Math576: Combinatorial Game Theory

Lecture note III

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Fall, 2020

Disclaimer

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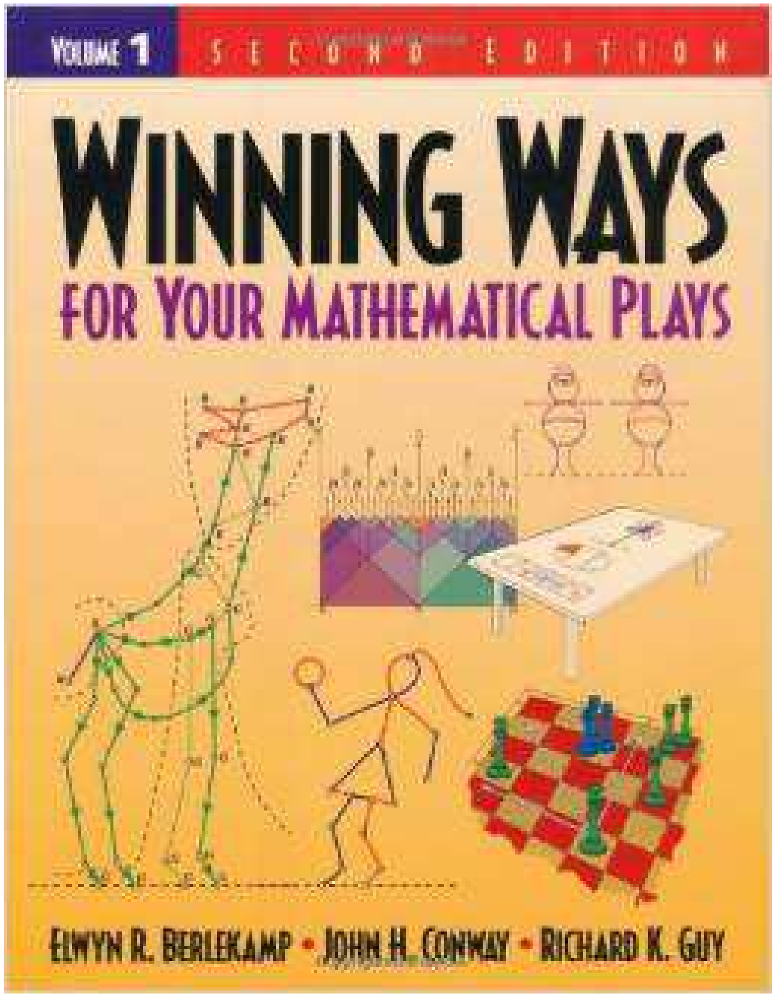
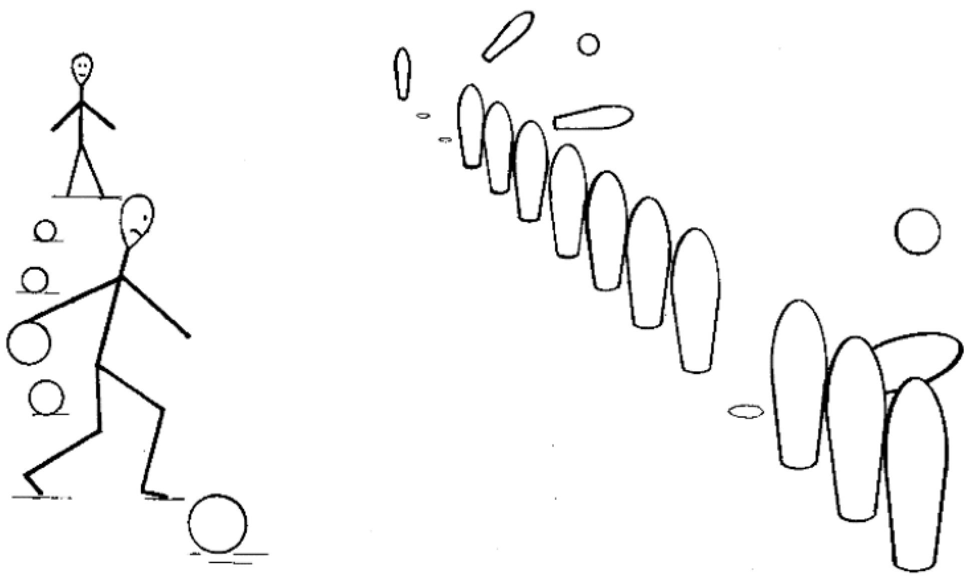


The Game of Kayles

- **Two players:** "Left" and "Right".
- **Game board:** a row of well-spaced pins.
- **Rules:** Two players take turns. Either player can knock down any desired pin or any two adjacent pins.
- **Ending positions:** Whoever gets stuck is the loser.



Analyse of Kayles

Since Kayles is an impartial game, the game values are $*m$ for some integer m. Let $\mathcal{G}(n)$ be the nim value of a row of n pins. It satisfies the following the cursive formula:

$$\mathcal{G}(n) = mex(\mathcal{G}(a) \stackrel{*}{+} \mathcal{G}(b))$$

where $0 \leq a, b$ and $a + b = n - 1$ or $n - 2$.

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where $0 \leq a, b$ and $a + b = n - 1$ or $n - 2$.

We have $\mathcal{G}(0) = 0$, $\mathcal{G}(1) = 1$, $\mathcal{G}(2) = 2$,

$$\begin{aligned} \mathcal{G}(3) &= mex(\mathcal{G}(0) \overset{*}{+} \mathcal{G}(2), \mathcal{G}(1) \overset{*}{+} \mathcal{G}(1), \mathcal{G}(1) \overset{*}{+} \mathcal{G}(0)) \\ &= mex(2, 0, 1) = 3. \end{aligned}$$

Kayles Game values

n	0	1	2	3	4	5	6	7	8	9	10	11
	0	1	2	**3**	1	4	**3**	2	1	**4**	2	**6**
12	4	1	2	**7**	1	4	**3**	2	1	**4**	**6**	7
24	4	1	2	8	**5**	4	7	2	1	8	**6**	7
36	4	1	2	**3**	1	4	7	2	1	8	2	7
48	4	1	2	8	1	4	7	2	1	**4**	2	7
60	4	1	2	8	1	4	7	2	1	8	**6**	7
72	4	1	2	8	1	4	7	2	1	8	2	7
84	4	1	2	8	1	4	7	2	1	8	2	7
96	4	1	2	8	1	4	7		...			

$\mathcal{P}$-position and $\mathcal{N}$-postition

Impartial games can only have two outcome classes:

- $\mathcal{P}$-positions (a value of 0): Previous player winning;
- $\mathcal{N}$-positions (values $*n$ $(n \neq 0)$): Next player winning.

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We say an impartial game has the **nim-sequence**

$$a.bcd\cdots$$

if

$$\mathcal{G}(0) = a, \mathcal{G}(1) = b, \mathcal{G}(2) = c, \mathcal{G}(3) = d, \ldots$$

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For example, Kayles has nim-equence

$$0.1231432142641271432146 7\cdots.$$

Substraction games

We may modify the game of Nim by requiring that in any move the number of beans taken away is at most three. This game is denoted by $S(1, 2, 3)$.

$$\mathcal{G}(n) = mex(\mathcal{G}(n-1), \mathcal{G}(n-2), \mathcal{G}(n-3)).$$

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In general, $S(1,2,\ldots,k)$ is the modified Nim game by requiring that in any move the number of beans taken away is at most k.

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In general, $S(1, 2, \ldots, k)$ is the modified Nim game by requiring that in any move the number of beans taken away is at most k. The nim sequence of $S(1, 2, \ldots, k)$ is

$$0.12\cdots k012\cdots k012\cdots k\ldots$$

General substraction games

In general, we can require that a heap may be reduced only by one of the numbers s_1,s_2, s_3,. . .. We call this a **substraction game** $S(s_1, s_2, s_3, \ldots)$.

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For example, consider the Game $S(2, 5, 6)$:

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For example, consider the Game $S(2, 5, 6)$:

$$\mathcal{G}(n) = mex(\mathcal{G}(n-2), \mathcal{G}(n-5), \mathcal{G}(n-6)).$$

The nim sequence of $S(2, 5, 6)$ is

$$0.0110213021001102130 21 \ldots.$$

It has a period 11.

Substraction Games

Nim-Sequences for Substraction Games:

Subtraction set (with optional extras)	nim-sequence	period
1(3 5 7 9 11 ...)	$\dot{0}\dot{1}01\ldots$	2
2(6 10 14 18 ...)	$\dot{0}01\dot{1}0011\ldots$	4
1 2(4 5 7 8 10 11 ...)	$\dot{0}1\dot{2}012\ldots$	3
3(9 15 21 27 ...)	$\dot{0}0011\dot{1}000111\ldots$	6
2 3(7 8 12 13 17 18 ...)	$\dot{0}011\dot{2}00112\ldots$	5
1 2 3(5 6 7 9 10 11 13 ...)	$\dot{0}12\dot{3}0123\ldots$	4
4(12 20 28 36 ...)	$\dot{0}000111\dot{1}00001111\ldots$	8
1 4(6 9 11 14 16 19 ...)	$\dot{0}101\dot{2}01012\ldots$	5
2 4(3 8 9 10 14 15 16 ...)	$\dot{0}0112\dot{2}001122\ldots$	6
3 4(10 11 17 18 24 25 ...)	$\dot{0}00111\dot{2}0001112\ldots$	7
1 3 4(6 8 10 11 13 15 17 ...)	$\dot{0}10123\dot{2}0101232\ldots$	7
1 2 3 4(6 7 8 9 11 12 13 14 ...)	$\dot{0}123\dot{4}01234\ldots$	5

Substraction Games

Nim-Sequences for Substraction Games:

Subtraction set (with optional extras)	nim-sequence	period
5(15 25 35 45 ...)	$\dot{0}00001111\dot{1}0000011111\ldots$	10
2 5(9 12 16 19 23 26 ...)	$\dot{0}01102\dot{1}0011021\ldots$	7
3 5(4 11 12 13 19 20 21 ...)	$\dot{0}001112\dot{2}00011122\ldots$	8
2 3 5(4 9 10 11 12 16 17 18 19 ...)	$\dot{0}01122\dot{3}0011223\ldots$	7
4 5(13 14 22 23 31 32 40 ...)	$\dot{0}0001111\dot{2}000011112\ldots$	9
1 4 5(3 7 9 11 12 13 15 17 19 ...)	$\dot{0}101232\dot{3}01012323\ldots$	8
2 4 5(3 9 10 11 12 16 17 18 19 ...)	$\dot{0}01122\dot{3}0011223\ldots$	7
1 2 3 4 5(7 8 9 10 11 13 14 15 16 ...)	$\dot{0}1234\dot{5}012345\ldots$	6

Substraction Games

Nim-Sequences for Substraction Games:

Subtraction set (with optional extras)	nim-sequence	period
6(18 30 42 54 ...)	$\dot{0}00000111111\dot{1}000000111111\ldots$	12
1 6(8 13 15 20 22 27 29 ...)	$\dot{0}101012\dot{0}101012\ldots$	7
1 2 6(5 8 9 12 13 15 16 19 20 ...)	$\dot{0}120123\dot{0}120123\ldots$	7
3 6(4 5 12 13 14 15 21 22 23 ...)	$\dot{0}00111222\dot{0}00111222\ldots$	9
1 3 6(8 10 12 15 17 19 21 24 ...)	$\dot{0}10101232\dot{0}10101232\ldots$	9
2 3 6(7 11 12 15 16 20 21 24 ...)	$\dot{0}01120312\dot{0}01120312\ldots$	9
4 6(5 14 15 16 24 25 26 34 ...)	$\dot{0}000111122\dot{0}000111122\ldots$	10
2 4 6(3 5 10 11 12 13 14 18 19 ...)	$\dot{0}0112233\dot{0}0112233\ldots$	8
1 2 4 6(7 9 10 12 14 15 17 18 20 ...)	$\dot{0}1201234\dot{0}1201234\ldots$	8
5 6(16 17 27 28 38 39 49 50 ...)	$\dot{0}0000111112\dot{0}0000111112\ldots$	11
1 5 6(3 8 10 12 14 16 17 19 21 ...)	$\dot{0}1010123232\dot{0}1010123232\ldots$	11
2 5 6(9 13 16 17 20 24 27 28 ...)	$\dot{0}0110213021\dot{0}0110213021\ldots$	11
2 3 5 6(4 10 11 12 13 14 18 19 ...)	$\dot{0}0112233\dot{0}0112233\ldots$	8
1 4 5 6(3 8 10 12 13 14 15 17 19 ...)	$\dot{0}10123234\dot{0}10123234\ldots$	9
1 2 4 5 6(8 9 11 12 14 15 16 18 19 ...)	$\dot{0}120123453\dot{0}120123453\ldots$	10
1 2 3 4 5 6(8 9 10 11 12 13 15 16 17 ...)	$\dot{0}123456\dot{0}123456\ldots$	7

Substraction Games

Nim-Sequences for Substraction Games:

Subtraction set (with optional extras)	nim-sequence	period
7(21 35 49 63 ...)	$\dot{0}000000111111\dot{1}0000000111111\ldots$	14
2 7(11 16 20 25 29 34 ...)	$\dot{0}0110011\dot{2}001100112\ldots$	9
3 7(13 17 23 17 33 37 ...)	$\dot{0}00111022\dot{1}0001110221\ldots$	10
4 7(5 6 15 16 17 18 26 27 28 ...)	$\dot{0}000111122\dot{2}00001111222\ldots$	11
1 4 7(9 12 15 17 20 23 25 28 ...)	$\dot{0}101201\dot{2}01012012\ldots$	8
2 4 7(10 13 16 19 22 25 28 31 ...)	$00112203\dot{1}0\dot{2}102\ldots$	3
3 4 7(5 6 13 14 15 16 17 23 24 ...)	$\dot{0}00111222\dot{3}0001112223\ldots$	10
1 3 4 7(5 9 11 12 13 15 17 19 20 ...)	$\dot{0}101232\dot{3}01012323\ldots$	8
2 3 4 7(8 9 13 14 15 18 19 20 24 ...)	$\dot{0}011220314\dot{2}00112203142\ldots$	11
5 7(6 17 18 19 29 30 31 41 ...)	$\dot{0}0000111112\dot{2}000001111122\ldots$	12
2 5 7(11 15 17 20 24 27 29 33 ...)	$\dot{0}01102132203100112233\dot{2}\ldots$	22
3 5 7(4 6 13 14 15 16 17 23 24 ...)	$\dot{0}00111222\dot{3}0001112223\ldots$	10
2 3 5 7(4 6 11 12 13 14 15 16 20 ...)	$\dot{0}0112233\dot{4}001122334\ldots$	9
2 4 5 7(3 6 11 12 13 14 15 16 20 ...)	$\dot{0}0112233\dot{4}001122334\ldots$	9

Substraction Games

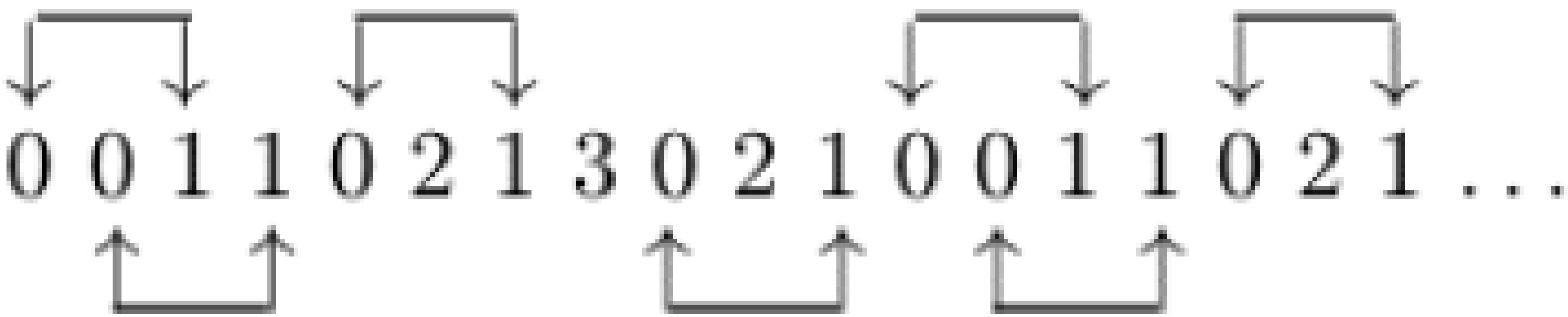
Nim-Sequences for Substraction Games:

Subtraction set (with optional extras)	nim-sequence	period
6 7(19 20 32 33 45 46 58 ...)	00000011111120000001111112...	13
1 6 7(3 5 7 9 11 13 15 17 18 19 ...)	010101232323010101232323...	12
2 6 7(11 15 19 20 24 28 32 33 ...)	00110011203120011001120312...	13
1 2 6 7(4 9 10 12 14 15 17 18 20 ...)	0120123401201234...	8
3 6 7(4 5 13 14 15 16 17 23 24 ...)	00011122230001112223...	10
1 4 6 7(9 12 14 17 19 20 22 25 ...)	01012012320120101201232012...	13
2 4 6 7(3 5 11 12 13 14 15 16 20 ...)	001122334001122334...	9
1 3 4 6 7(5 9 11 13 14 15 16 17 19 ...)	01012323450101232345...	10
2 5 6 7(10 14 17 18 19 22 26 29 ...)	001102132233001102132233...	12
1 2 5 6 7(4 9 10 12 13 15 16 17 18 ...)	0120123453401201234534...	11
1 4 5 6 7(3 9 11 13 14 15 16 17 19 ...)	01012323450101232345...	10
1 2 3 4 5 6 7(9 10 11 12 13 14 15 17 ...)	0123456701234567...	8

Ferguson's Pairing Property

$\mathcal{G}(n) = 1$ if and only if $\mathcal{G}(n - s_1) = 0$, where s_1 is the least member of the substraction set.

For example, the nim-sequence for $S(2, 5, 6)$ has its zeros and ones paired as:



Proof

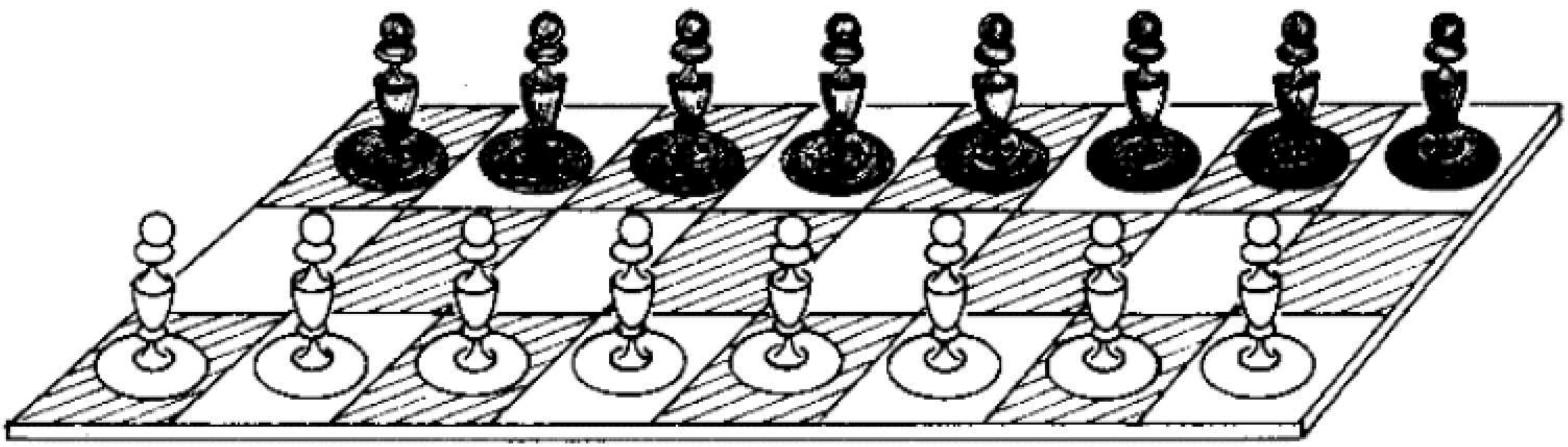
$\mathcal{G}(n) = 1$ and $\mathcal{G}(n - s_1) \neq 0$	or	$\mathcal{G}(n - s_1) = 0$ and $\mathcal{G}(n) \neq 1$

respectively imply

$\mathcal{G}(n - s_1 - s_k) = 0$ for some s_k, which implies inductively $\mathcal{G}(n - s_k) = 1$, which implies $\mathcal{G}(n) \neq 1$.	or	$\mathcal{G}(n - s_k) = 1$ for some s_k, which implies inductively $\mathcal{G}(n - s_k - s_1) = 0$, which implies $\mathcal{G}(n - s_1) \neq 0$.

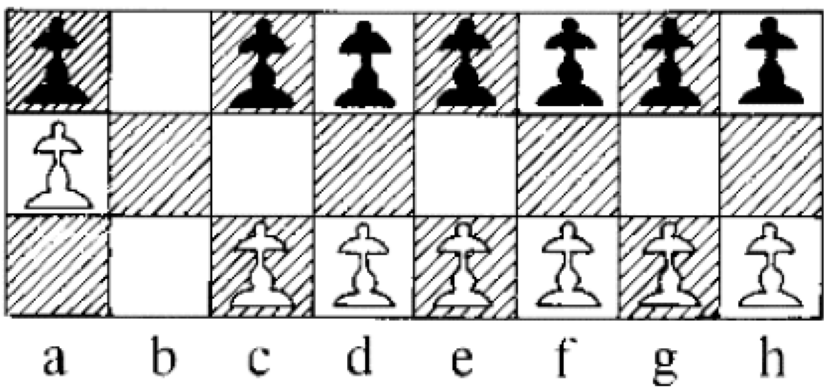
Dawson's Chess

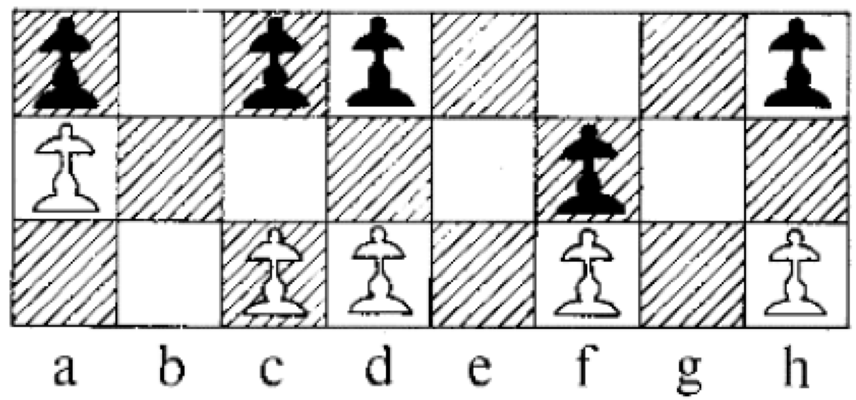
- **Two players:** "White" and "Black".
- **Game board:** a $3 \times n$ chessboard with White pawns on the first rank and Black pawns on the third.
- **Rules:** Two players take turns. Pawns move (forwards) and capture (forwards) and capture (diagonally) as in Chess. If a pawn of the opponent can be captured, then it must be captured immediately.
- **Ending positions:** Whoever gets stuck is the loser.



Analysis of Dawson's Chess

Observe that "queening" can never arise in this game. After White moves a-pawn, Black must capture this with b-pawn, White must then recapture it. If Black now advances his f-pawn, White captures it, Black recaptures it, and White recaptures it.





Dawson's chess is similar to Kayles. It is a kind of take-and-break game. So the game values are $*m$.

Values of Dawson's Chess

Let $\mathcal{G}(-1) = \mathcal{G}(0) = 0$. Then

$$\mathcal{G}(n) = mex(\mathcal{G}(a) \overset{*}{+} \mathcal{G}(b) \colon -1 \leq a, b \text{ and } a + b = n - 3.$$

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$$\mathcal{G}(n) = mex(\mathcal{G}(a) \overset{*}{+} \mathcal{G}(b) \colon -1 \leq a, b \text{ and } a + b = n - 3.$$

$\mathcal{G}(1) = 1$, $\mathcal{G}(2) = mex(\mathcal{G}(0) \overset{*}{+} \mathcal{G}(1)) = mex(0) = 1$, and

$$\mathcal{G}(3) = mex(\mathcal{G}(-1) + \mathcal{G}(1), \mathcal{G}(0) + \mathcal{G}(0)) = mex(0, 1) = 2.$$

$$\mathcal{G}(4) = mex(\mathcal{G}(-1) + \mathcal{G}(2), \mathcal{G}(0) + \mathcal{G}(1)) = mex(1, 1) = 0.$$

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n	0	1	2	3	4	5	6	7	8	9		11		13		15		17		19		21		23		25		27		29		31		33
	0	1	1	2	0	3	1	1	0	3	3	2	2	4	**0**	5	**2**	**2**	3	3	0	1	1	3	0	2	1	1	0	4	5	**2**	7	4
34	**0**	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	**2**	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
68	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
102	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
136	8	1	1	...																														

Other take-and-break games

Dawson's Chess can be turned into a game with heaps:

- A *single* pin may be removed.
- *Two* pins at the end of a longer row may be removed.
- Any *three* adjacent pins may be removed and leave two shorter rows.

Dawson's Chess can be written symbolically as $.\mathbf{137}$. Here

$\mathbf{1}$	2^0	for removal of one bean,
$\mathbf{3}$	$2^1 + 2^0$	for removal of two beans,
$\mathbf{7}$	$2^2 + 2^1 + 2^0$	for removal of three beans.

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Kayles Game can be coded as $.\mathbf{77}$ since we can remove 1 or 2 beans in any way.

Code digits interpretation

A take-and-break game is coded by $.\mathbf{d_1 d_2 d_3} \ldots$ where the code digit

$$d_k = 2^a + 2^b + 2^c + \cdots \text{ for removal of k beans}$$

Value of $\mathbf{d}_k$	Conditions for removal of k beans from a single heap.
0	Not permitted.
1	If the beans removed are the whole heap.
2	Only if some beans remain and are left as a single heap.
3	Provided the remaining beans, if any, are left in one heap.
4	Only if some beans remain and are left as exactly two non-empty heaps.
5	Provided the remaining beans, if any, are left as two non-empty heaps.
6	Only if some beans remain and are left as one or two heaps.
7	Provided the remaining beans are left in at most two heaps.
8	Only if some beans remain and are left in just three non-empty heaps.
etc.	

Dawson's Kayles

Dawson's Kayles is the take-and-break game $.\mathbf{07}$, which means you are allowed to take any two adjacent beans from a row of size 2, one end of the row, or in the middle.

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$$\mathcal{G}(n) = mex(\mathcal{G}(a) \overset{*}{+} \mathcal{G}(b) \colon a + b = n - 2).$$

$$\mathcal{G}(\mathbf{.07}) = 0.0112031103322405223301130211045274 \cdots$$

The value is the same as that of the Dawson's Chess game with $n-1$ pairs of pawns.

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The value is the same as that of the Dawson's Chess game with $n-1$ pairs of pawns.

$$\mathcal{G}(\mathbf{.17}) = 0.1102130113223415322311031201144264\cdots$$

The values are obtained from Dawson's Kayles by mim-adding 1 when n is odd.

Guiles

Guiles: to remove a heap of 1 or 2 beans completely, or to take two beans from a sufficient large heap and partition what remains into two smaller non-empty heaps. Guiles is just the game $\mathbf{.15}$.

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$$\mathcal{G}(n) = 0.\dot{1}10112212\dot{2}.$$

Guiles has a period 10.

Treblecross

Treblecross: a Tic-Tac-Toe game played on a $1 \times n$ strip in which both player use the same symbol (X). The first person to complete a line of three consecutive crosses wins.

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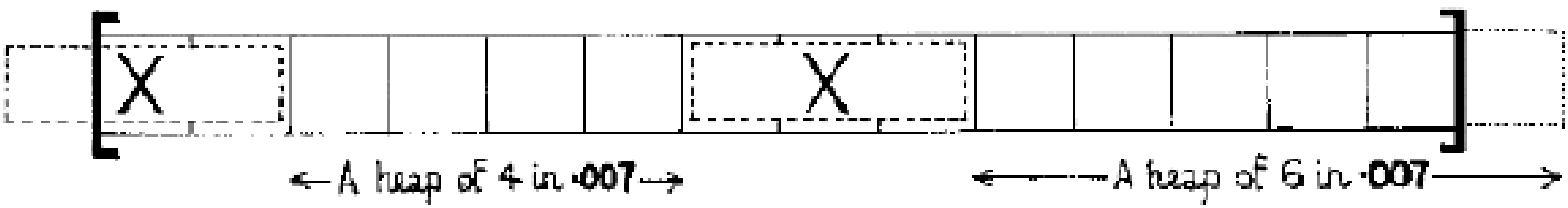
Analysis: It is stupid to move next to or next but one to a pre-existing cross, since your opponent wins immediately. If we consider only sensible moves we can therefore regard each X as also occupying the two neighbors of the square in which it lies (one of which may be off the board), and no two of these 3-square regions may overlap.

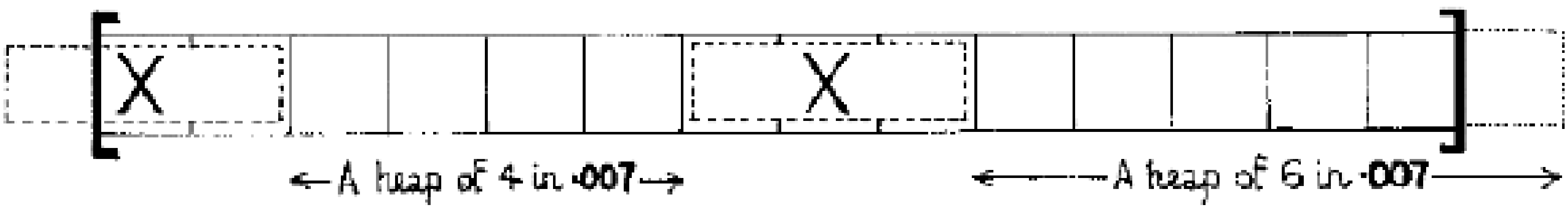


Treblecross is just the game $.\mathbf{007}$.

Treblecross

Treblecross: a Tic-Tac-Toe game played on a $1 \times n$ strip in which both player use the same symbol (X). The first person to complete a line of three consecutive crosses wins.

Analysis: It is stupid to move next to or next but one to a pre-existing cross, since your opponent wins immediately. If we consider only sensible moves we can therefore regard each X as also occupying the two neighbors of the square in which it lies (one of which may be off the board), and no two of these 3-square regions may overlap.



Treblecross is just the game $.\mathbf{007}$.

$$\mathcal{G}(.007) = 0.0011122033111043332224405522223305\cdots$$

Grundy's Game

Grundy's Game is a breaking game in which the only legal move is to split a single heap into two smaller ones of different sizes. The game ends when all the heaps will have size 1 or 2. The player who splits the last heap is the winner.

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$n =$ 0–19	0	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3
20–39	0	4	3	0	4	3	0	4	1	2	3	1	2	4	1	2	4	1	2	4
40–59	1	5	4	1	5	4	1	5	4	1	0	2	1	0	2	1	5	2	1	3
60–79	2	1	3	2	4	3	2	4	3	2	4	3	2	4	3	2	4	3	2	4
80–100	5	2	4	5	2	4	3	7	4	3	7	4	3	7	4	3	5	2	3	5