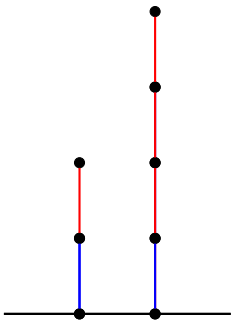
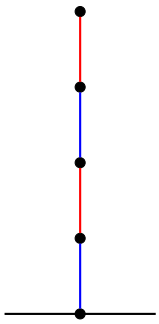
Math576 Combinatorial Game Theory
Solution of homework 1

1. Draw two red-blue Hackenbush games with the same game value $\frac{5}{8}$.

   Solution:



$\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$.



$\{0,\frac{1}{2}\mid\frac{3}{4},1\}=\frac{5}{8}$.

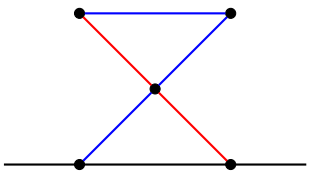
2. State the simplicity rule. Use it to find the following game values:

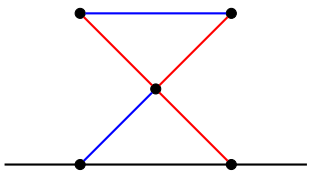
   $(a)\ \{\frac{3}{4}\mid 2\frac{1}{4}\}$ $(b)\ \{-2,-1\mid -\frac{1}{2}\}$ $(c)\ \{0\mid\frac{3}{4},1\}$ $(d)\{0,1\mid 1\frac{1}{2},2\}$

   Solution: The simplicity rule: If there is a number that fits, the answer is the simplicity itself.

   $(a)\ \{\frac{3}{4}\mid 2\frac{1}{4}\}=1$ $(b)\ \{-2,-1\mid -\frac{1}{2}\}=-\frac{3}{4}$ $(c)\ \{0\mid\frac{3}{4},1\}=\frac{1}{2}$ $(d)\{0,1\mid 1\frac{1}{2},2\}=1\frac{1}{4}$.

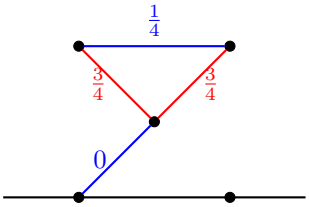
3. Find the values of the following Hackenbush games.



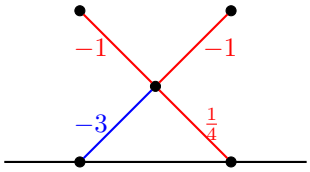


   Solution: In the left Hackenbush game, both red edges and blue edges are connected to the ground via their own edges. Thus, the value is $3-2=1$.

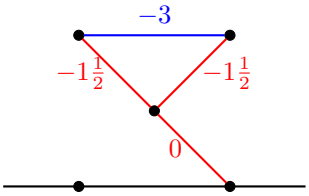
   For the right Hackenbush game, we first compute the game values of some subgraphs:



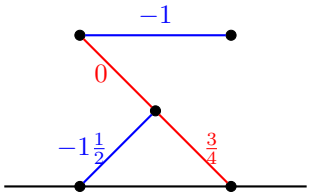
$\{0, \frac{1}{4} \mid \frac{3}{4}\} = \frac{1}{2}$



$\{-3 \mid -1, \frac{1}{4}\} = -2$

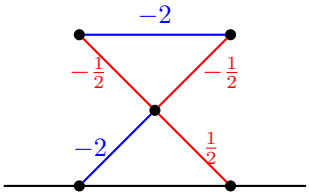


$\{-3 \mid -1\frac{1}{2}, 0\} = -2$



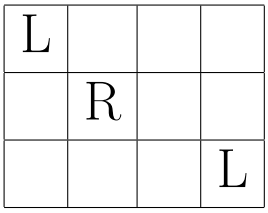
$\{-1\frac{1}{2}, -1 \mid 0, \frac{3}{4}\} = -\frac{1}{2}$

With calculation above, we are ready to compute the game value:

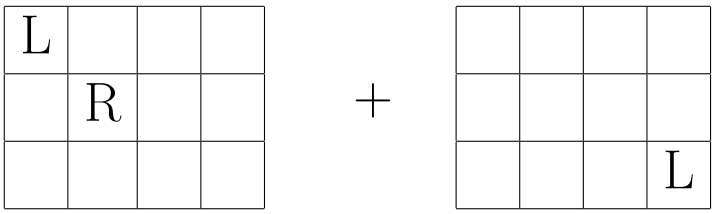


$\{-2 \mid -\frac{1}{2}, \frac{1}{2}\} = -1$

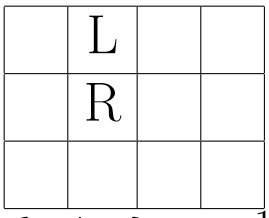
4. Find the value of the following ski-jump game.



Soution: This game can be viewed the sum of two games:

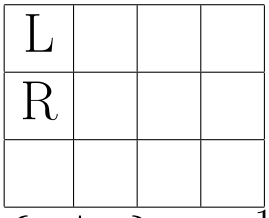


The value of the second board is 1 while the value of the first board can be computed as follows. The left option of the first board is



$$\{1 \mid 2\} = 1\frac{1}{2}.$$

The right option of the first board is



$$\{3 \mid 4\} = 3\frac{1}{2}.$$

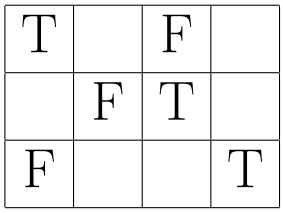
So the value of the first board is

$$\{1\frac{1}{2} \mid 2\frac{1}{2}\} = 2.$$

The value of the board is

$$2 + 1 = 3.$$

5. Find the value of the following Toad-and-Frog game. If ~~Right~~ Left starts first, what is his best first move? (Left moves Toads (T) eastwards and Right moves Frogs (F) westwards.)



Solutioln: This is a sum of Toad-Frog games on three rows. The first row has game value $\frac{1}{2}$; the second row has game value 0; the third row has game value 0. Thus the value of game is

$$\frac{1}{2} + 0 + 0 = \frac{1}{2}.$$

If Left plays on the first row, it moves to the position with value

$$0 + 0 + 0 = 0;$$

Left wins. If Left plays on the second row, it moves to the position with value

$$\frac{1}{2} + (-1) + 0 = -\frac{1}{2};$$

Right wins. Left cannot move on the third row. Left can win only he plays on the first row. Thus, his best move is to move the Toad on the first row one unit eastwards.