Math777: Graph Theory (II)
Spring, 2018
Homework 3, solution

1. [page 289, #10] Prove the following result of Schur: for every \( k \in \mathbb{N} \) there is an \( n \in \mathbb{N} \) such that, for every partition of \( \{1, 2, \ldots, n\} \) into \( k \) sets, at least one of the subsets contains numbers \( x, y, z \) such that \( x + y = z \).

**Proof:** Choose \( n = R_k(3) \), the Ramsey number such that any \( k \)-coloring of \( K_n \) contains a monochromatic triangle. Given a color \( c: [n] \to [k] \), we construct an edge coloring of \( K_n \): color edge \( ij \) by \( c(|i - j|) \). By the Ramsey theorem for graphs, there is a monochromatic triangle \( \{i, j, k\} \); assume \( i < j < k \). Then we set \( x = ji \), \( y = kj \) and \( z = ki \). We have \( c(x) = c(y) = c(z) \) and \( x + y = z \).

2. [page 289, #11] A family of sets is called a \( \Delta \)-system if every two of the sets have the same intersection. Show that every infinite family of sets of the same finite cardinality contains an infinite \( \Delta \)-system.

**Proof:** This is the Erdos-Rado’s theorem: There is a function \( f(k, r) \) so that every family \( F \) of \( k \)-sets with more than \( f(k, r) \) members contains a \( \Delta \)-system of size \( r \).

Let \( F \) be a family of \( k \)-sets without a \( \Delta \)-system of size \( r \). Let \( A_1, A_2, \ldots, A_t \) be a maximum subfamily of pairwise disjoint sets in \( F \). Since a family of pairwise disjoint sets is a \( \Delta \)-system, we must have \( t < r \). Now let \( A = \cup_{i=1}^t A_i \). For every \( a \in A \) consider the family \( F_a = \{ S \setminus \{a\} : S \in F, a \in S \} \). Now, the size of \( A \) is at most \( (r-1)k \) and the size of each \( F_a \) is at most \( f(k-1, r) \). We get that \( f(k, r) \leq (r-1)kf(k-1, r) \). This gives \( f(k, r) \leq (r-1)^k \times k! \).

3. [page 290, #14] Prove that \( 2^c < R(2, c, 3) \leq 3c! \) for every \( c \in \mathbb{N} \).

**Proof:** Lower bound: let \( n = 2^c \) and consider the \( c \)-coloring of \( K_n \) so that an edge \( ij \) receives the color \( l \) if \( 2^{l-1} \leq |i - j| < 2^l \). Since each interval \( [2^{l-1}, 2^l - 1] \) contains no triple \( x < y < z \) so that \( x + y = z \). There is no monochromatic triangle in this coloring. Thus, \( R(2, c, 3) > 2^c \).

Upper bound: For each vertex \( v \) and a fixed color \( i \), the neighbors of \( v \) in the color \( i \) can have at most \( R(2, c - 1, 3) \) vertices. Thus we have a recursive formula:

\[
R(2, c, 3) \leq c R(2, c - 1, 3).
\]

Since \( R(2, 1, 3) = 3 \), we have

\[
R(2, c, 3) \leq 3c!.
\]

4. [page 290, #18] Show that any Kuratowski set \( \{P_1, \ldots, P_k\} \) of a given collection \( \mathcal{C} \) of non-trivial graph properties is unique up to equivalence.
Proof: Let \{Q_1, \ldots, Q_l\} be another Kuratowski set. Assume \(Q_l\) is a minimum element under the partial ordering. By the definition, there is a \(P_i\) with \(P_i \leq Q_l\). For this \(P_i\), there is a \(Q_j\) such that \(Q_j \leq P_i\). We get \(Q_j \leq P_i \leq Q_l\). Since \(Q_1\) is minimum, we must have \(j = 1\). Thus \(P_i\) and \(Q_1\) are equivalent. Delete these two elements from the two sets and do the induction. We conclude that \{\(P_1, \ldots, P_k\}\} are equivalent to \{\(Q_1, \ldots, Q_l\}\}.

5. Let us 3-color the points of the plane. Prove that there will be two points at distance 1 with the same color.

Proof: The Moser Spindle graph \(G\) is the 7-node unit-distance graph shown below:

![Moser Spindle Graph](image)

It is know that \(\chi(G) = 4\). Thus any 3-color of the 7-nodes contains a monochromatic edge, which has distance 1 in the plane.

6. Let us \(k\)-color all non-empty subsets of an \(n\)-element set. Prove that if \(n\) is large enough, there are two disjoint non-empty subsets \(X\) and \(Y\) such that \(X, Y, X \cup Y\) have the same color.

Proof: Let \(n = R(2, k, 3)\). Assume that all non-empty subsets of \([n]\) are \(k\)-colored with colors 1, 2, \ldots, \(k\). Now we construct a \(k\)-edge coloring of \(K_n\). Color each edge \(ij\) by the color of the interval \([i, j - 1]\). By the definition of Ramsey number \(R(2, k, 3)\), there is a monochromatic triangle \(ijl\) with \(i < j < l\). Let \(X = [i, j - 1]\), \(Y = [j, l - 1]\), and \(X \cup Y = [i, l - 1]\). These three sets are in the same color.