Math576 Combinatorial Game Theory
Homework 5 Solutions

1. Who wins the following games?

   (a) $\{2|-1\} + \{2|0\} + \{-2|-3\} + *$

   (b) $\{*|-1\} + \{1|\downarrow\} + *$

   (c) $\pm 1 + +_2 + \uparrow$

   (d) $-_2 + +_1 + *$

   Solution: (a) The temperature of $\{2|-1\} + \{2|0\} + \{-2|-3\} + *$ are $\frac{3}{2}, 1, \frac{1}{2}$, and 0. If Left plays first, after 4 moves, the game eventually looks like

   $$2 + 0 - 2 + 0 = 0.$$

   In this case, Right wins.

   If Right plays first, after 4 moves, the game eventually looks like

   $$-1 + 2 - 3 + 0 = -2.$$

   In this case, Right wins.

   Right can win in both cases. It is a negative game.

   (b) Both temperature of $\{*|-1\}$ and $\{1|\downarrow\}$ are $\frac{1}{2}$.

   After 2 moves, the game looks like

   $$-1 + 1 + * = * \quad \text{or} \quad * + \downarrow + * = \downarrow.$$

   The Left plays first, Left will choose the first option and win the game. If Right plays first, Right can take any of the two options and win the game. Thus, the first player can win this game. This is a fuzzy game.

   (c) If Left plays first, Left moves to $1 + +_2 + \uparrow > 0$ and wins the game. If Right plays first, Right moves to $-1 + +_2 + \uparrow < 0$ and wins the game. The first player can always win. This is a fuzzy game.

   (d) If Left plays first, Left can move from * to 0 and leave the game value $-_2 + +_1 + 0 > 0$. So Left wins.

If Right plays first, Right can move from $+_1$ to $\{0|-1\}$. Left has three cases, we will show Right can win for each case.

Case 1, Left moves from $-_2$ to $\{2|0\}$. Right responses from $\{2|0\}$ to 0. Then Left moves from $\{0|-1\}$ to 0. Now the game value is $*$. Right wins.

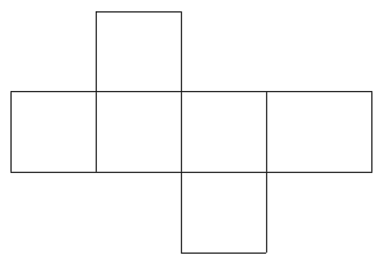
Case 2, Left moves from $\{0|-1\}$ to 0. Now the game is $-_2 + * < *$. Right wins.

Case 3, If Left moves from $*$ to 0, the Right move from $\{0|-1\}$ to -1. The game becomes $-_2 + (-1) < 0$. Right wins the game.

Thus, Right can win if Right plays first.

Hence, this is a fuzzy game and the first player can always win.

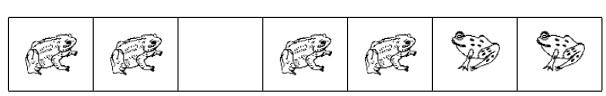
2. Find the values of the following domineering game.



Solution: The game value is

$$\{*|0,*\} = \{*|0\} = \downarrow .$$

3. Find the game value of the following Toads-and-Frogs game:



Solution: Frogs are always stucked. There are two free moves for Toads. Thus the game value is 2.

4. For the game of seating Boys and Girls, find the game values for $L7L$, $L7R$, and $R7R$. For a round table of 8 seats, who will the game?

Solution:

$$\begin{aligned} L7L &= \{L0L + L6L, L1L + L5L, L2L + L4L, L3L + L3L \\ &\qquad \mid L1R + R5L, L2R + R4L, L3R + R3L\} \\ &= \{\{3 \mid *\} \pm 1, 1 + \{4 \mid 0, \pm 1\}, 2 + \{3 \mid *\}, \{2 \mid 0\} + \{2 \mid 0\} \\ &\qquad \mid \pm 2*, * + \pm 2, \pm 1 + \pm 1\} \\ &= \{\{5 \mid 2*\} \mid \pm 2*, 0\}. \end{aligned}$$

This implies

$$R7R = -L7L = \{\pm 2*, 0 \mid \{-2* \mid -5\}\}.$$

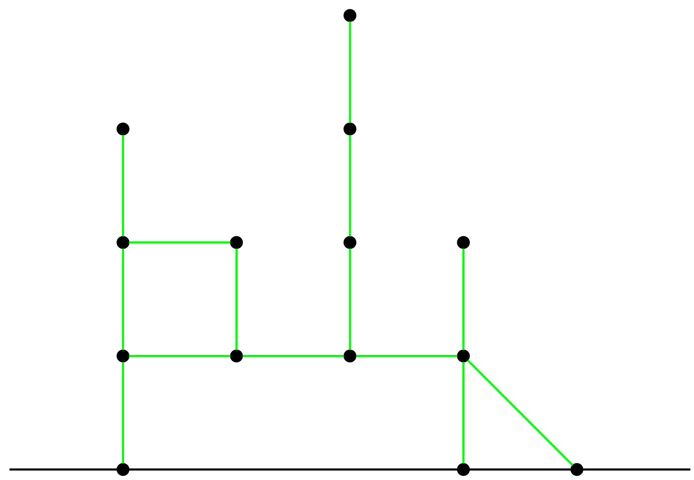
$$\begin{aligned} L7R &= \{L0L + L6R, L1L + L5R, L2L + L4R, L3L + L3R, L4L + L2R, L5L + L1R \\ &\qquad \mid L1R + R5R, L2R + R4R, L3R + R3R, L4R + R2R, L5R + R1R, L6R + R0R\} \\ &= \{\pm 2 \pm 1, 1 + \pm 2*, 2 + \pm 2, \{2 \mid 0\} + \pm 1, \{3 \mid *\} + *, \{4 \mid 0, \pm 1\} \\ &\qquad \mid \pm 2 \pm 1, -1 + \pm 2*, -2 + \pm 2, \{0 \mid -2\} + \pm 1, \{* \mid -3\} + *, \{0, \pm 1 \mid -4\} \\ &= \{1, \{4 \mid 0\} \mid -1, \{0 \mid -4\}\}. \end{aligned}$$

For the round table of size 8, the game value is

$$\begin{aligned} G &= \{L7L \mid R7R\} \\ &= \{\{\{5 \mid 2*\} \mid \pm 2*, 0\} \mid \{\pm 2*, 0 \mid \{-2* \mid -5\}\}\} \\ &= 0. \end{aligned}$$

This is because that the second player can always move to 0 and wins the game.

5. Find the game value for the following green Hackenbush game.



Solution: After applying fusion and colon principle, the value is $*2$.