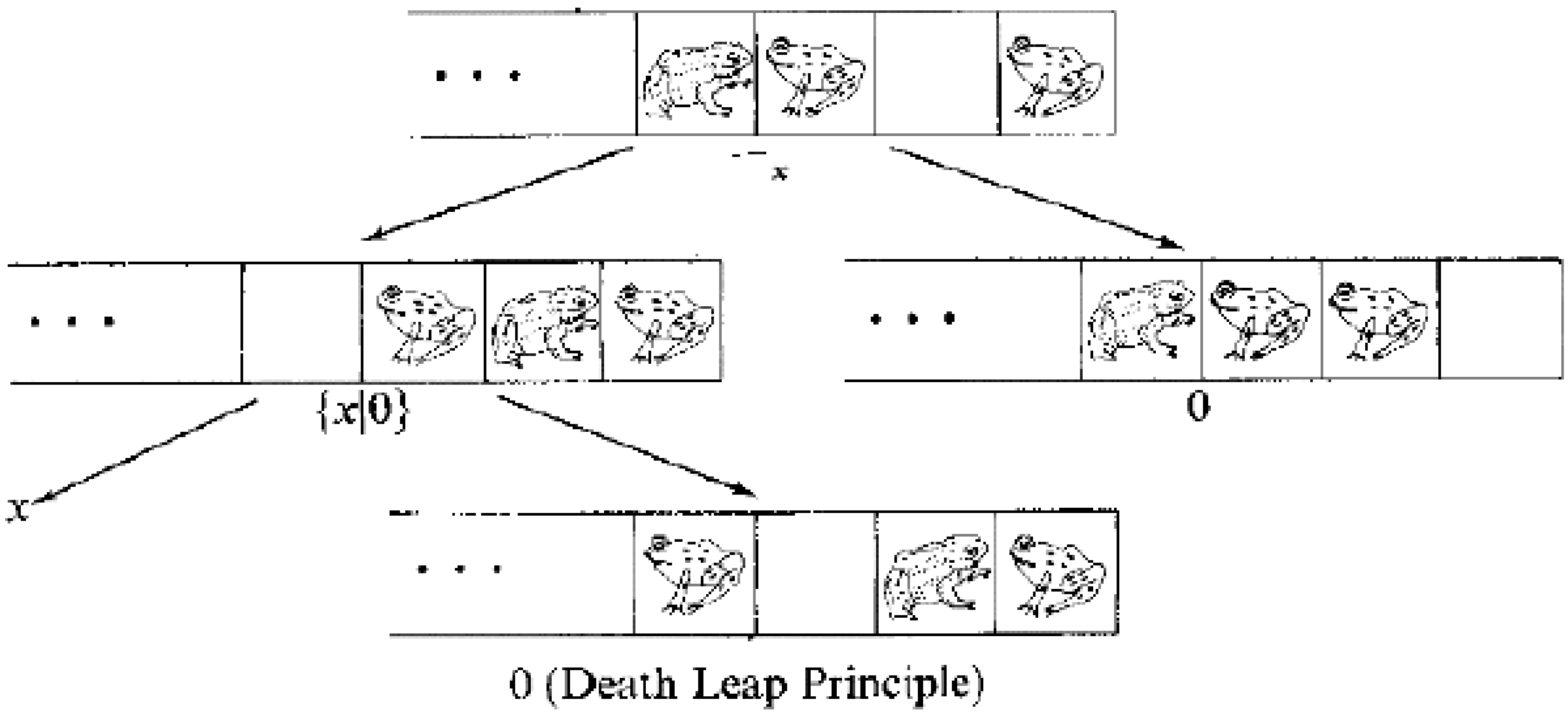
is $-_x$, where x is the value of the position obtained by making two toad moves, or is $-\frac{1}{2}$ $(= -_{-2})$ if only one toad move can be made.

For example,

 $= -_{\frac{1}{4}}$.

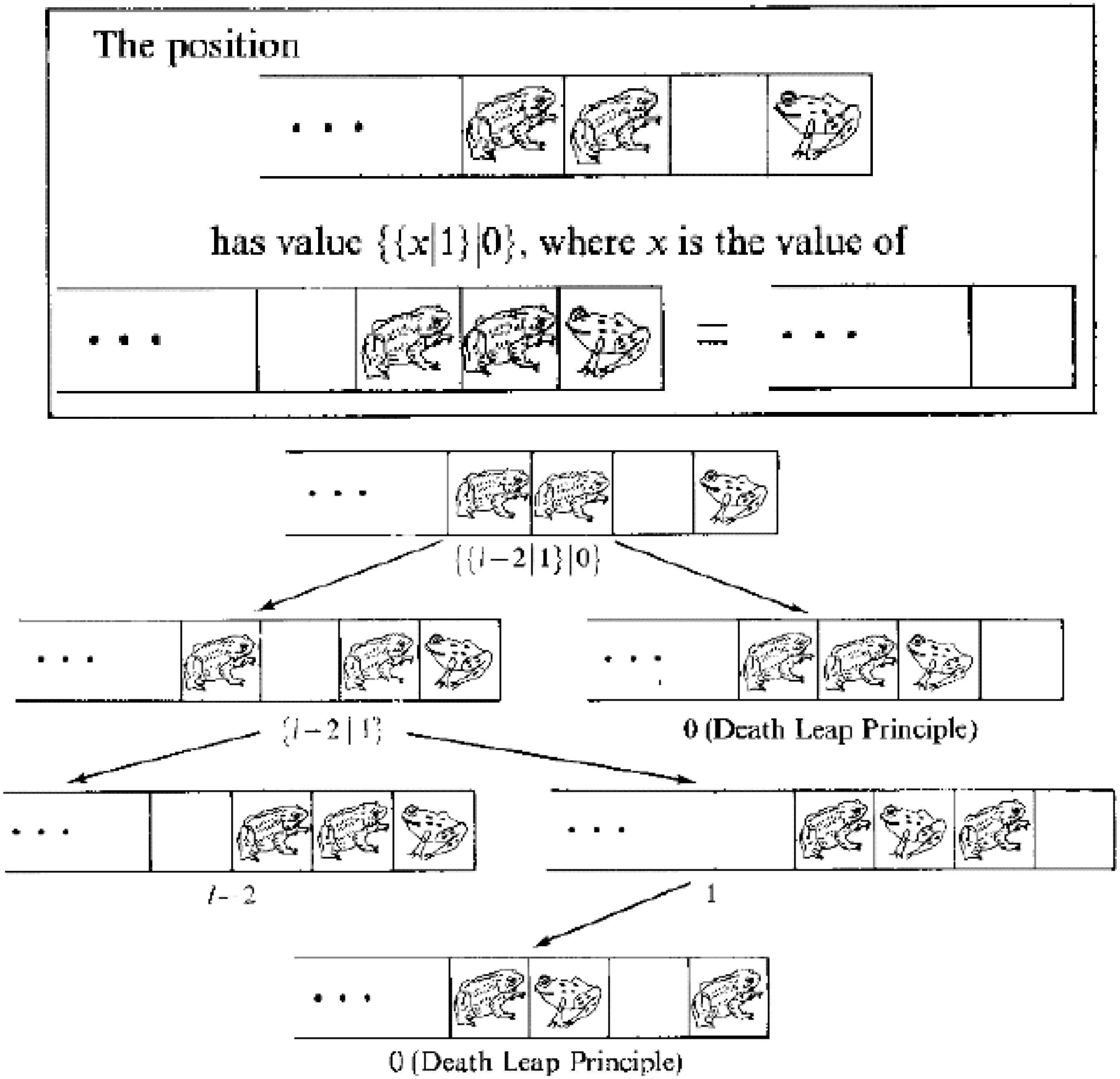
Miny Toads-and-Frogs

The occurrence of $-_x$:



Death leap principle: In a Toads-and-Frogs game, if the only legal moves from some position are jumps, the value is 0.

More Toads-and-Frogs



If Left moves, who wins?

						value	temperature $\frac{1}{2}(x-y)$
T	T	F		F	F	$* \mid -1$	$\frac{1}{2}$
	F	T	T		F	$-\frac{1}{2} \mid -1$	$\frac{1}{4}$
T	F	T		F	F	$0 \mid -\frac{1}{4}$	$\frac{1}{8}$
T	T		F	F		$\frac{1}{4} \mid \downarrow$	$\frac{1}{8}$
F	T	T		T	F	$1 \mid 1 = 1*$	0
T		T	F	F	T	$0 \mid * = \uparrow$	0

If Left moves, who wins?

T	T	F		F	F
	F	T	T		F
T	F	T		F	F
T	T		F	F	
F	T	T		T	F
T		T	F	F	T

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$* \mid -1$	$\frac{1}{2}$
$-\frac{1}{2} \mid -1$	$\frac{1}{4}$
$0 \mid -\frac{1}{4}$	$\frac{1}{8}$
$\frac{1}{4} \mid \downarrow$	$\frac{1}{8}$
$1 \mid 1 = 1*$	0
$0 \mid * = \uparrow$	0

After two rounds, it seems that Right win:

$$* - 1 + 0 + \uparrow + 1* + \uparrow = 0$$

If Left moves, who wins?

							value	temperature $\frac{1}{2}(x-y)$
T	T	F		F	F		$* \mid -1$	$\frac{1}{2}$
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F	T	T		T	F		$1 \mid 1 = 1*$	0
T		T	F	F	T		$0 \mid * = \uparrow$	0

After two rounds, it seems that Right win:

$$* - 1 + 0 + \uparrow + 1 * + \uparrow = 0$$

But $\{\frac{1}{4} \mid \downarrow\}$ is slightly hotter than $\{0 \mid -\frac{1}{4}\}$. The correct values after two rounds is: $* - 1 + \frac{1}{4} - \frac{1}{4} + 1 * + \uparrow = \uparrow$. So Left wins.

Latent heat

If left starts, who wins this game?

	T	T		F	T
T	F	T		F	F
T		T	F	F	F
	T	T	F		F

value	temperature
$\frac{1}{2} \mid 0$	$\frac{1}{4}$
$0 \mid -\frac{1}{4}$	$\frac{1}{8}$
$+_{\frac{1}{4}}$	0
$-\frac{1}{4}$	0

Latent heat

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T	F	T		F	F
T		T	F	F	F
	T	T	F		F

value	temperature
$\frac{1}{2} \mid 0$	$\frac{1}{4}$
$0 \mid -\frac{1}{4}$	$\frac{1}{8}$
$+_{\frac{1}{4}}$	0
$-\frac{1}{4}$	0

According to the temperature policy, after two moves:

$$\frac{1}{2} - \frac{1}{4} + +_{\frac{1}{4}} - \frac{1}{4} = +_{\frac{1}{4}}.$$

It seems that Left wins.

Latent heat

However, Right can responded to Left's opening by moving on the third row. The result is

$$\frac{1}{2} + \{0 \mid -\frac{1}{4}\} + \{0 \mid -\frac{1}{4}\} - \frac{1}{4}.$$

After two more moves: $\frac{1}{2} + 0 - \frac{1}{4} - \frac{1}{4} = 0$. Right wins.

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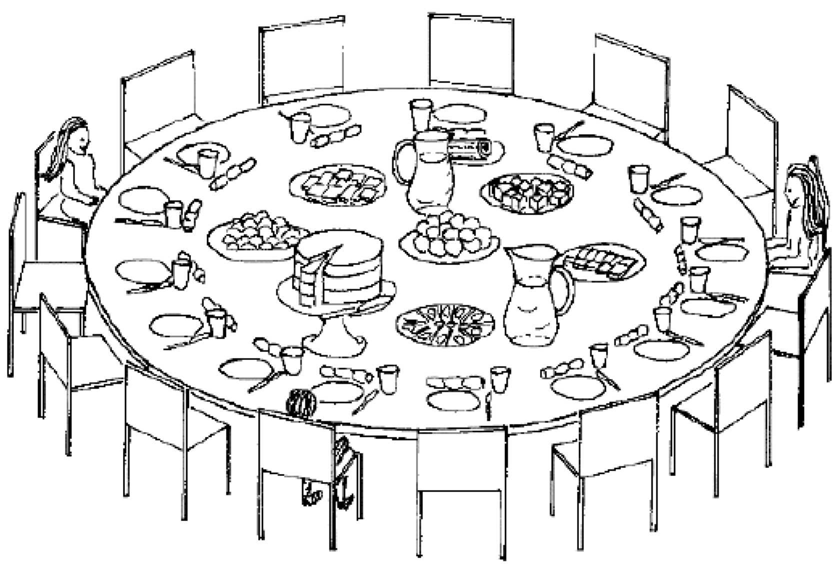
The temperature policy works with games whose options are like

$$x, x + *, x + \uparrow, x + *2, x + \uparrow + *$$

for any number x, since these have no latent heat.

Seating Boys and Girls

- **Two players:** "Left" and "Right".
- **Game board:** some dinning tables of various sizes.
- **Rules:** Two players take turns to seat boys and girls. Left will seat the boys and Right the girls. No child may be seated next to another of the opposite sex.
- **Ending positions:** Whoever gets stuck is the loser.



Values of seating-boys-girls

LnL, a row of n empty chairs between two boys,
RnR, a row of n empty chairs between two girls, and
LnR or RnL, a row of n empty chairs between a boy and a girl.

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Recursive formula: where $a + b = n - 1$, $L0R$ is not allowed.

$$LnL = \{LaL + LbL \mid LaR + RbL\}$$
$$RnR = \{RaL + LbR \mid RaR + RbR\} = -LnL$$
$$LnR = \{LaL + LbR \mid LaR + RbR\} = RnL.$$

Values of seating-boys-girls

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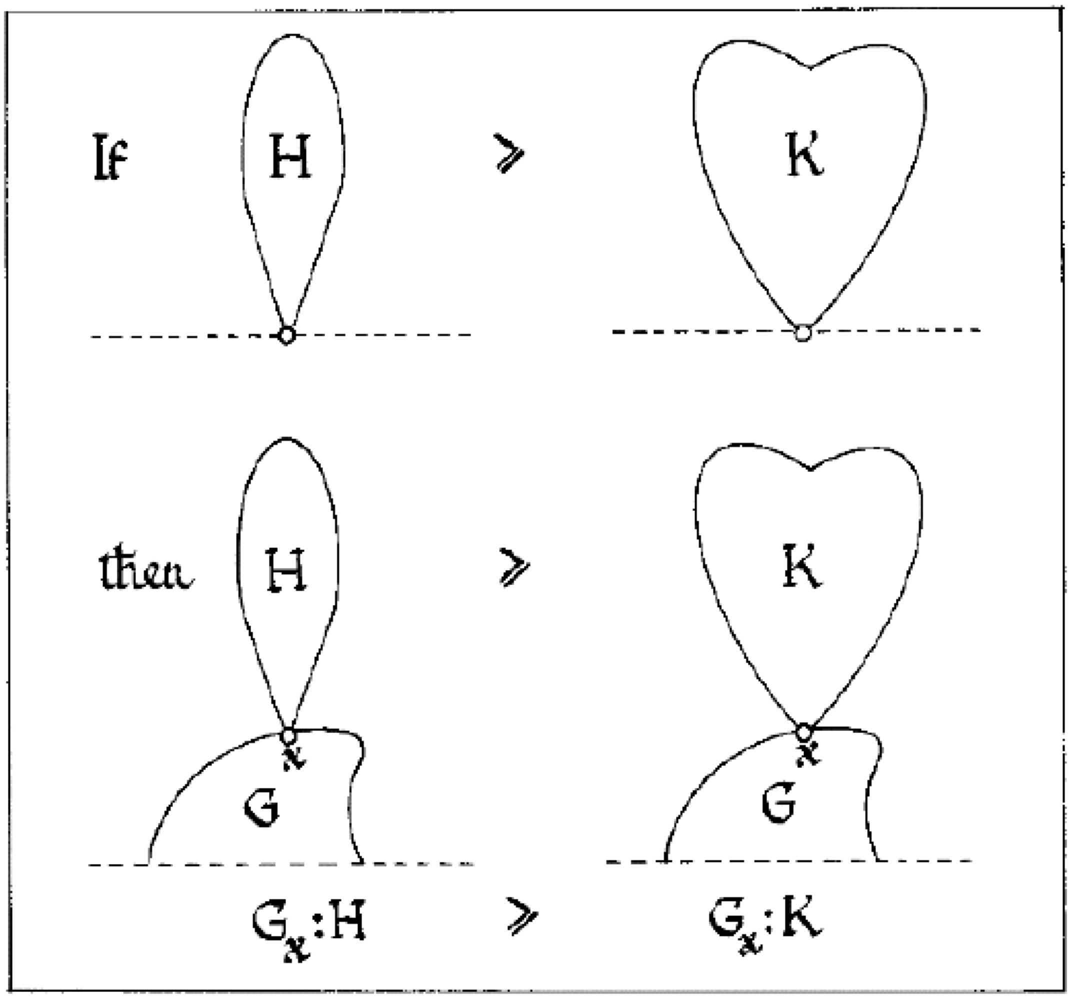
$$RnR = \{RaL + LbR \mid RaR + RbR\} = -LnL$$
$$LnR = \{LaL + LbR \mid LaR + RbR\} = RnL.$$

n	0	1	2	3	4	5	6
LnL	0	1	2	$2 \mid 0$	$3 \mid *$	$\{4 \mid 0, \pm 1\}$	$\{3 \mid *\} \pm 1$
LnR	–	0	$*$	± 1	± 2	$\pm 2*$	$\pm 2 \pm 1$
RnR	0	-1	-2	$0 \mid -2$	$* \mid -3$	$\{\pm 1, 0 \mid -4\}$	$\{* \mid -3\} \pm 1$

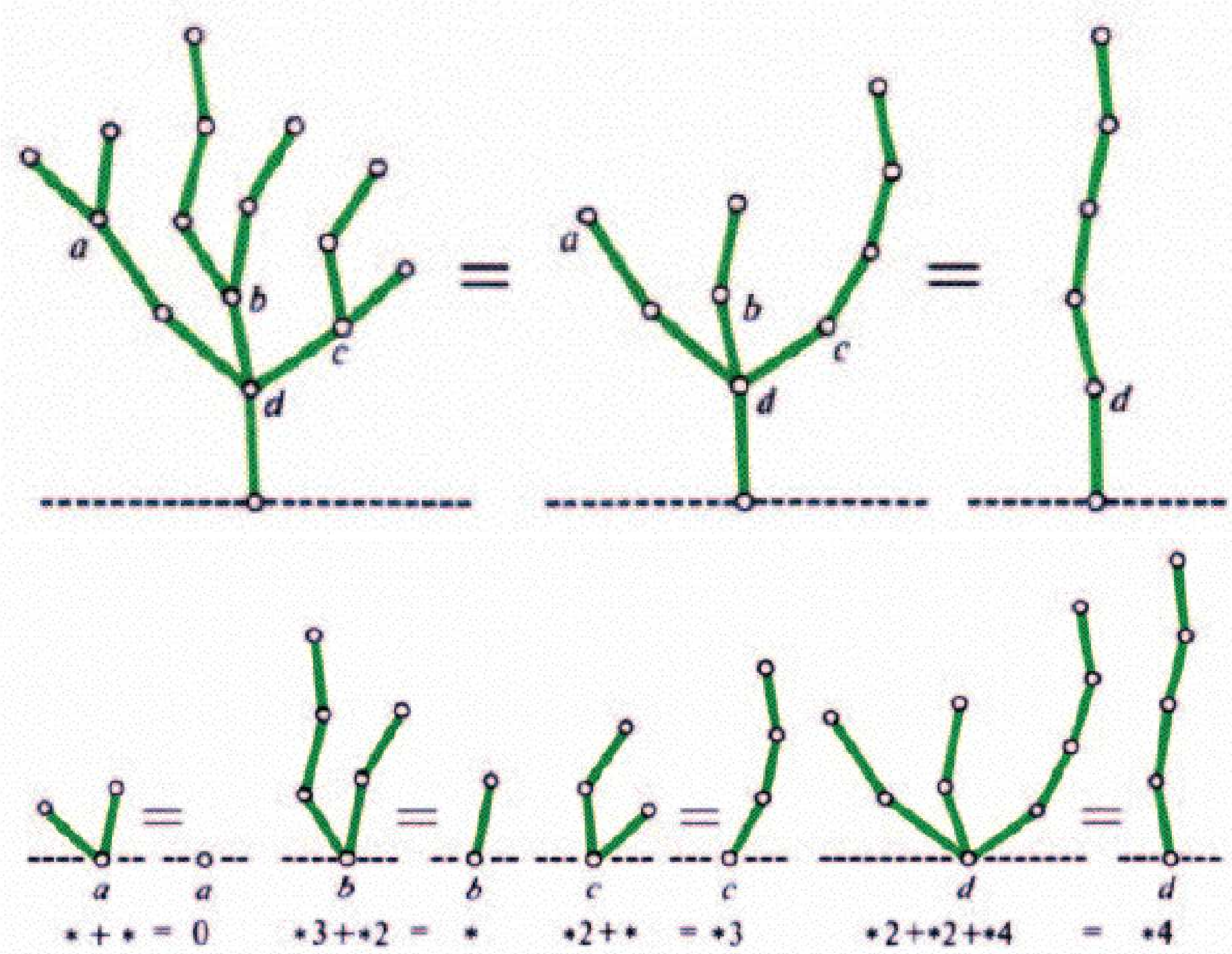
Colon Principle

In Hackenbush game, we have the following important tool:



If $H = K$, then $G_x : H = G_x : K$.

Work out Green Tree



The parity Principle

The nim value of any sum of green trees has the same parity as the total number of edges.

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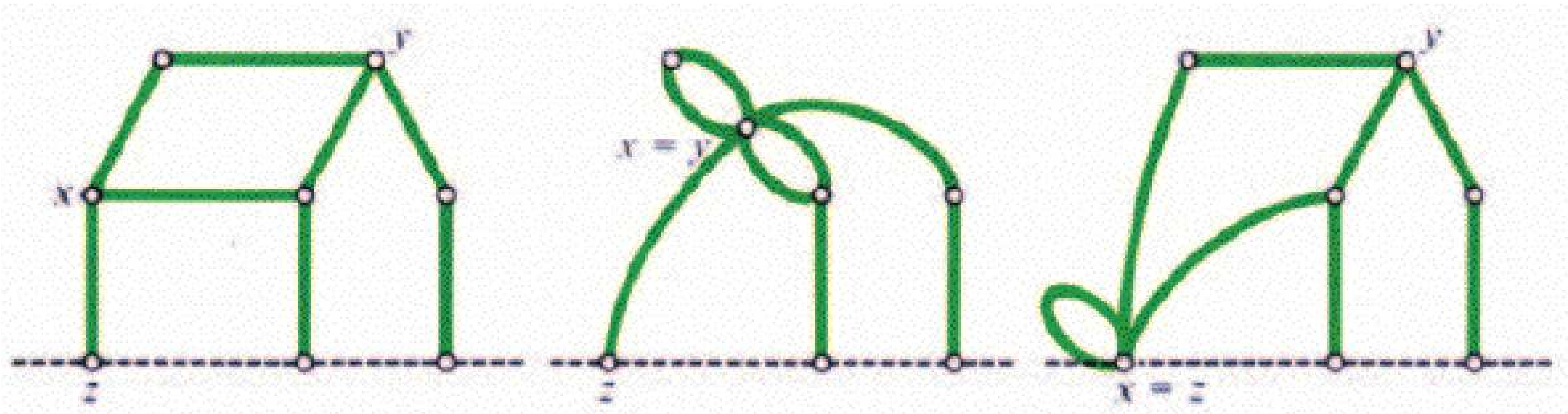
The nim value of any sum of green trees has the same parity as the total number of edges.

This is because the nim sum $a \overset{*}{+} b$ has the same parity as the ordinary sum $a + b$.

Fusion Principle

You can fuse all the nodes in any cycle of a green Hackenbush game without changing its value.



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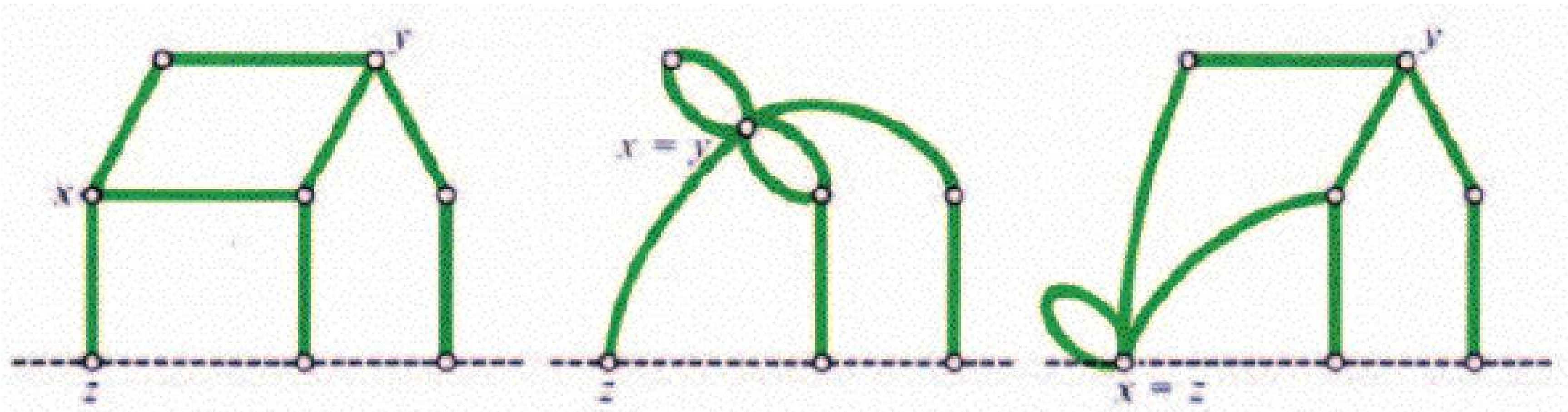
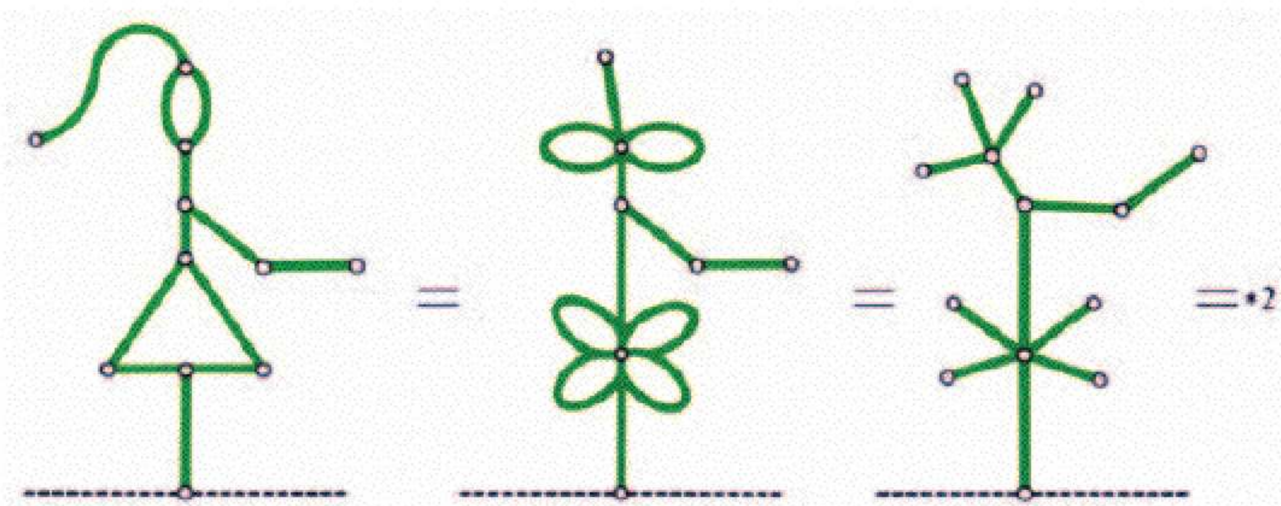


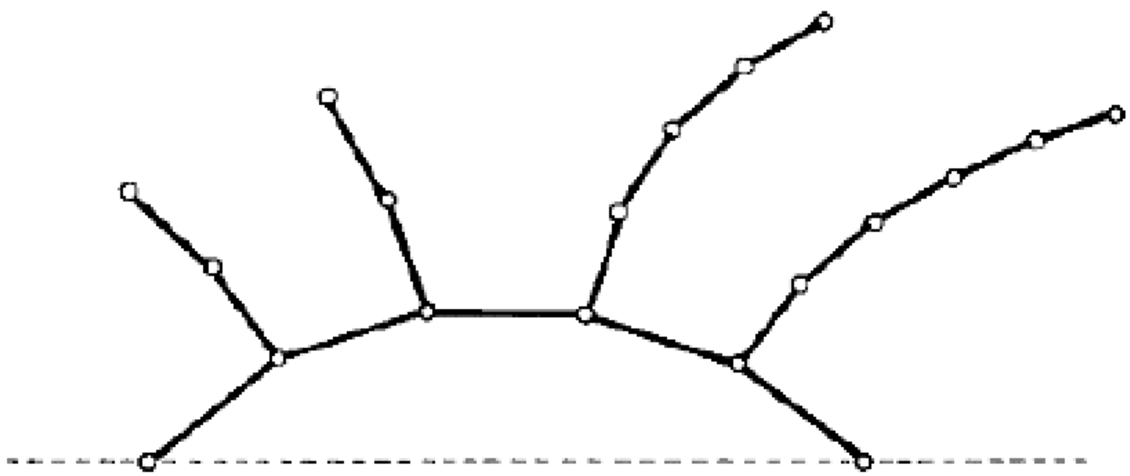


(a) (b) (c)

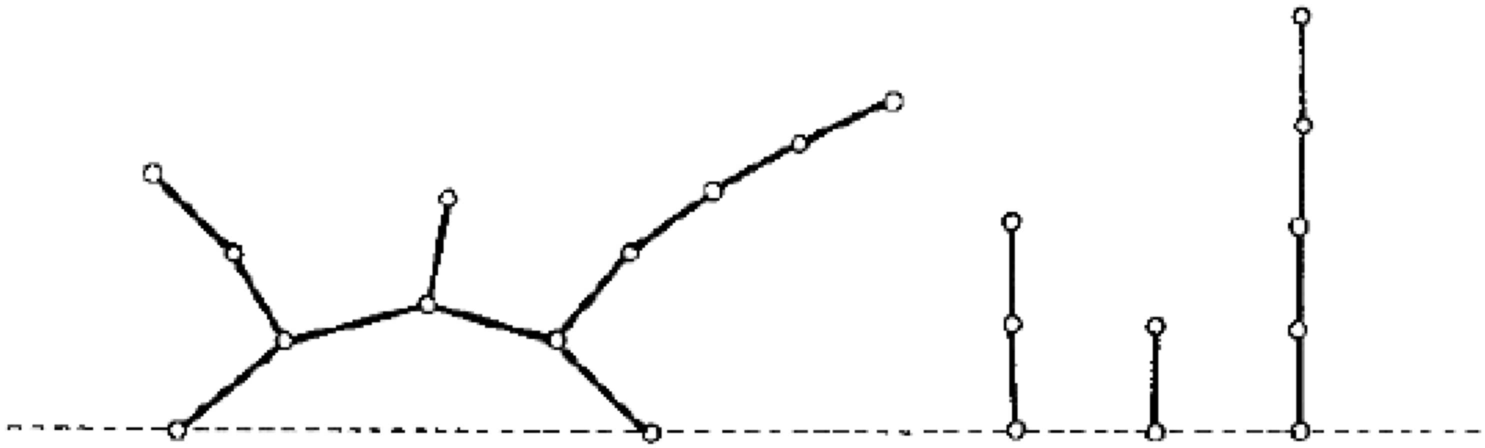
Proof of Fusion Principle

If there is a counter-example, choose the one with minimum number of edges and then with minimum number of vertices. The minimum counter example has the following properties:

- G has only one vertex on the ground.
- For any two vertices a, b, there is no three edge-dependent paths from a to b.
- No cycle can exclude the ground.
- G contains one cycle including the ground.

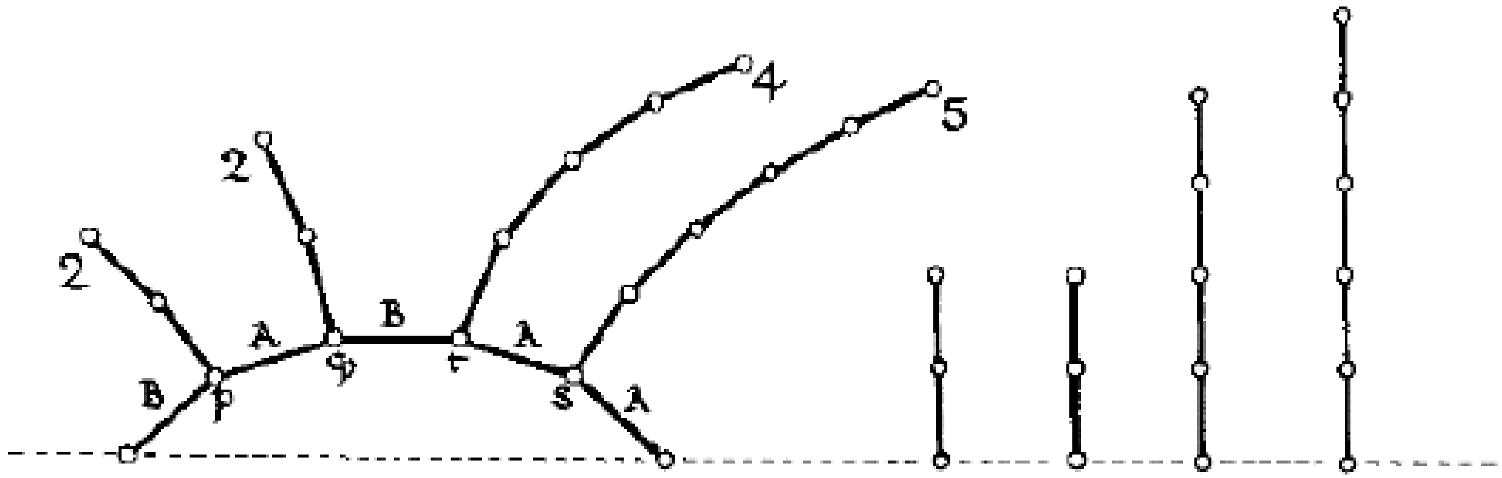


Even Bridge



The number of edges in this bridge is even. The sum of this bridge and copies of all its strings is a zero game. If not, there must a edge on the bridge so that removing it results a zero game. By the parity principle, this is a nonzero game. Contradiction.

Odd Bridge

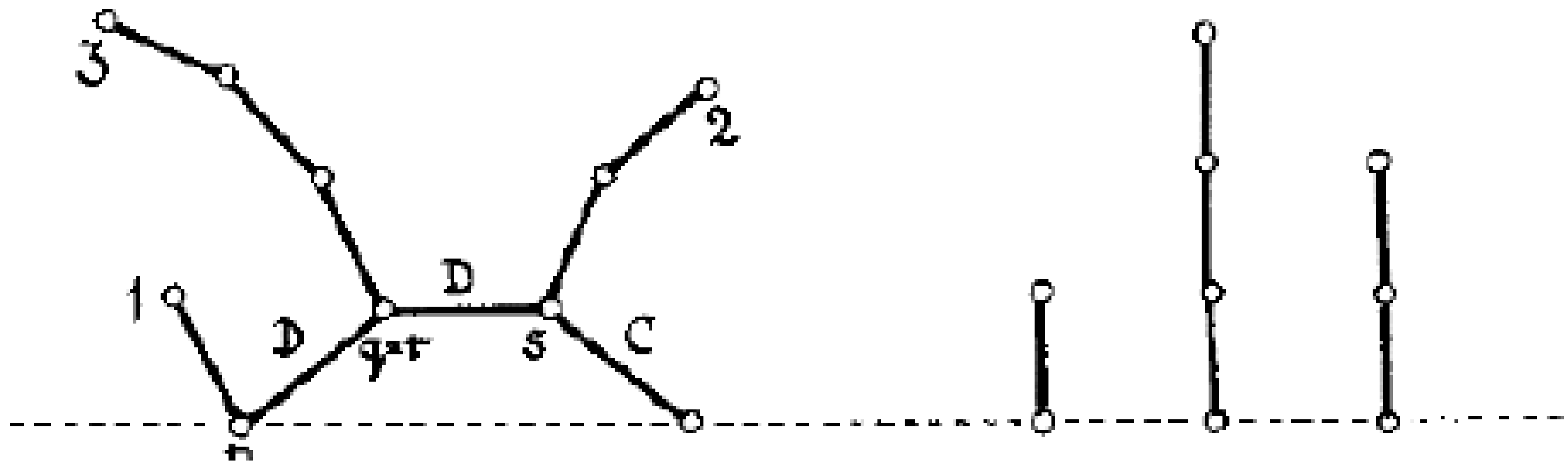


The number of edges in this bridge is odd. The sum of this bridge and copies of all its strings has game value $*$ because no option has the value $*$. It will sufficient to find an option with value 0.

Label the bridge edges by A or B so that adjacent edges have the same label if with odd string between them and different labels if with even string between them.

Half graph

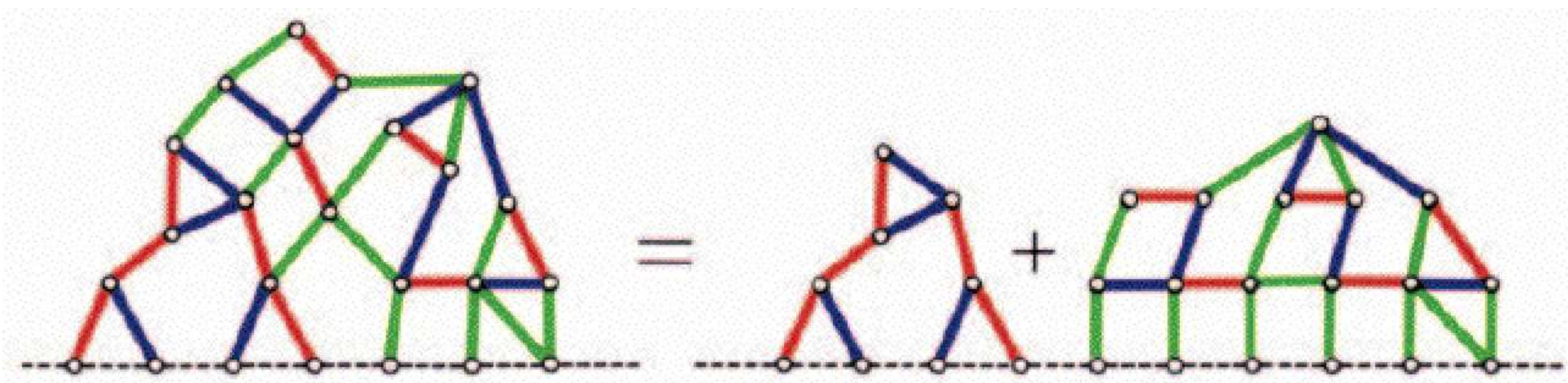
Since B appears even times, contract B-edges and half the strings. We get the following half graph.



It can show that this reduction halfs the nim value. There is one edge labeled in C. This edge is the winning move to 0 in the original graph.

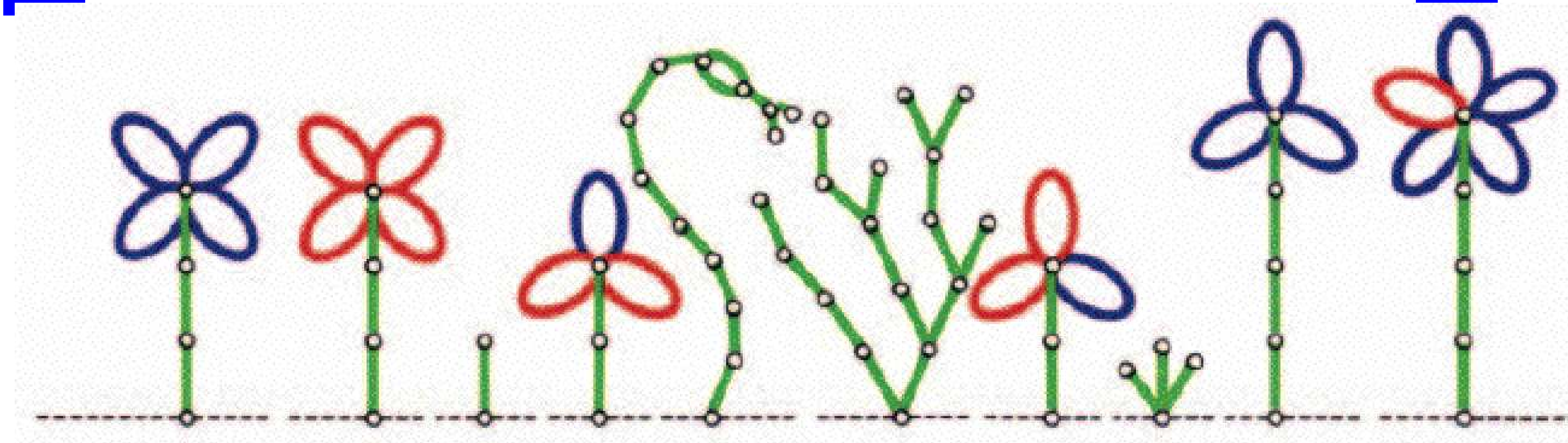
Purple mountain

In red-blue-green Hackenbush game, the part of the picture made up red and blue edges, which are connected to the ground by other red or blue edges, is called purple mountain; the rest of the picture is called green jungle.



If you know the values of purple mountain and the green jungle, then you know the value of Hackenbush game.

Flower Gardens

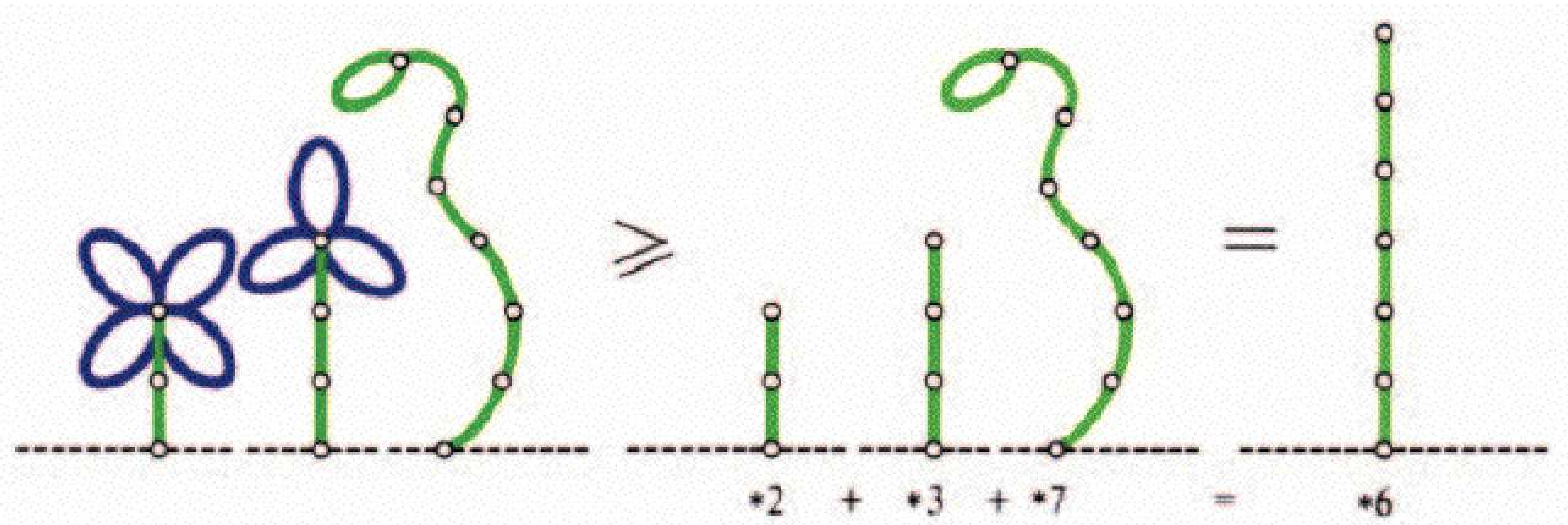


A flower has a green stem supporting a blossom of blue or red petals.

> If there are no red flowers, at least one blue flower, and any amount of greenery, then Left has a winning move.

Two-head Rule

If there are no red flowers, at least two blue flower, and any amount of greenery, then Left wins even Right starts first.



Atomic weights

In a sum of flowers and nimbers, Left will prefer any move which cuts a red flower than any move which cuts the blue flower.

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All blue flowers have atomic weight $+1$ while all red flowers have atomic weight -1.

If atomic weights ≥ 2, Left wins. If atomic weights ≤ -2, Right wins.

In Hackenbush flowers, quantity is much important than quality!