Math576 Combinatorial Game Theory
Homework 3 Solution

1. Simplify the following game values.

   (a) $\{0, *, *3, *4 \mid 0, *, *3, *4\}$

   (b) $*3 + *5 + *7$

   (c) $\{0, 1 \mid 1, 2\}$

   (d) $\{-\frac{1}{2}, * \mid *\}$

   (e) $\{-1 + * \mid -1 + *\}$

   Solution:

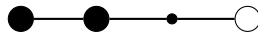 $$\{0, *, *3, *4 \mid 0, *, *3, *4\} = *2, \quad \text{by Mex rule.}$$

   $$*3 + *5 + *7 = *, \quad \text{by nim addtion}$$
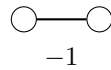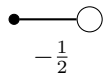
   $$\{0, 1 \mid 1, 2\} = \{1 \mid 1\} = 1 + * = 1*$$

   $$\{-\frac{1}{2}, * \mid *\} = \{* \mid *\} = 0. \quad \text{delete dominated option}$$

   $$\{1 + * \mid 1 + *\} = 1 + \{* \mid *\} = 1 + 0 = 1.$$

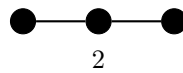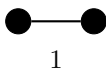2. Find the value of the following Col game:

   Solution: Left colors one of the first three vertices and results three Left options:

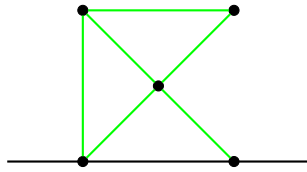 $$-\frac{1}{2} \qquad\qquad -1 \qquad 1 + (-1) = 0$$

   Right can color the third or fouth vertex and results two Right options:

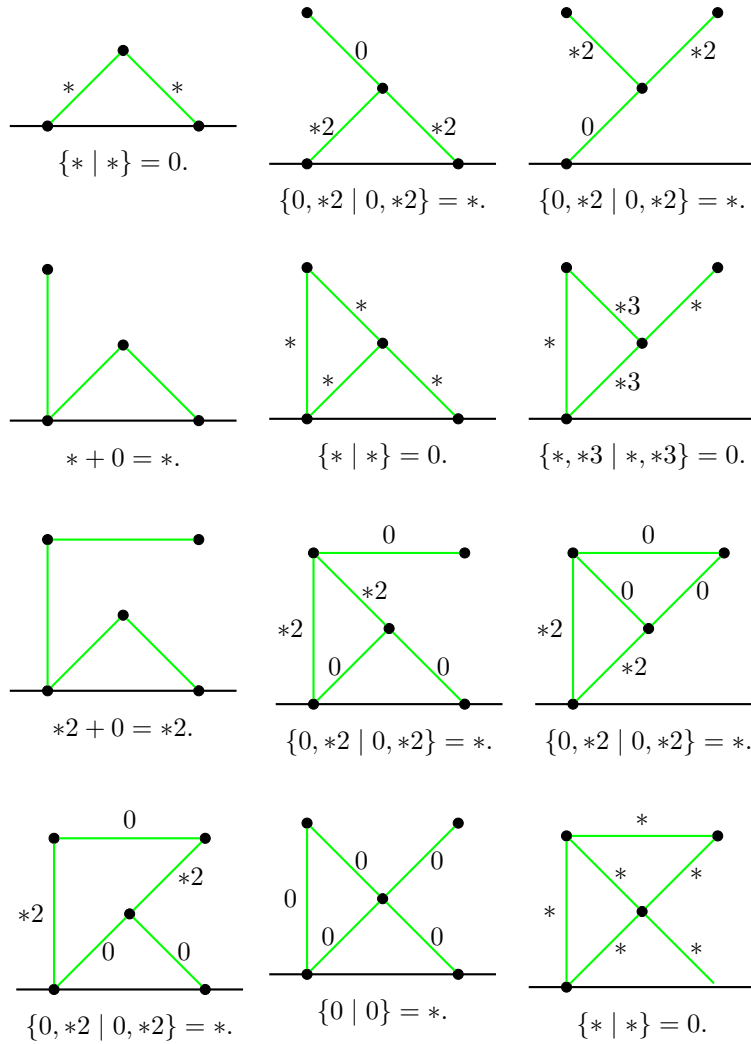 $$1 \qquad\qquad 2$$

   Thus, this Col game value is

   $$\{-1, -\frac{1}{2}, 0 \mid 1, 2\} = \{0 \mid 1\} = \frac{1}{2}.$$

3. Find the value of the following green Hackenbush game.

Solution:

$\{* \mid *\} = 0.$

$\{0, *2 \mid 0, *2\} = *.$

$\{0, *2 \mid 0, *2\} = *.$

$* + 0 = *.$

$\{* \mid *\} = 0.$

$\{*, *3 \mid *, *3\} = 0.$

$*2 + 0 = *2.$

$\{0, *2 \mid 0, *2\} = *.$

$\{0, *2 \mid 0, *2\} = *.$

$\{0, *2 \mid 0, *2\} = *.$

$\{0 \mid 0\} = *.$

$\{* \mid *\} = 0.$

The final answer is 0.

4. Two players are playing the Nim game with the following heaps:

$$2, 5, 6, 11.$$

2

- What is the game value of the current position?
- What is the winning move of the first player?

Solution: Write the numbers in base 2 and compute the nim sum.

$$
\begin{array}{rrrrcr}
 & & 1 & 0 & = & 2 \\
 & 1 & 0 & 1 & = & 5 \\
 & 1 & 1 & 0 & = & 6 \\
1 & 0 & 1 & 1 & = & 11 \\
\hline
1 & 0 & 1 & 0 & = & 10
\end{array}
$$

The game value is
$$*2 + *5 + *6 + *11 = *10.$$

The winning move for the first player is $11 \to 1$, which restores the game value to 0.

5. In the White Knight game, the Knight is at position g4 with a baggage of a Nim-heap of height 1.

   - What is the game value of the current position?
   - What is the winning move of the first player?

Solution: The g4 position contributes $*2$ so the total value is

$$*2 + * = *3.$$

The first player needs to restore this value to 0. He moves the Knight to h2 so the new game value becames

$$* + * = 0.$$