Math576 Combinatorial Game Theory
Homework 4 Solution

1. Simplify the following game values.

   (a) $\{0, \downarrow \mid 0, *, \uparrow\}$
   (b) $\{* \mid 0, *\}$
   (c) $\{\downarrow, 0 \mid 0, \uparrow *\}$
   (d) $\{\uparrow \mid \Uparrow\}$
   (e) $\{0, * \mid *2, \uparrow\}$

   Solution:

   $$\{0, \downarrow \mid 0, *, \uparrow\} = \{0 \mid 0, *\} = \downarrow *, \quad \text{deleting dominated option.}$$

   $$\{* \mid 0, *\} = \{* \mid 0\} = \downarrow \text{ bypassing right reversible move } *.$$

   $$\{\downarrow, 0 \mid 0, \uparrow *\} = \{0 \mid 0, \uparrow *\} = *, \quad \text{adding gifted horse } \uparrow * \text{ to } * = \{0, 0\}.$$

   $$\{\uparrow \mid \Uparrow\} = \{0, \uparrow \mid \Uparrow\} = \{0 \mid \Uparrow\} = 3.\uparrow *, \quad \uparrow \text{ is a gifted horse to } \{0 \mid \Uparrow\} = 3.\uparrow *.$$

   $$\{0, * \mid *2, \uparrow\} = \{0, * \mid *2\} = \{0 \mid *2\} = \uparrow *3, \quad * \text{ is the gift horse for } \uparrow *3.$$

2. Prove that $\{*2 \mid 0\} = \downarrow *3$.

   Proof $\downarrow *3 = \{* \mid 0\} + \{0, *, *2 \mid 0, *, *2\}$
   $= \{*2, \downarrow, \downarrow *, \downarrow *2 \mid *3, \downarrow, \downarrow *, \downarrow *2\}$
   $= \{*2 \mid \downarrow, \downarrow *, \downarrow *2\}$ deleting dominated options
   $= \{*3 \mid 0\}$ deleting two gift horse options or bypassing them.

3. Two player play the Kayles game. There are three blocks of pins with sizes $7, 8, 9$ respectively. What is the game value? What is the winning move for the first player?

   Solution: The game value is

   $$*2 + * + *4 = *7.$$

   The first player can win by knocking down the second pins in the block of 9, i.e, breaking into two blocks of size 1 and size 7. Now the game value is

   $$*2 + * + (* + *2) = 0.$$

   So this player wins.

4. Find the nim sequence for the substraction game $S(2, 3, 6, 8)$. What is the period of this nim sequence?

   Solution: The subtraction game $S(2, 3, 6, 8)$ has the nim sequence

   $$\overline{00112031220312}$$

   with period 14.

5. Explain what rules is for the Take-and-Break Games .34, then find the nim sequence.

   Solution: The Take-and-Break Game .34 means

   - The player can take away the heap consisting of one bean.
   - The player can take away one bean away from the top of the heap more than one bean.
   - The player can take away 2 beans from any heap more than 4 beans and split the remaining of the heap into two non-empty heaps.

   The nim sequence is

   $$0.101201\overline{03121203}$$

   with period 8 starting at $n = 7$.