1. [page 195, #4] Determine the value of \( \text{ex}(n, K_{1,r}) \) for all \( r, n \in \mathbb{N} \).

**Solution:** We would like to determine how many edges a graph \( G \) on \( n \) vertices can have before a \( K_{1,r} \) subgraph is forced. Note that \( G \) has a \( K_{1,r} \) subgraph if and only if \( \Delta(G) \geq r \). Thus, we must determine the maximum number of edges \( G \) can have and still maintain \( \Delta(G) < r \). We consider two cases:

**Case 1:** \( n \leq r \). Clearly, if \( n \leq r \), we must have \( \Delta(G) < r \). In this case, a complete graph on \( n \) vertices has no \( K_{1,r} \) subgraph and \( \text{ex}(n, K_{1,r}) = \binom{n}{2} \).

**Case 2:** \( n > r \). In this case, we try to draw edges on the vertices of \( G \) so that \( d(v) = r - 1 \) for all \( v \in V(G) \). While it may not be possible to draw an \( r - 1 \)-regular graph on \( n \) vertices, we are able to draw \( \lfloor \frac{(r-1)n}{2} \rfloor \) edges. Thus \( \text{ex}(n, K_{1,r}) = \lfloor \frac{(r-1)n}{2} \rfloor \).

2. [page 195, #5] Given \( k > 0 \), determine the extremal graphs without a matching of size \( k \).

**Solution:** Let \( n \in \mathbb{N} \) and \( k > 0 \). We will consider two cases:

Suppose \( n < 2k \). The complete graph \( K_{2n} \) will certainly have no matching of size \( k \). This graph \( K_{2n} \) has \( \binom{n}{2} \) edges, and we certainly cannot better. Thus \( K_{2n} \) is the extremal graph in this case.

Suppose \( n \geq 2k \). We construct the extremal graph in the following way: First, construct a \( K_{k-1} \). Then, draw an edge from each of the remaining \( (n - k + 1) \) vertices to each of the edges in the \( K_{k-1} \). Then every edge in a maximal matching will be incident to a vertex in the \( K_{k-1} \). Thus, the size of the maximal matching on this graph is \( k - 1 \). This graph has \( \binom{k-1}{2} + (n - k + 1)(k - 1) \) edges, and it is the extremal graph.

3. [page 195, #9] Show that deleting at most \( (m-s)(n-t)/s \) edges from a \( K_{m,n} \) will never destroy all its \( K_{s,t} \) subgraphs.

**Solution:** Let \( M \cup N \) be the partition of the graph \( G \) obtained from \( K_{m,n} \) by deleting these vertices. On the average, a vertex in \( M \) is losing \( (m-s)(n-t)/(sm) \) edges. Picking a set \( S \) of \( s \) vertices with most degrees from \( M \). Consider the induced subgraph \( G[S \cup N] \). We have

\[
|E(G[S \cup N])| \geq s(n-(m-s)(n-t)/(sm)) = (s-1)n+t-s \frac{(n-t)}{m} \geq (s-1)n+t.
\]

Thus in \( G[S \cup N] \), there are a set \( T \) of \( t \) vertices from \( N \) with degree equal \( s \). The induced subgraph \( G[S \cup T] \) is a complete bipartite graph \( K_{s,t} \).
4. Let $1 \leq r \leq n$ be integers. Let $G$ be a bipartite graph with bipartition $\{A, B\}$, where $|A| = |B| = n$, and assume that $K_{r,r} \not\subseteq G$. Show that

$$\sum_{x \in A} \binom{d(x)}{r} \leq (r-1) \binom{n}{r}.$$ 

Use it to deduce $ex(n,K_{r,r}) \leq cn^{2-1/r}$.

**Solution:** Let $1 \leq r \leq n$ be integers. Let $G$ be a bipartite graph with bipartition $\{A, B\}$, where $|A| = |B| = n$. Assume $K_{r,r} \not\subseteq G$. Let $d(x)$ denote the degree of vertex $x \in A$, and let $N(x)$ denote the neighborhood of $x$. Note that $N(x)$ contains $\binom{d(x)}{r}$ $r$-tuples of vertices. If we take the sum of all such $r$-tuples over the neighborhoods of all $x \in A$, we get $\sum_{x \in A} \binom{d(x)}{r}$. Note that any $r$-tuple can be counted at most $r-1$ times. Otherwise, we would get a $K_{r,r}$ subgraph. Thus,

$$\sum_{x \in A} \binom{d(x)}{r} \leq (r-1) \binom{n}{r}.$$ 

Due to the convexity of $\binom{d(x)}{r}$ (for $d(x) > r-1$), $\sum_{x \in A} \binom{d(x)}{r}$ is minimized if the degrees of $x \in A$ are as even as possible. Thus,

$$\sum_{x \in A} \binom{d(x)}{r} \geq n \cdot \binom{|E(G)|/n}{r} \geq n \cdot \frac{|E(G)| (n-r)^r}{r!}$$

Also,

$$(r-1) \binom{n}{r} \leq (r-1) \frac{n^r}{r!}$$

Therefore,

$$n \cdot \frac{|E(G)| (n-r)^r}{r!} \leq (r-1) \frac{n^r}{r!}$$

If we solve this for $|E(G)|$, we conclude that $ex(n,K_{r,r}) \leq cn^{2-1/r}$.

5. Given a graph $G$ with $\epsilon(G) \geq k \in \mathbb{N}$, find a minor $H \prec G$ such that $\delta(H) \geq k \geq |H|/2$.

**Solution:** Let $k = 1$. Let $G$ be a graph with $\epsilon(G) \geq 1 = k$. That means $G$ has at least one edge. If we let $H$ be a path $P_1$, $H$ is certainly a minor of $G$, and $\delta(H) = k = |H|/2 = 1$. 


We proceed by induction. Let \( n \in \mathbb{N} \). Suppose for all \( k \leq n - 1 \), for every graph \( G \) with \( \epsilon(G) \geq k \), we can find a minor \( H \prec G \) such that \( \delta(H) \geq k \geq |H|/2 \). Let \( G \) be a graph with \( \epsilon(G) \geq n \). Pick the minimal minor \( H \prec G \) such that \( \delta(H) \geq k \), and let \( x \in H \). Let us create a new graph \( H' \) from \( H \) by removing \( x \). Since \( \delta(H) \geq k \), \( x \) is not isolated, and the neighbors of \( x \) will have degree at least \( k - 1 \) when \( x \) is removed. Since \( \epsilon(H') \geq k - 1 \), by the inductive hypothesis, we can find a minor \( H'' \) of \( H' \) that satisfies \( \delta(H'') \geq k - 1 \). When we add \( x \) back in, we still get \( \delta(H) \geq k \). Since we are adding only one vertex, \( |H''| \) goes up by at most \( \frac{1}{2} \), so \( k \geq |H|/2 \).

6. If a graph \( G_n \) contains no \( K_4 \) and only contains \( o(n) \) independent vertices, then \( |G_n| < \left( \frac{1}{8} + o(1) \right)n^2 \). (Hint: apply Szemerédi’s Regularity Lemma.)

**Solution:** For any \( \epsilon > 0 \), we apply Szemerédi’s Regularity Lemma to \( G \) to get a regularity partition \( V = V_0 \cup V_1 \cup \ldots \cup V_k \). We define an auxiliary graph \( R \) with the vertex set \( \{V_1, \ldots, V_k\} \). A pair \((V_i, V_j)\) forms an edge of \( R \) if it is a regular pair with edge density at least \( 3\epsilon \). We claim:

**Claim a:** No regularity pair has density \( d > \frac{1}{2} + 2\epsilon \).

**Claim b:** \( R \) is triangle-free.

**Proof of Claim a:** If a regular pair \((V_i, V_j)\) has density \( d > \frac{1}{2} + 2\epsilon \). We claim that we can find a \( K_4 \) in \( G \). Call a vertex \( v \in V_i \) good if for any \( B \subset V_j \) with \( |B| > \epsilon|V_j| \), \( v \) has at least \( (d - \epsilon)|V_j| \) neighbors in \( B \). All vertices in \( V_i \) but a \( \epsilon \)-fraction are good. Since the independent number of \( G \) is \( o(n) \), there is an edge \( xy \) in \( V_i \) so that both \( x \) and \( y \) are good. This implies \( |N(x) \cap N(y) \cap V_j| > (d - \epsilon)^2|V_j| \). Thus inside \( N(x) \cap N(y) \cap V_j \) contains an edge \( st \). The induced graph on \( \{x, y, s, t\} \) is a \( K_4 \). Contradiction.

**Proof of Claim b:** Suppose that \( R \) contains a triangle \( V_iV_jV_s \). We can define a vertex \( v \in V_i \) is good in a similarly way. At least \( (1 - 2\epsilon)|V_j| \) vertices are good. Pick an edge \( xy \) so that both \( x \) and \( y \) are good in \( V_i \). Consider \( N(x) \cap N(y) \cap V_j \) and \( N(x) \cap N(y) \cap V_s \). Both sets have size at least \( (d - \epsilon)^2|V_j| \). Thus we can select an edge \( zw \) so that \( z \in N(x) \cap N(y) \cap V_j \) and \( w \in N(x) \cap N(y) \cap V_s \). Once again, we found a \( K_4 \). Contradiction.

Let \( l = |V_i| \approx \frac{n}{2} \). Since \( R \) is triangle-free, \( R \) has at most \( k^2/4 \) edges. The total number of edge in \( G \) can be bounded by

\[
|G_n| \leq |R|\left(\frac{1}{2} + 2\epsilon\right)l^2 + \left(\frac{k}{2} - |R|\right)3kl^2 + \epsilon kl^2 \\
\leq \left(\frac{1}{8} + 20\epsilon\right)n^2.
\]

Now let \( \epsilon \to 0 \), we have \( |G_n| < \left(\frac{1}{8} + o(1)\right)n^2 \).