Math576: Combinatorial Game Theory

Lecture note II

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Disclaimer

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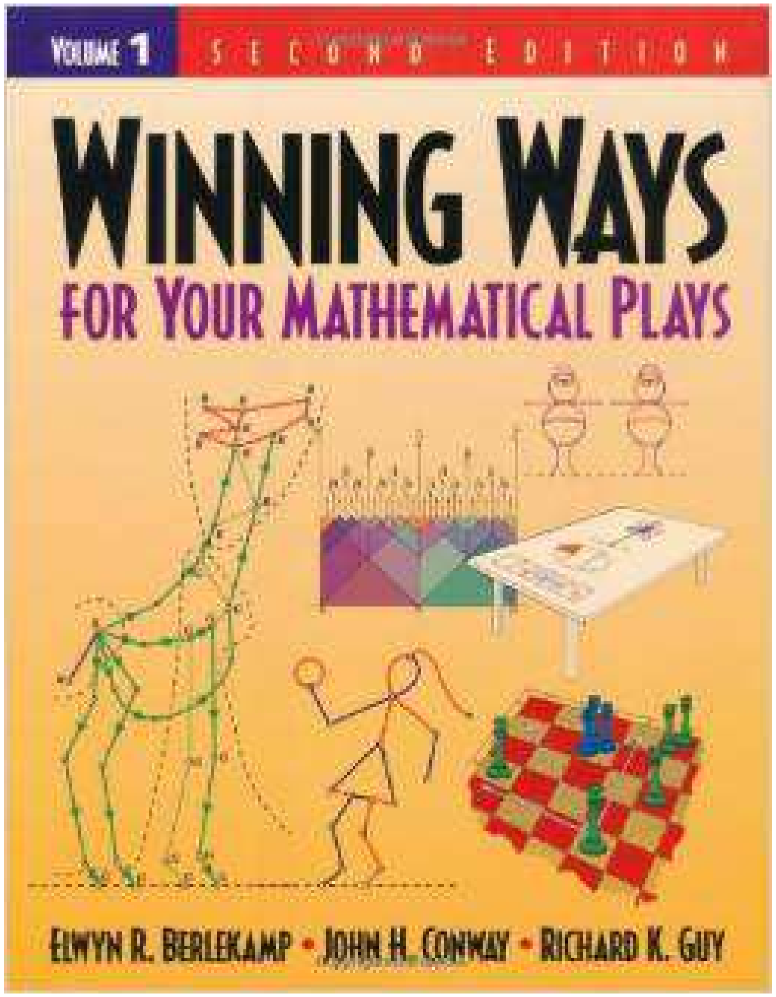
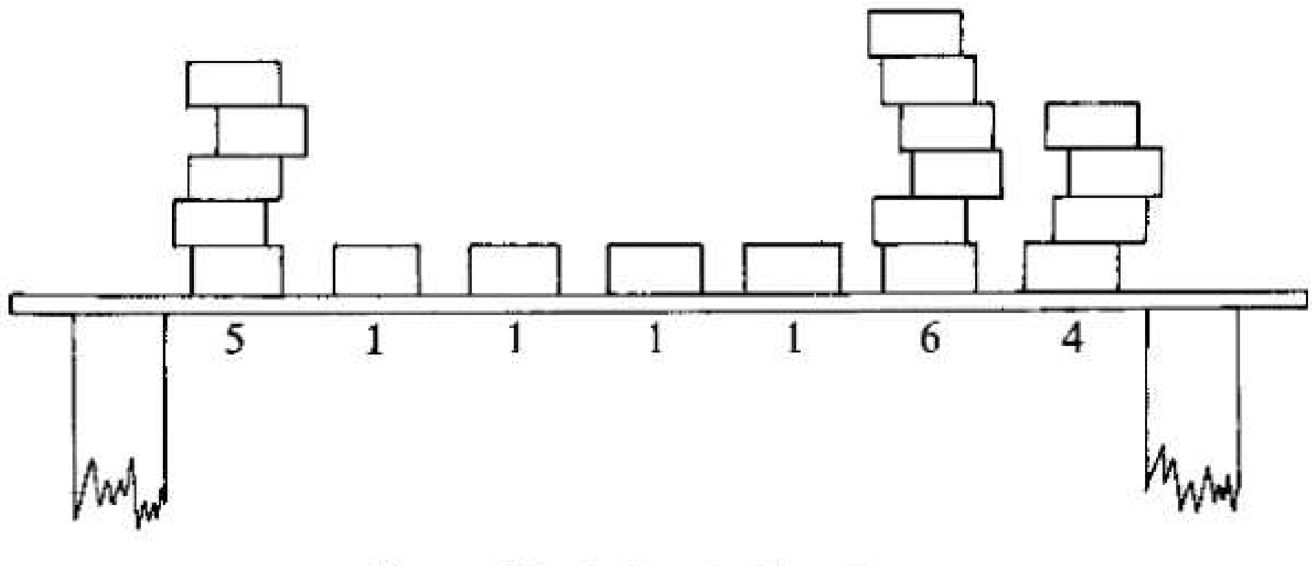


The Game of Nim

- **Two players:** "Left" and "Right".
- **Game board:** a number of heaps of counters.
- **Rules:** Two players take turns. Either player can remove any positive number of counters from any one heap.
- **Ending positions:** Whoever gets stuck is the loser.



Nimbers

The game value of a heap of size n is denoted by $*n$. It can defined recursively as follows:

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\begin{aligned}
* &= \{0 \mid 0\}; \\
*2 &= \{0, * \mid 0, *\}; \\
*3 &= \{0, *, *2 \mid 0, *, *2\}; \\
&\vdots \\
*n &= \{0, *, *2, \cdots, *(n-1) \mid 0, *, *2, \cdots, *(n-1)\}.
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So the previous nim game has the game value

$$*5 + * + * + * + * + *6 + *4.$$

Identities of nimbers

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What is the value of $*3 + *5$?

Nimber addition: $a \overset{*}{+} b = c$ if $*a + *b = *c$.

Nim-Addition Table

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Nimber addition rule

Property 1: If $a < 2^n$, then

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Property 2: For any a, b, $*a + *b$ is the sum of two binary numbers without carrying.

Property 3: If $a, b < 2^n$, then $a \overset{*}{+} b < 2^n$.

$$\begin{array}{r|ccc} & 4 & 2 & 1 \\ \hline 5 = & 1 & 0 & 1 \\ 3 = & & 1 & 1 \\ \hline 6 = & 1 & 1 & 0 \end{array} \qquad \begin{array}{r|cccccc} & 32 & 16 & 8 & 4 & 2 & 1 \\ \hline 11 = & & & 1 & 0 & 1 & 1 \\ 22 = & & 1 & 0 & 1 & 1 & 0 \\ 33 = & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 60 = & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \qquad \begin{array}{r|cccccc} & 32 & 16 & 8 & 4 & 2 & 1 \\ \hline 9 = & & & 1 & 0 & 0 & 1 \\ 25 = & & 1 & 1 & 0 & 0 & 1 \\ 49 = & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 33 = & 1 & 0 & 0 & 0 & 0 & 1 \end{array}$$

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Inductive step: Assume $2^n \overset{*}{+} a' = 2^n + a'$ for all $a' < a$, and $2^k \overset{*}{+} b = 2^k + b$ for all $k < n$ and $b < 2^k$.

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Consider the Nim game of three heaps of size $(2^n + a)$, 2^n, and a. It suffices to show that this is a zero game.

- If $2^n + a \to 2^n + a'$, then $a \to a'$.
- If $2^n + a \to b$ $(b < 2^n)$, then $2^n \to a \overset{*}{+} b$.
- If $2^n \to b$, then $2^n + a \to a \overset{*}{+} b$.
- If $a \to a'$, then $2^n + a \to 2^n + a'$.

So the second player can always reduce to the zero position and thus wins the game. Thus $2^n \overset{*}{+} a = 2^n + a$.

Winning strategy of Nim

If the current position is non-zero, then the current player can win the game by resetting the game value to zero and wins the game.

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Consider the Nim Game

$$*3 + *4 + *6 + *9$$

Note that the current position has the game value $*8 \neq 0$. The only winning move is $9 \to 1$ so that the new game value is

$$*3 + *4 + *6 + * = 0.$$

Sprague-Grundy Theory

An **impartial game** is a game so that Left and Right have exactly the same options at any game position.

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The Minimal-Excluded Rule: If $G = \{*a, *b, *c, \ldots \mid *a, *b, *c, \ldots$ is an impartial game, then $G = *n$ where n is the minimal-excluded number from the list $a, b, c, \ldots$.

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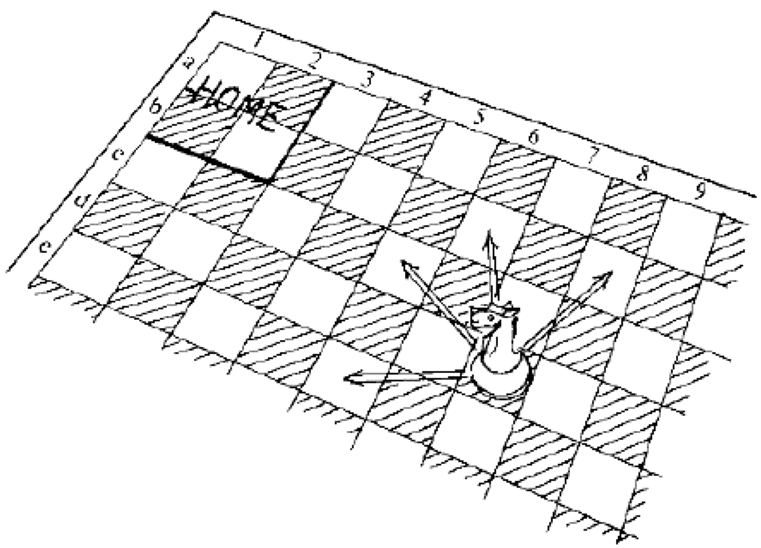
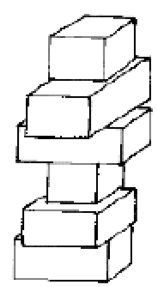
$$\{0, *, *2, *4 \mid 0, *, *2, *4\} = *3$$

since 3 is the minimal-excluded number from $0, 1, 2, 4$.

Game of White Knight

- **Two players:** "Left" and "Right".
- **Game board:** a chess board and a Nim-heap of size 6.
- **Rules:** Two players take turns. Either player can either move the Knight on the chess board North-West, or lose some of his belonging— to remove any positive number of chips from the Nim-heap.
- **Ending positions:** Whoever gets stuck is the loser.



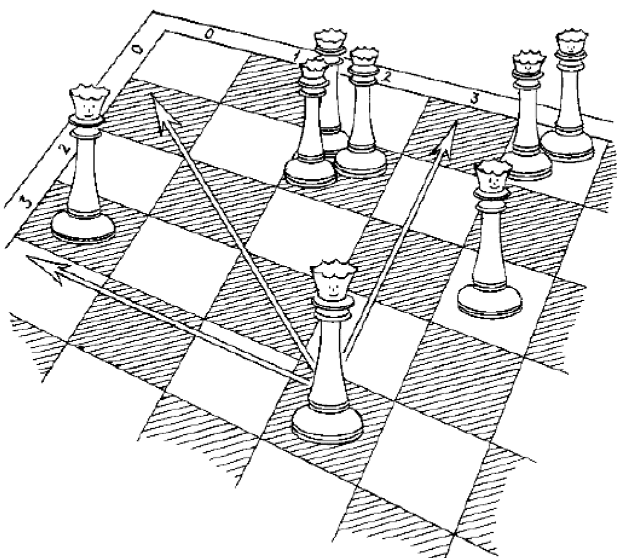


Value of Knight only

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1
b	0	0	*2	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1
c	*1	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	
d	*1	*1	*2	*1	*4	*3	*2	*3	*3	*3	*2	*3	*3	*3	*2	*3	*3	*3	*2	
e	0	0	*3	*4	0	0	*1	*1	0	0	*1	*1	0	0	*1	*1	0	0		
f	0	0	*2	*3	0	0	*2	*1	0	0	*1	*1	0	0	*1	*1	0	0		
g	*1	*1	*2	*2	*1	*2	*2	*2	*3	*2	*2	*2	*3	*2	*2	*2	*3			
h	*1	*1	*2	*3	*1	*1	*2	*1	*4	*3	*2	*3	*3	*3	*2	*3	*3			
i	0	0	*3	*3	0	0	*3	*4	0	0	*1	*1	0	0	*1	*1				
j	0	0	*2	*3	0	0	*2	*3	0	0	*2	*1	0	0	*1	*1				
k	*1	*1	*2	*2	*1	*1	*2	*2	*1	*2	*2	*2	*3	*2						
l	*1	*1	*2	*3	*1	*1	*2	*3	*1	*1	*2	*1								
m	0	0	*3	*3	0	0	*3	*3	0	0										
n	0	0	*2	*3	0	0	*2	*3												
o	*1	*1	*2	*2	*1	*1														
p	*1	*1	*2	*3																
q	0	0																		

Game of Wyt Queen

- **Two players:** "Left" and "Right".
- **Game board:** several queens on a chess board.
- **Rules:** Any number of Queens can be on the same square and each player when its her turn to move, can move any single queen an arbitrary distance North, West, North-West as indicated, even jumping over other queens.
- **Ending positions:** Whoever gets stuck is the loser.

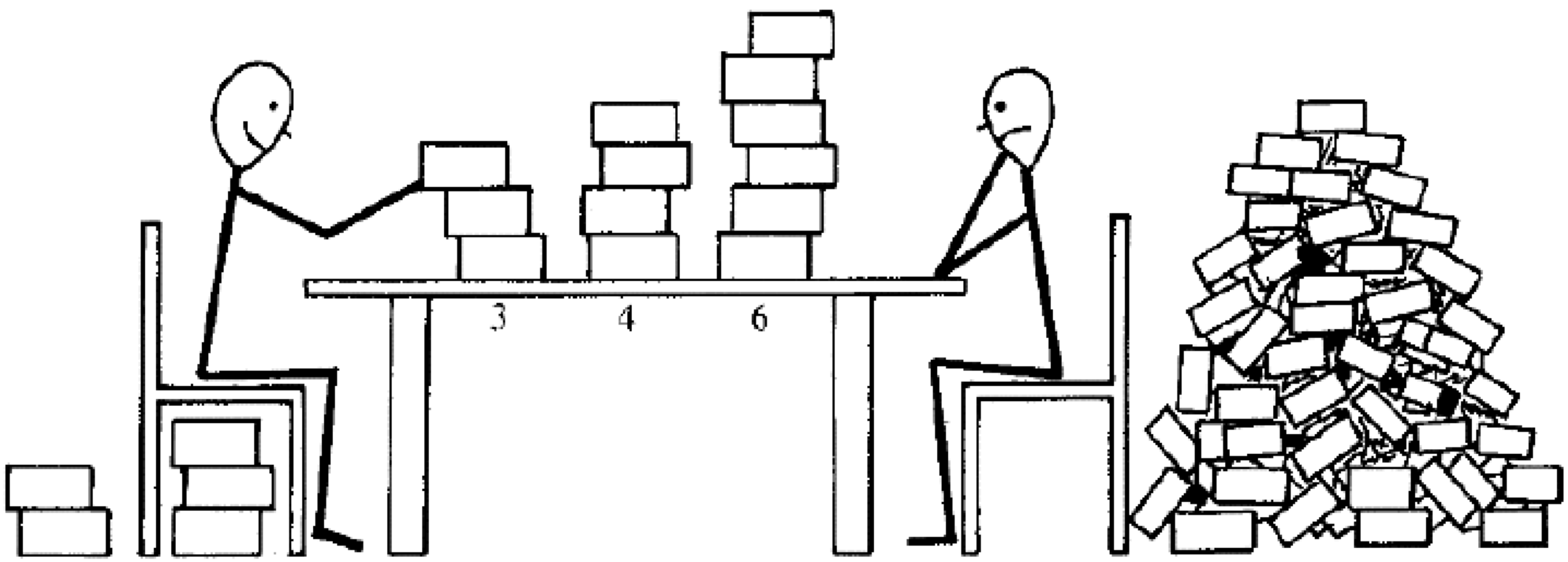


Value of one Queen

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	*1	*2	*3	*4	*5	*6	*7	*8	*9	*10	*11	*12	*13	*14	*15	*16	*17
1	*1	*2	0	*4	*5	*3	*7	*8	*6	*10	*11	*9	*13	*14	*12	*16	*17	*15
2	*2	0	*1	*5	*3	*4	*8	*6	*7	*11	*9	*10	*14	*12	*13	*17	*15	*16
3	*3	*4	*5	*6	*2	0	*1	*9	*10	*12	*8	*7	*15	*11	*16	*18	*14	*13
4	*4	*5	*3	*2	*7	*6	*9	0	*1	*8	*13	*12	*11	*16	*15	*10	*19	*18
5	*5	*3	*4	0	*6	*8	*10	*1	*2	*7	*12	*14	*9	*15	*17	*13	*18	*11
6	*6	*7	*8	*1	*9	*10	*3	*4	*5	*13	0	*2	*16	*17	*18	*12	*20	*14
7	*7	*8	*6	*9	0	*1	*4	*5	*3	*14	*15	*13	*17	*2	*10	*19	*21	*12
8	*8	*6	*7	*10	*1	*2	*5	*3	*4	*15	*16	*17	*18	0	*9	*14	*12	*19
9	*9	*10	*11	*12	*8	*7	*13	*14	*15	*16	*17	*6	*19	*5	*1	0	*2	*3
10	*10	*11	*9	*8	*13	*12	0	*15	*16	*17	*14	*18	*7	*6	*2	*3	*1	*4
11	*11	*9	*10	*7	*12	*14	*2	*13	*17	*6	*18	*15	*8	*19	*20	*21	*4	*5

Poker-Nim

- **Two players:** "Left" and "Right".
- **Game board:** a number of heaps of counters.
- **Rules:** Two players take turns. Either player can either remove any positive number of chips from any one heap or **add** to it some of his chips that he acquired in earlier moves.
- **Ending positions:** Whoever gets stuck is the loser.



Reversible move

If a player can win a Nim position, then he can win the Poker-Nim game as well. If the opponent add some chips to a heap, he can reverse the opponent's move by taking the same amount of chips away for the heap.

$$\text{Poker-Nim} = \text{Nim}$$

The new move will only prolong the time of playing game but not affect who will win this game.

General reversible move

Consider a general game

$$G = \{A, B, C, \ldots \mid D, E, F, \ldots\}.$$

We say Right's move to D is reversible if there is some move for Left from D to D^L which is at least as good for Left as G was, i.e. $D^L \geq G$.

General reversible move

Consider a general game

$$G = \{A, B, C, \ldots \mid D, E, F, \ldots\}.$$

We say Right's move to D is reversible if there is some move for Left from D to D^L which is at least as good for Left as G was, i.e. $D^L \geq G$.

Suppose

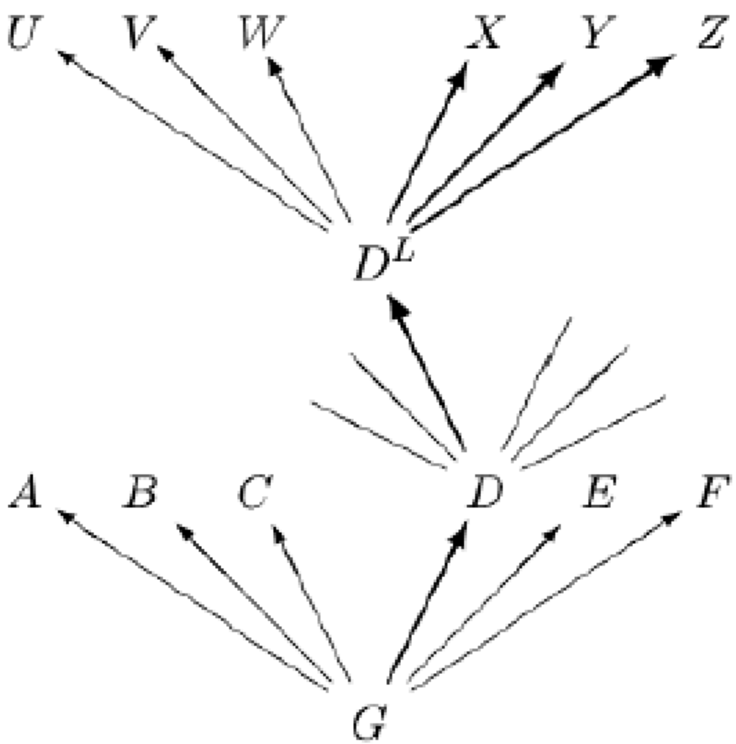
$$D^L = \{U, V, W, \ldots, \mid X, Y, Z, \ldots\}.$$

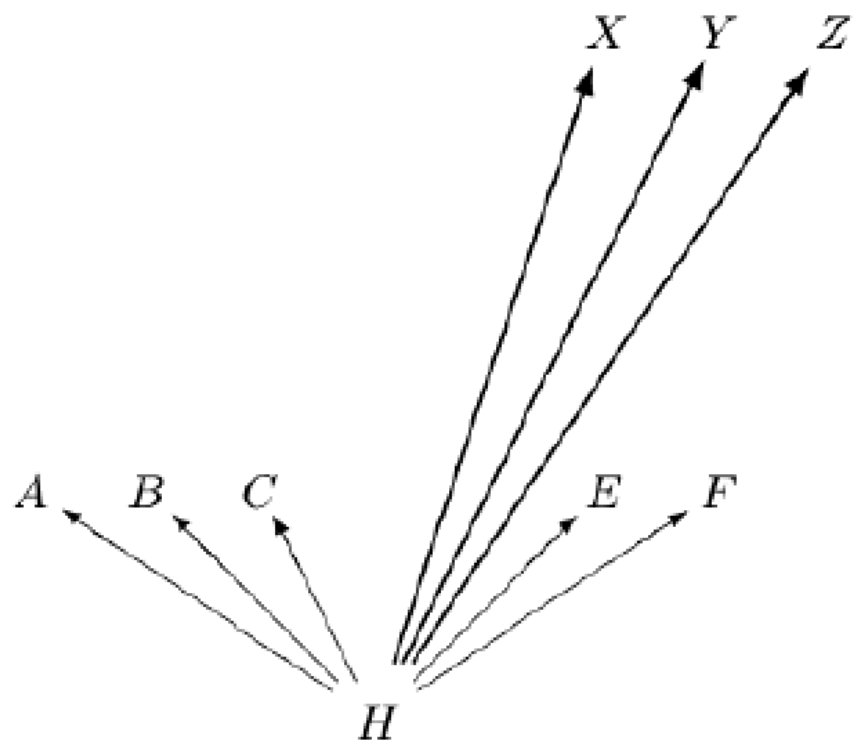
Now whenever Right plays from G to D, Left will reverse from D to D^L. We may view it as a new game

$$H = \{A, B, C, \ldots \mid X, Y, Z, \ldots, E, F, \ldots\}.$$

Bypassing reversible move

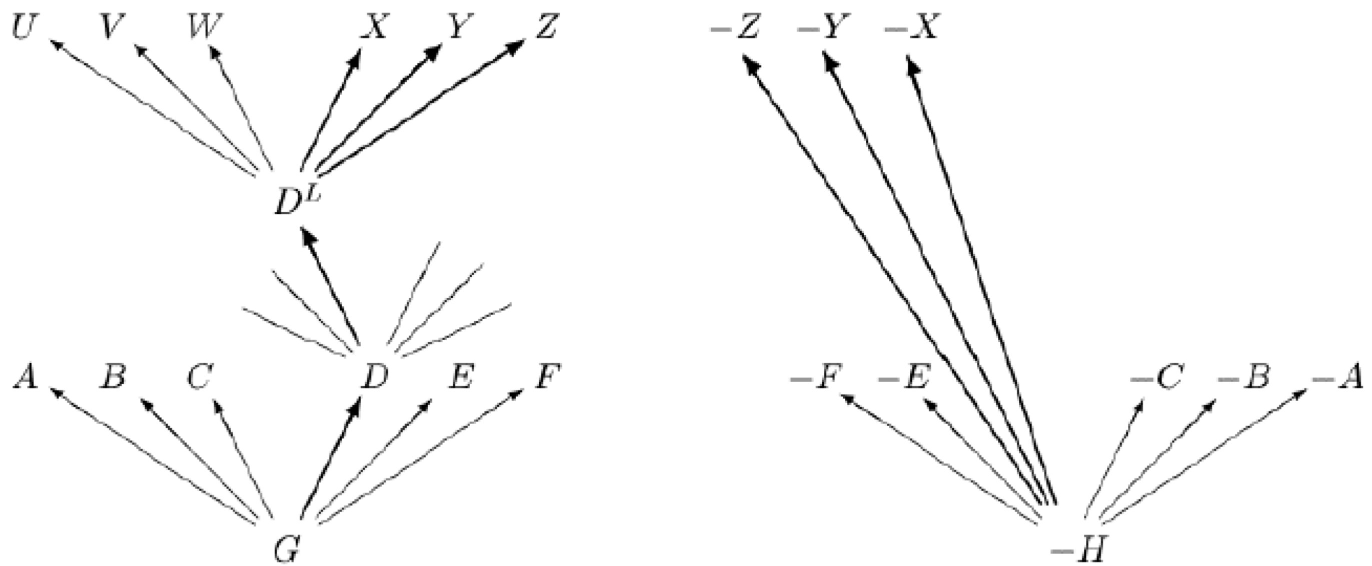
If any Right option D of G has itself a Left option $D^L \geq G$, then it will not affect the value of G if we replace D as a Right option of G by all the Right options $X, Y, Z, \ldots$ of that D^L.





Bypassing reversible move

Proof: It is suffice to show $G + (-H) = 0$.



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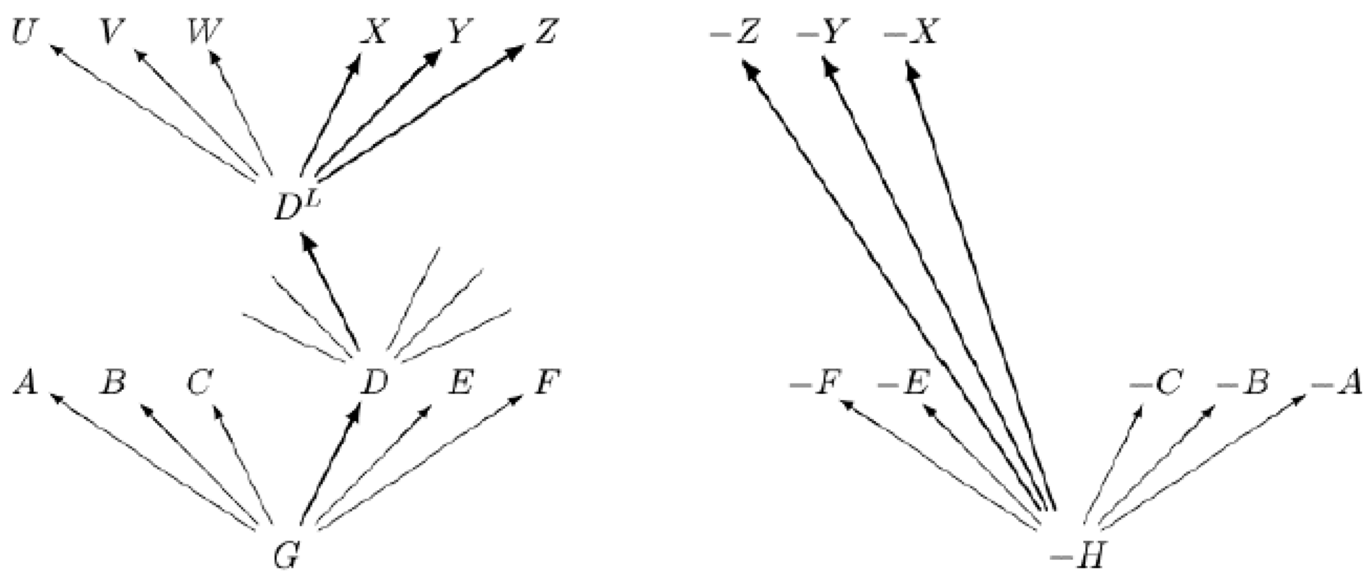
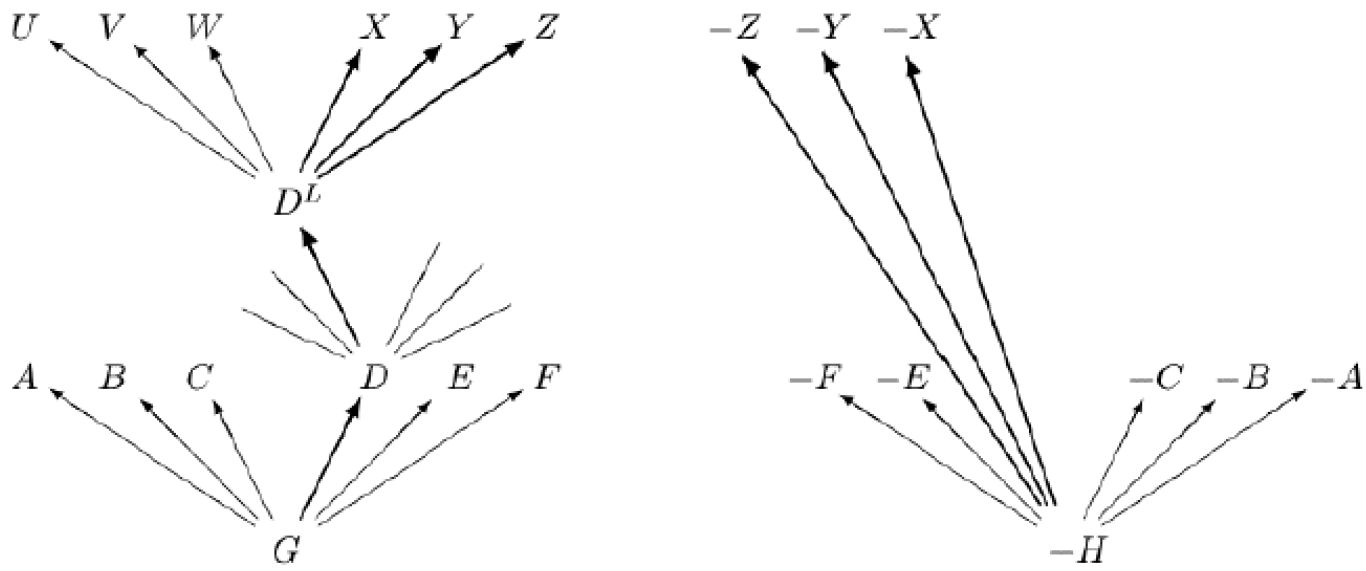


Left plays first. If Left moves from G to A, B, or C, Right can move from $-H$ to $-A$, $-B$, or $-C$ and win the game. If Left moves from $-H$ to $-X$, then the value is $G - X \leq D^L - X$, which Right can also win.

continue



Now Right plays first. If Right moves other than G to D, Left can always move the zero position and win the game. If Right moves from G to D, then Left can move from D to D^L. Now the value is $D^L - H$. Right cannot move from D^L to X, Y, Z. His only hope is to move from $-H$ to $-A$, $-B$, or $-C$. But then Left can win since $D^L - A \geq G - A$ and Left wins $G - A$.

Bypassing reversible moves

Bypassing Right's reversible move:

If any Right option D of G has itself a Left option $D^L \geq G$, then it will not affect the value of G if we replace D as a Right option of G by all the Right options $X, Y, Z, \ldots$ of that D^L.

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If any Right option D of G has itself a Left option $D^L \geq G$, then it will not affect the value of G if we replace D as a Right option of G by all the Right options $X, Y, Z, \ldots$ of that D^L.

Bypassing Left's reversible move:

If any Left option C of G has itself a Left option $C^R \leq G$, then it will not affect the value of G if we replace C as a Left option of G by the list of all the Left options of that C^R.

Deleting dominated options

In the game

$$G = \{A, B, C, \ldots \mid D, E, F, \ldots\},$$

if $A \leq B$ we say that A is **dominated** by B, and if $D \leq E$, that E is dominated by D.

Deleting dominated options:

It won't affect the value of G if we delete dominated options but retain the options that dominated them.

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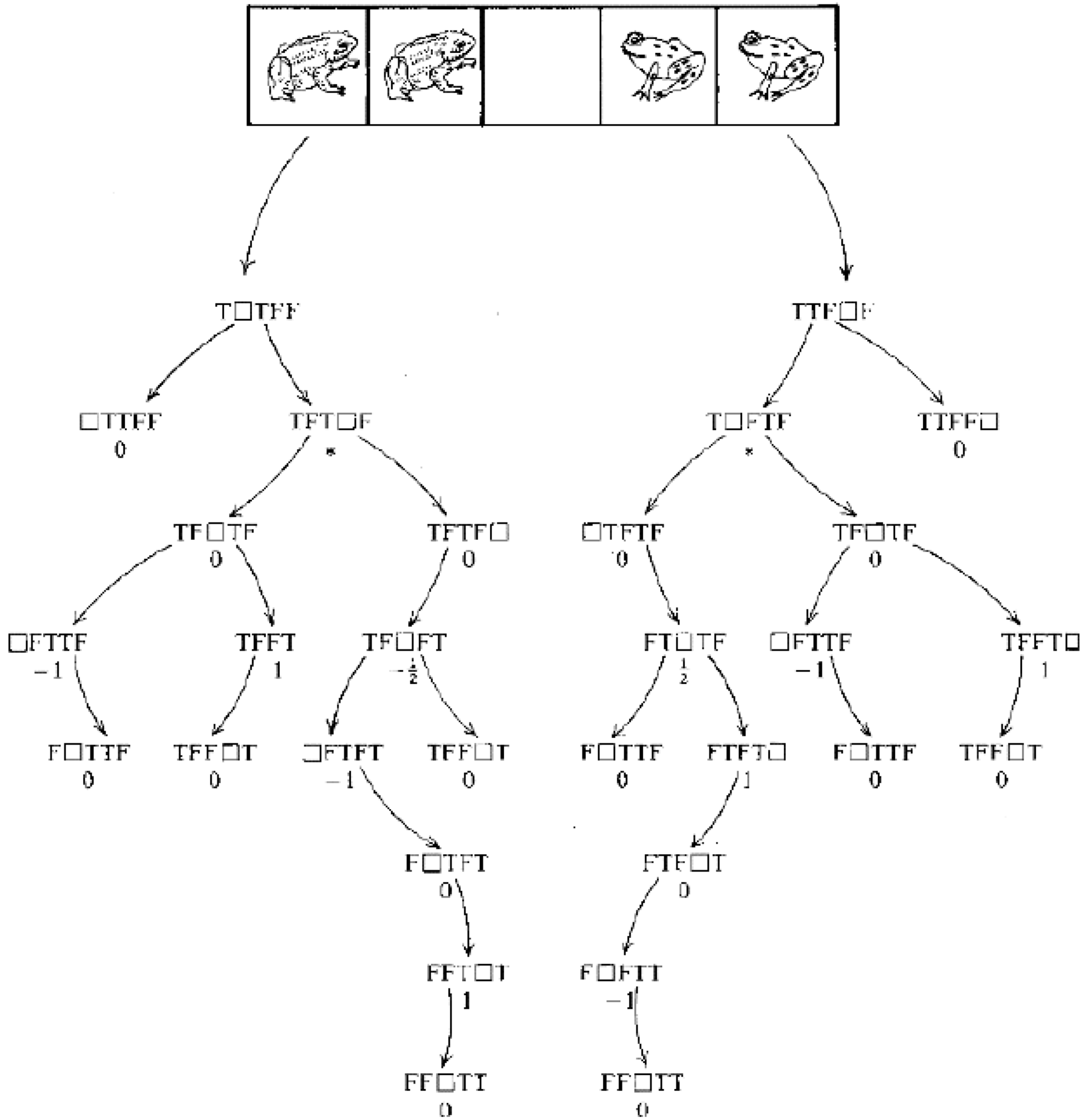
Deleting dominated options:

It won't affect the value of G if we delete dominated options but retain the options that dominated them.

Proof: Let $K = \{B, C \ldots \mid D, F, \ldots\}$. We show $G + (-K) = 0$. Any move from G to A or E are countered by those from $-K$ to $-B$ or $-D$, and all other moves in either component are countered by moves to their negatives from the other.

2 Toads and 2 Frogs



Up and Down

Define

$$\uparrow = \{0 \mid *\},$$

$$\downarrow = \{* \mid 0\}.$$

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Property:

- $-\frac{1}{2^n} < \downarrow < 0 < \uparrow < \frac{1}{2^n}$ for any integer $n > 0$.

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- $\downarrow = -\uparrow$.

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- $\downarrow = -\uparrow$.
- Both $\uparrow$ and $\downarrow$ are fuzz to $*$.
- $\{\uparrow \mid \downarrow\} = \{\uparrow \mid 0\} = \{0 \mid \downarrow\} = \{0 \mid 0\} = *$.

Up and Down

Define

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Property:

- $-\frac{1}{2^n} < \downarrow < 0 < \uparrow < \frac{1}{2^n}$ for any integer $n > 0$.
- $\downarrow = -\uparrow$.
- Both $\uparrow$ and $\downarrow$ are fuzz to $*$.
- $\{\uparrow \mid \downarrow\} = \{\uparrow \mid 0\} = \{0 \mid \downarrow\} = \{0 \mid 0\} = *$.

The last equality can be proved by showing $\downarrow$ (or) $\uparrow$) is a Right's (or Left's) reversible move for $G = \{\uparrow \mid \downarrow\}$. This can be done by showing $G + * = 0$.

The value $\uparrow *$

Define $\uparrow * = \uparrow + *$. We have

$$
\begin{aligned}
\uparrow * &= \uparrow + * \\
&= \{0 \mid *\} + \{0 \mid 0\} \\
&= \{\uparrow + 0, 0 + * \mid \uparrow + 0, * + *\} \\
&= \{\uparrow, * \mid 0, \uparrow\} \\
&= \{\uparrow, * \mid 0\} \quad \text{remove dominated option} \\
&= \{0, * \mid 0\} \quad \text{bypassing reversible move.}
\end{aligned}
$$

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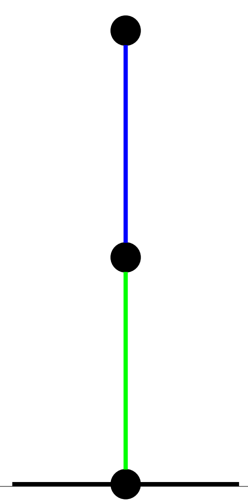
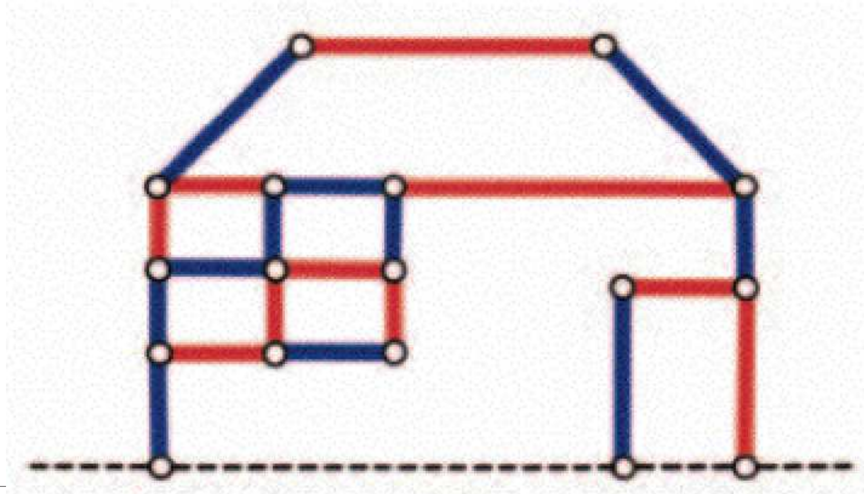
&= \{0, * \mid 0\} \qquad \text{bypassing reversible move.}
\end{aligned}$$



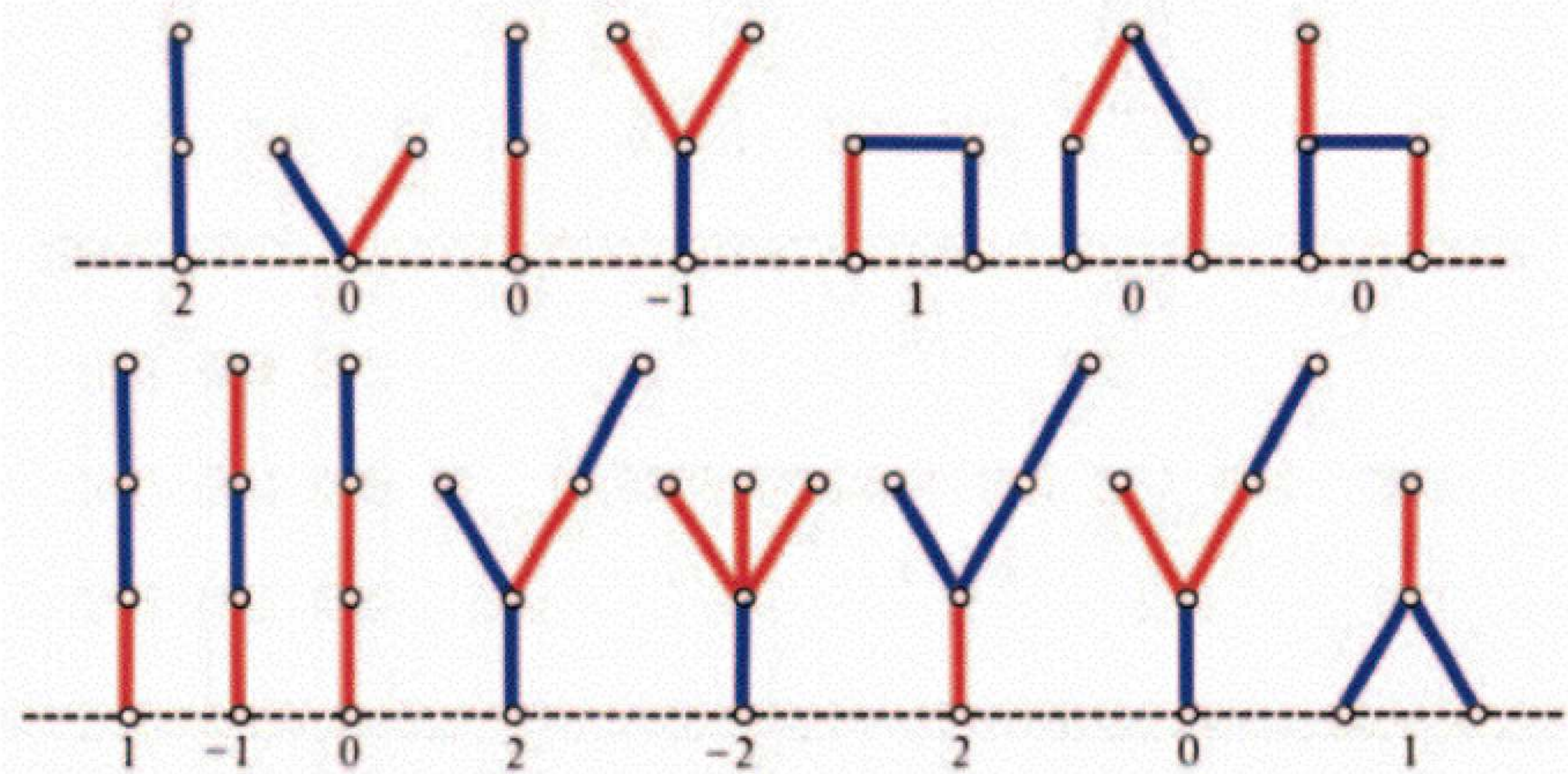
The blue flower has value $\uparrow *$.

Childish Hackenbush

- **Two players:** “Left” and “Right”.
- **Game board:** blue-red graphs connected to the ground.
- **Rules:** Two players take turns. Right deletes one red edge **provided deleting this red edge does not result any piece disconnected from the ground.** Left does the similar move but deletes one blue edge.
- **Ending positions:** Whoever gets stuck is the loser.



Childish Hackenbush Values



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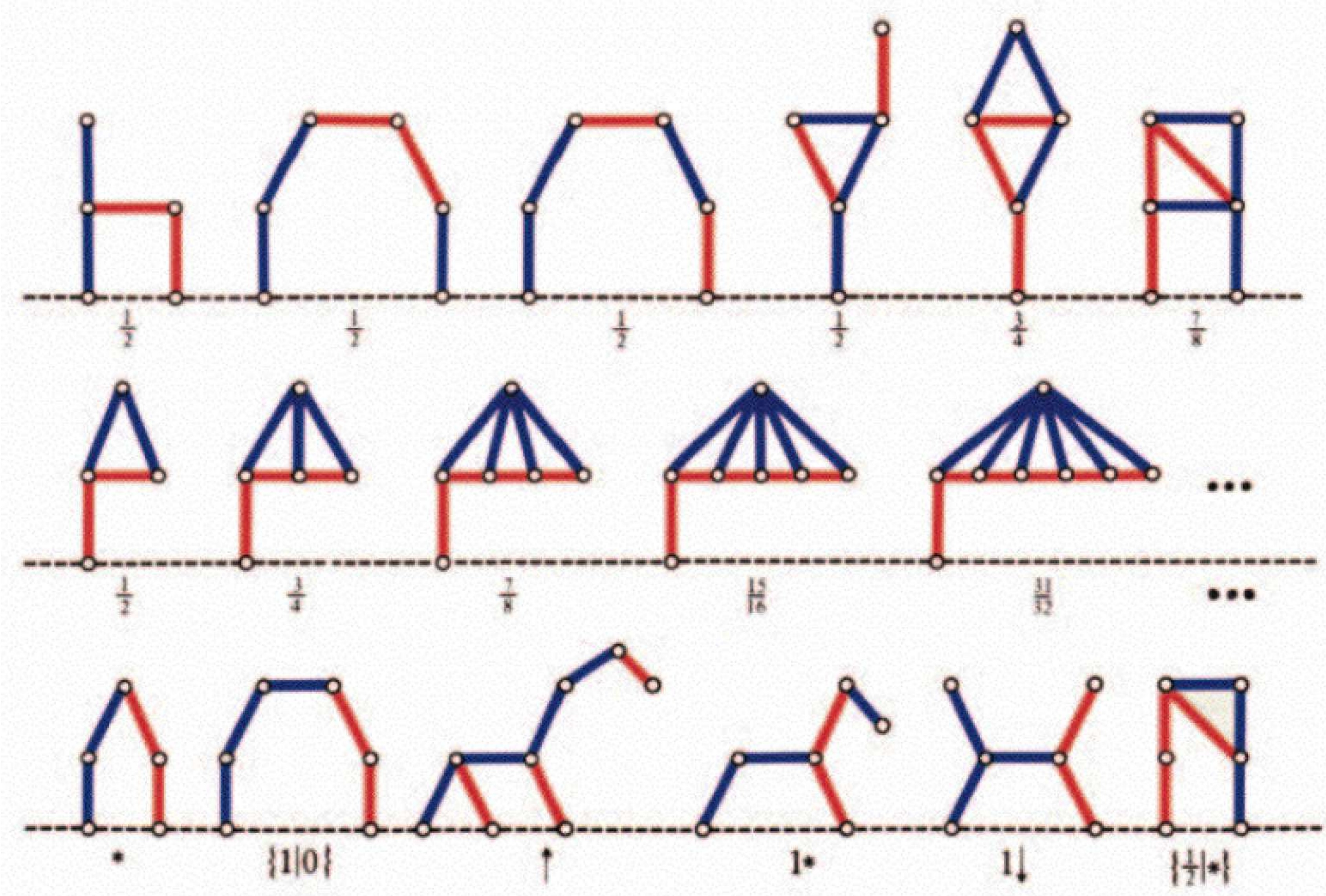

Whereabout of $*$

Define

$$\Uparrow = \uparrow + \uparrow,$$

$$\Uparrow * = \uparrow + \uparrow + * \,.$$

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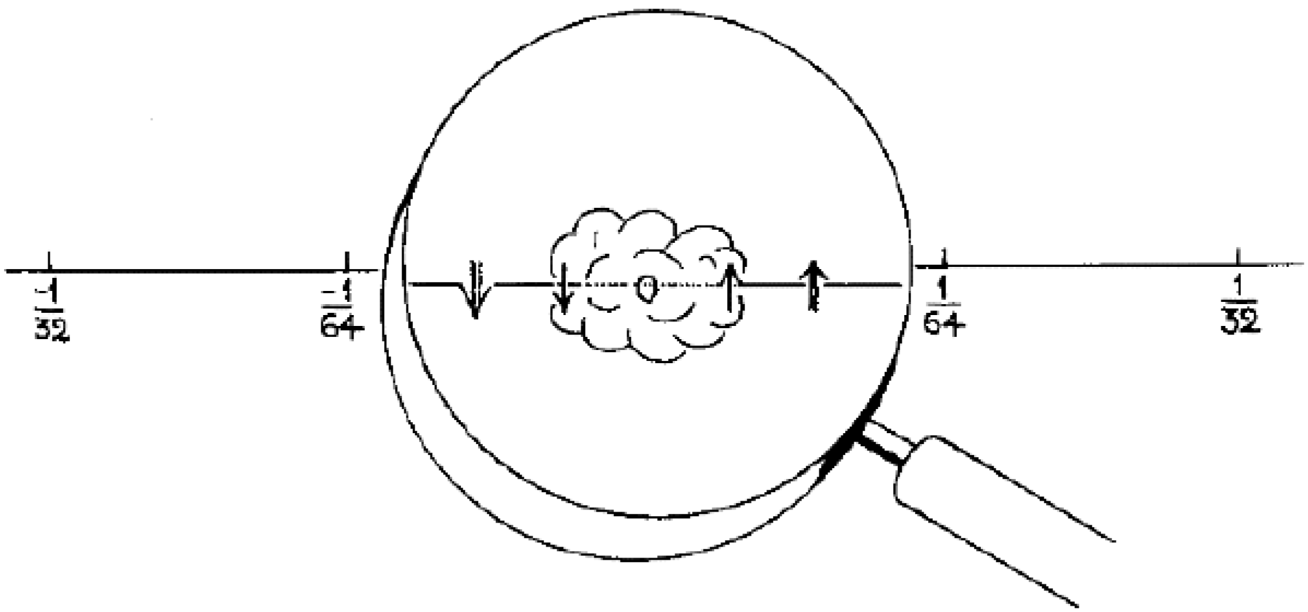
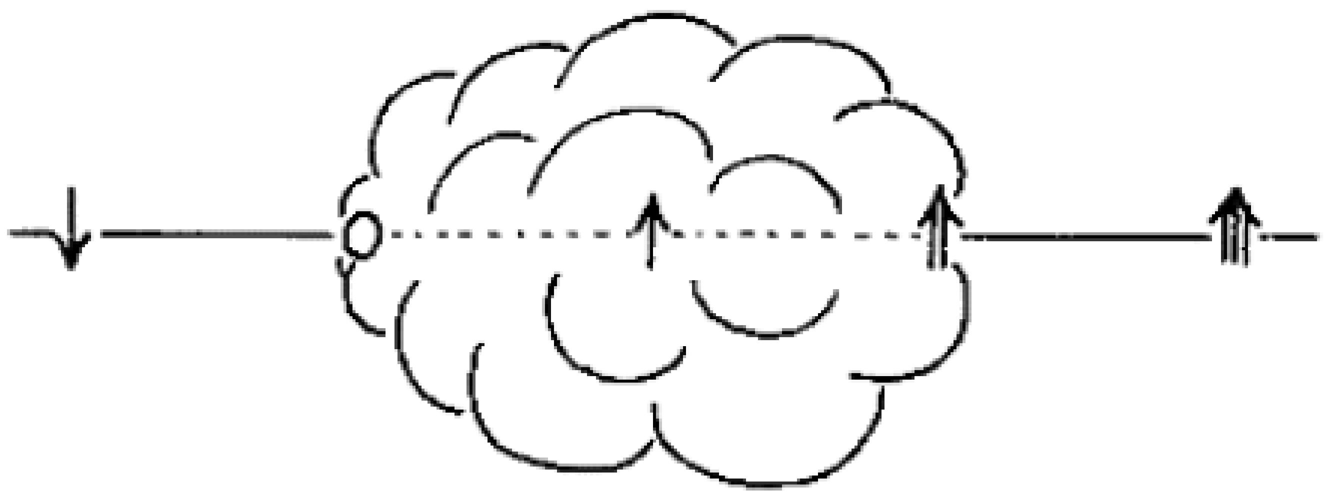
Similarly we can define $\Downarrow$ and $\Downarrow *$. We have $\Downarrow * < 0$.



Whereabouts of $\uparrow *$ and $*n$

$\uparrow *$ is fuzz to 0 and $\Uparrow$, is greater than $\downarrow$ and less than $3.\uparrow$.



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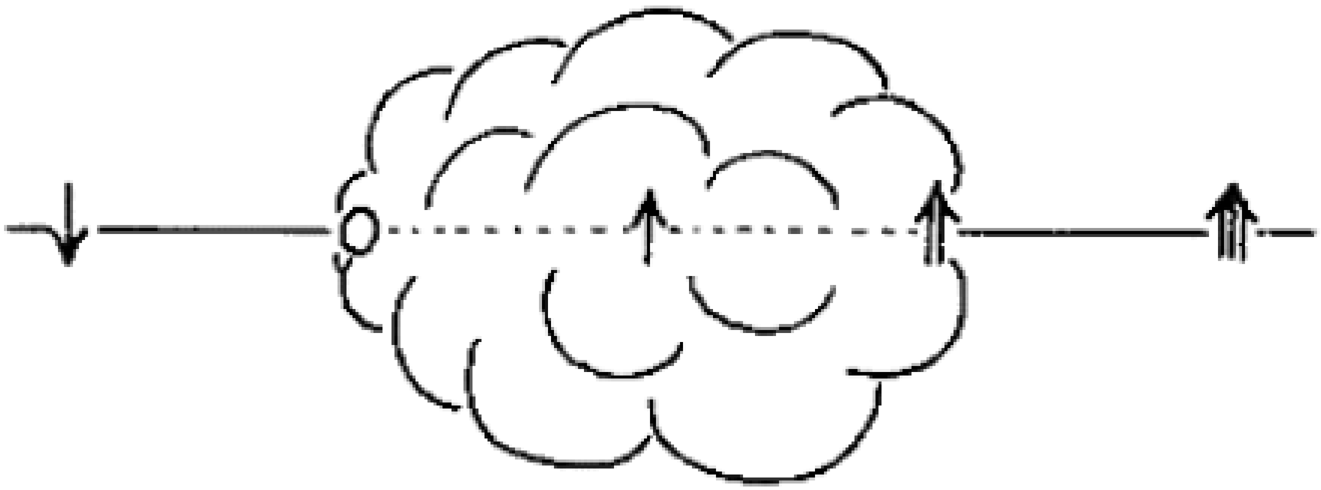
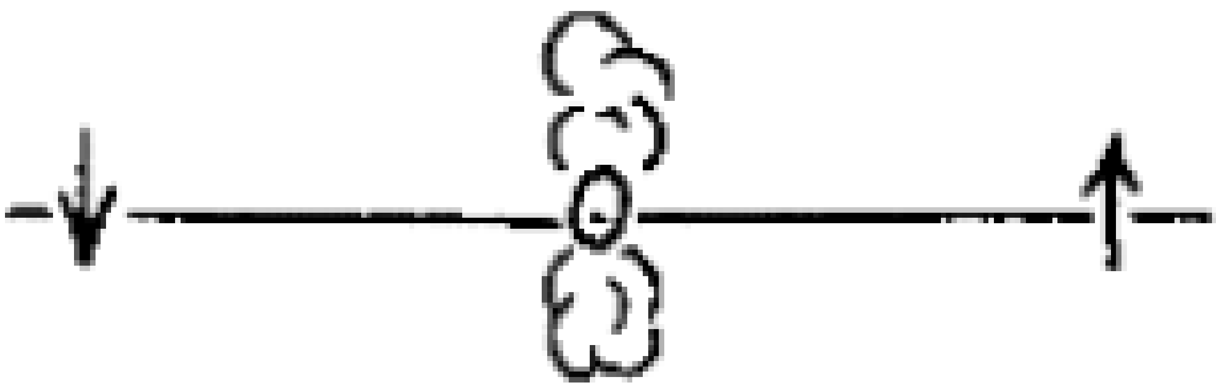
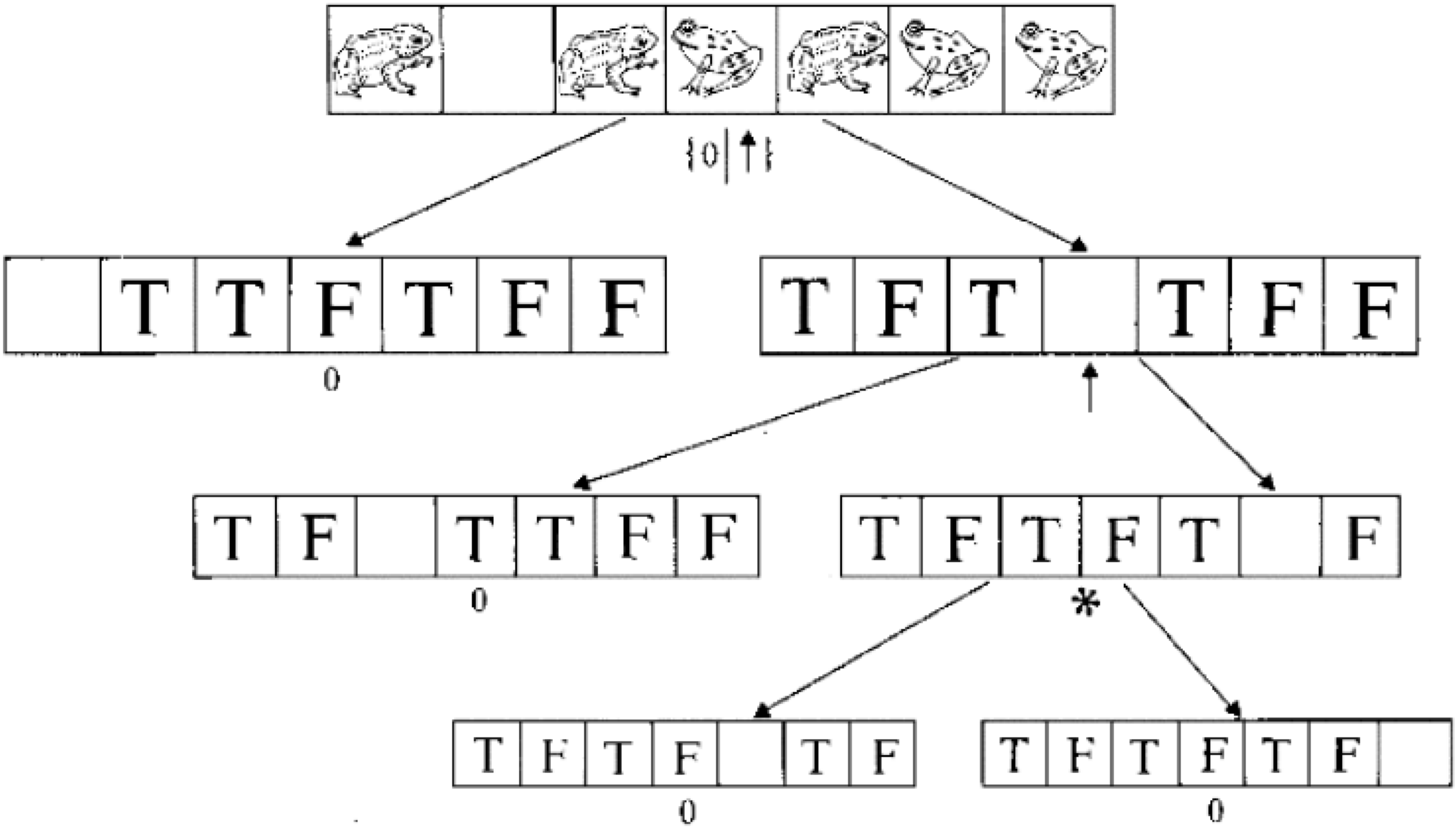


For $n \geq 2$. $*n$ is greater than $\downarrow$ and less than $\uparrow$.



Up in Toads-Frogs



Interesting identity

$$\boxed{\{\uparrow|\uparrow\} = \{0\ |\uparrow\} = \Uparrow *.}$$

Bypassing Left's reversible move $\uparrow$, we get

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Now let $X = \{0\ |\uparrow\}$. Consider the sum of Games:

$$X + \downarrow + \downarrow * = \{0\ |\uparrow\} + \{*\ |\ 0\} + \{0\ |\ 0, *\}.$$

Left first: if $X \to 0$ then the value is $*$ and Right wins. If Left plays on $\downarrow$ or $\downarrow *$, then Right $X \to \uparrow$ and Right wins. Right first: if $X \to \uparrow$ then the value is $\downarrow *$ so Right wins. If Right plays on $\downarrow$ or $\downarrow *$, then Left responses on $\downarrow *$ or $\downarrow$ respectively. The game value becomes X or $X + *$. Both of them are positive so Left wins. Thus $X + \Downarrow * = 0$.

Ups and Stars

Here are the simplest forms for Ups and Stars.

$$
\begin{array}{rlrlrl}
\vdots & & \vdots & & \vdots & \\
3.{\downarrow} &= \{{\Downarrow}* \mid 0\} & 3.{\downarrow}* &= \{{\Downarrow} \mid 0\} & 3.{\downarrow}{*n} &= \{{\Downarrow}{*m} \mid 0\} \\
{\Downarrow} &= \{{\downarrow}* \mid 0\} & {\Downarrow}* &= \{{\downarrow} \mid 0\} & {\Downarrow}{*n} &= \{{\downarrow}{*m} \mid 0\} \\
{\downarrow} &= \{* \mid 0\} & {\downarrow}* &= \{0 \mid 0, *\} & {\downarrow}{*n} &= \{{*m} \mid 0\} \\
0 &= \{\mid\} & * &= \{0 \mid 0\} & & \\
{\uparrow} &= \{0 \mid *\} & {\uparrow}* &= \{0, * \mid 0\} & {\uparrow}{*n} &= \{0 \mid {*m}\} \\
{\Uparrow} &= \{0 \mid {\uparrow}*\} & {\Uparrow}* &= \{0 \mid {\uparrow}\} & {\Uparrow}{*n} &= \{0 \mid {\uparrow}{*m}\} \\
3.{\uparrow} &= \{0 \mid {\Uparrow}*\} & 3.{\uparrow}* &= \{0 \mid {\Uparrow}\} & 3.{\uparrow}{*n} &= \{0 \mid {\Uparrow}{*m}\} \\
\vdots & & \vdots & & \vdots &
\end{array}
$$

Here $n \geq 2$ and $m = n \overset{*}{+} 1$.

Gift horse principal

It does not affect the value of G if we add a new Left option H provided $H \lhd\!\!| \, G$, or a new right option $\bar{H}$ provided $\bar{H} \, |\!\!\rhd G$.

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Proof: Consider the game

$$\{G^L, H \mid G^R\} + \{-G^R \mid -G^L\}.$$

If Left moves to H, then the game value is $H - G \lhd\!\!| \, 0$ so Right wins. Any other move can be reversed to 0 by the next player. So this is a zero game. Thus

$$\{G^L, H \mid G^R\} = G.$$

Examples

Since $\{0 \mid \uparrow\} = \Uparrow *$, so it is fuzz to $\uparrow$, $\Uparrow$, and $3.\uparrow$. Those options are gift horses for left. We have

$$\{0 \mid \uparrow\} = \{0, \Uparrow \mid \uparrow\} = \{\Uparrow \mid \uparrow\}.$$

$$\{0 \mid \uparrow\} = \{0, 3.\uparrow \mid \uparrow\} = \{3.\uparrow \mid \uparrow\}.$$

Thus,

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Thus,

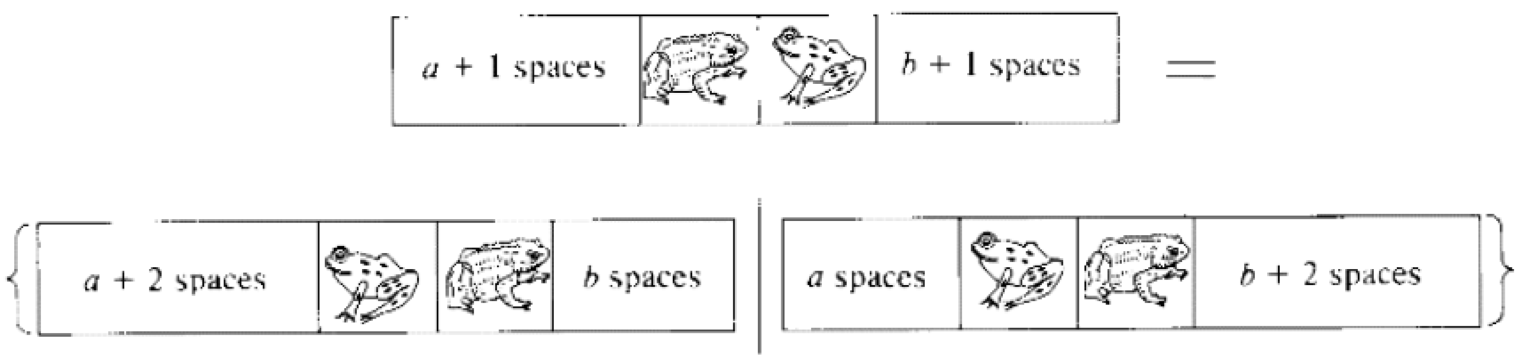
$$\{3.\uparrow \mid \uparrow\} = \{\Uparrow \mid \uparrow\} = \{\uparrow \mid \uparrow\} = \{0 \mid \uparrow\} = \Uparrow *.$$

Note that the above arguments fail for $4.\uparrow$. In fact

$$\{4.\uparrow \mid \uparrow\} > \Uparrow *.$$

Toad versus Frog

One toad meet one frog:



$= \{d-2 \mid d+2\}$ where $d = b - a$ is the difference.

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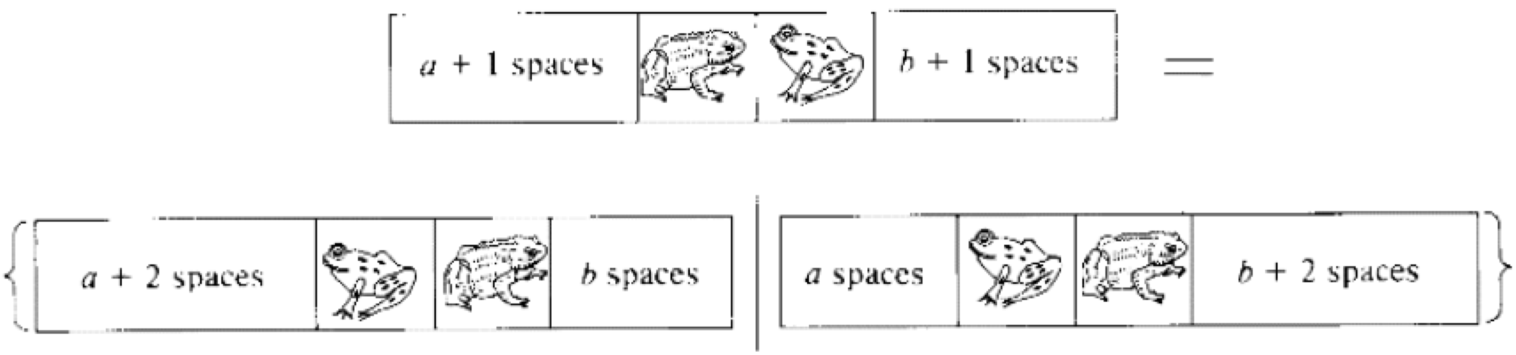

$= \{d - 2 \mid d + 2\}$ where $d = b - a$ is the difference. In general, if there is c spaces between two creatures, we have

$$f(c, d) = \{f(c-1, d-1) \mid f(c-1, d+1)\}.$$

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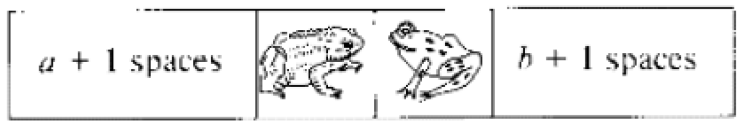
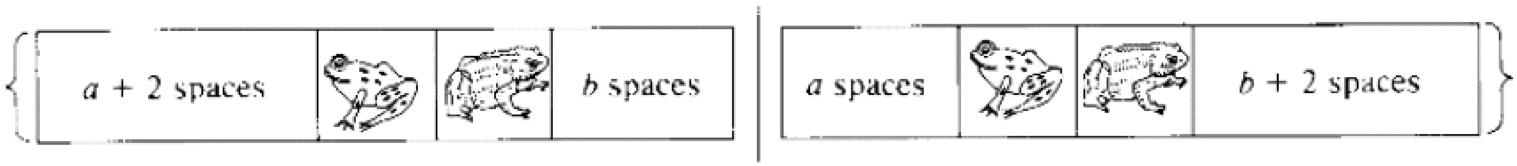


$=$



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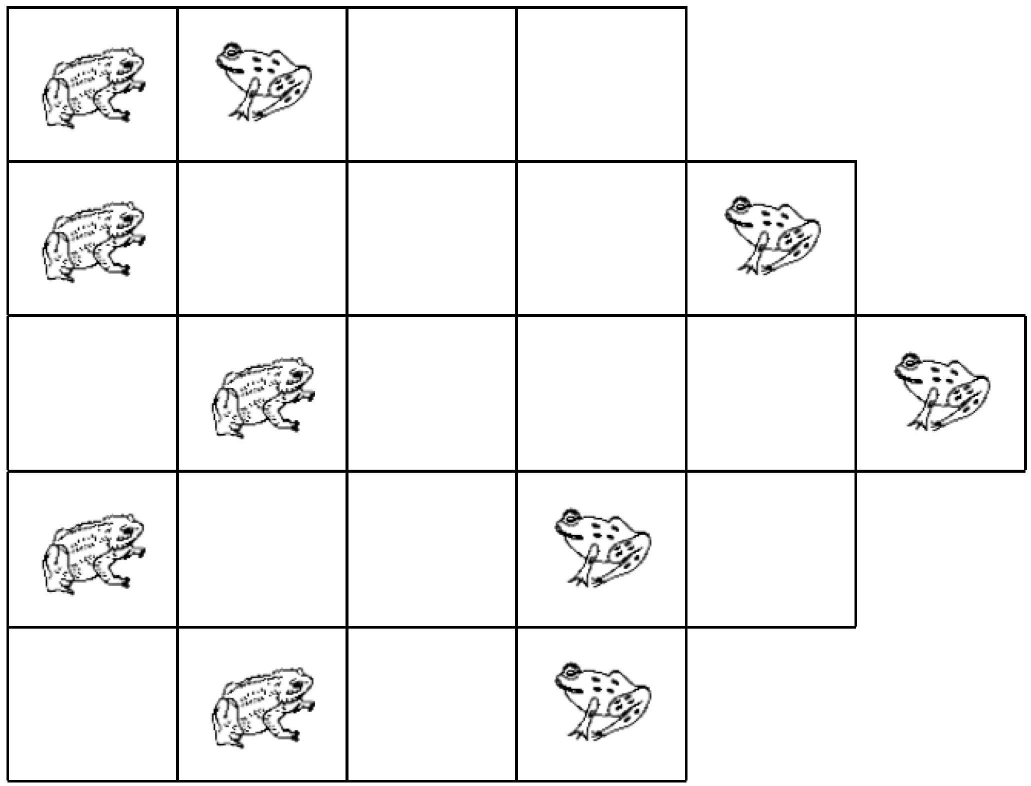
d	$\ldots$	-4	-3	-2	-1	0	1	2	3	4	$\ldots$
$c=0$	$\ldots$	-3	-2	-1	0	0	0	1	2	3	$\ldots$
$c=1$	$\ldots$	-3	-2	-1	$-\frac{1}{2}$	$*$	$\frac{1}{2}$	1	2	3	$\ldots$
$c=2$	$\ldots$	-3	-2	-1	0	0	0	1	2	3	$\ldots$
$c=3$	$\ldots$	-3	-2	-1	$-\frac{1}{2}$	$*$	$\frac{1}{2}$	1	2	3	$\ldots$
$\ldots$											

An example

Who wins the following game? What are Left's winning moves?



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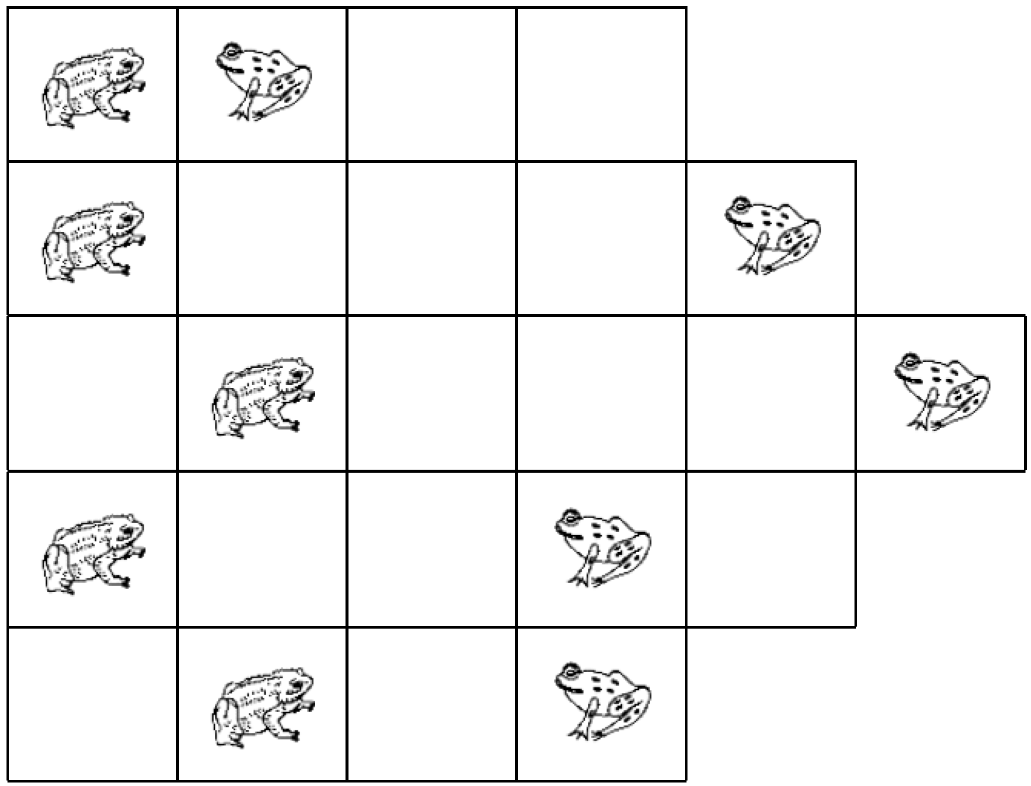

$$1 + * + (-\frac{1}{2}) + 0 + (-\frac{1}{2}) = *.$$

First player wins by moving in the second lane.

More Toads-and-Frogs

Theorem: The Toads-and-Frogs game $(TF)^x T\Box(TF)^n F$ has the value $n.\uparrow + (n+1).*$ for all $n \geq 0$, independent of x.

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Proof: We will prove by induction on n. Initial case $n=0$, it is easy to show that the value is $*$ (by another induction on x).

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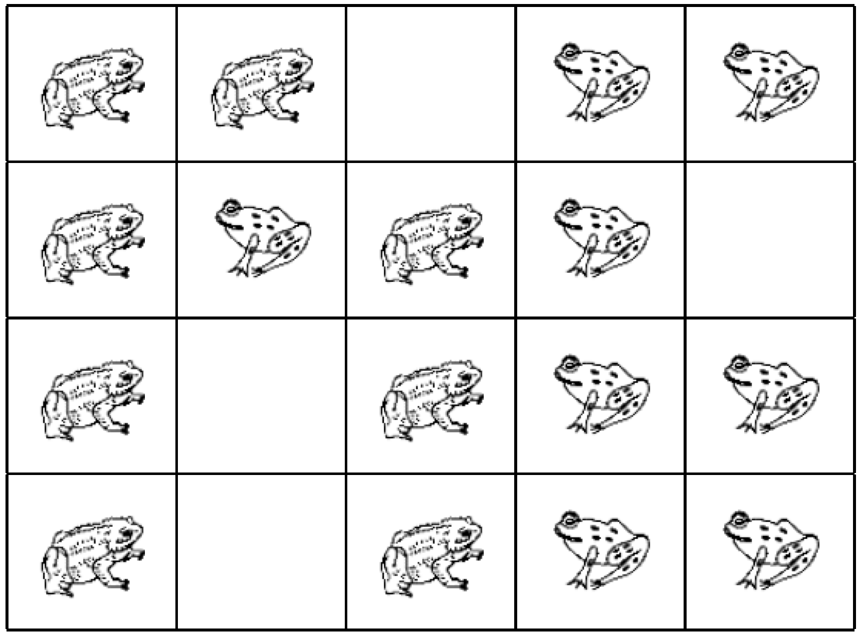
Inductive step:

$$\begin{aligned}
(TF)^x T\Box(TF)^n F &= \{0 \mid (TF)^{x+1} T\Box(TF)^{n-1} F\} \\
&= \{0 \mid (n-1).\uparrow +n.*\} \\
&= n.\uparrow +(n+1).*\,.
\end{aligned}$$

Another example

Who wins the following game? What are Left's winning moves?



Another example

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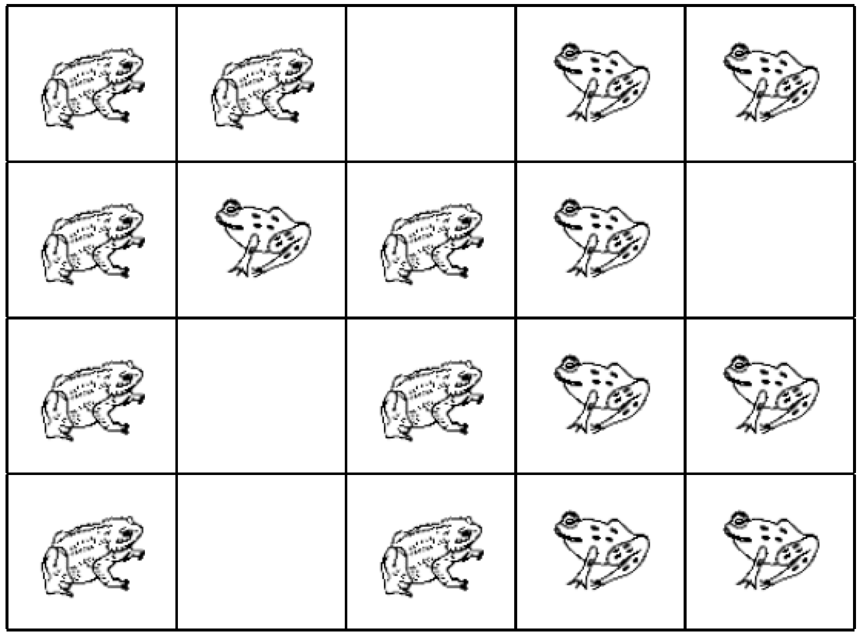

The value are

$$* + 0 + \uparrow + \uparrow = \Uparrow *.$$

Left wins. Left's winning move is moving in the first lane.