Math576 Combinatorial Game Theory
Homework 2 solutions

1. Let $C(2,n)$ be the game value of the rectangle $2\times n$ in the Cut Cake game. Prove $C(2,n) = \lfloor \frac{n}{2} \rfloor - 1$ for all $n \geq 2$.

   Proof: We use induction on n.

   Initial cases, $n = 1, 2$. We have

   $$C(2,1) = \{\mid 2C(1,1)\} = \{\mid 0\} = -1;$$

   $$C(2,2) = \{2C(2,1) \mid 2C(1,2)\} = \{-2 \mid 2\} = 0.$$

   Now we assume that $C(2,k) = \lfloor \frac{k}{2} \rfloor - 1$ for all $1 \leq k \leq n+1$. For $n+1$, the left option for $C(2,n+1)$ are

   $$C(2,k) + C(2,n+1-k)$$

   for some $1 \leq k \leq n$. By inductive hypothesis, it has the value

   $$\lfloor \frac{k}{2} \rfloor - 1 + \lfloor \frac{n+1-k}{2} \rfloor - 1 = \begin{cases} \lfloor \frac{n+1}{2} \rfloor - 2 & \text{if } k \text{ is even ;} \\ \lfloor \frac{n+1}{2} \rfloor - 3 & \text{if } k \text{ is odd .} \end{cases}$$

   The maximum value of left options is $\lfloor \frac{n+1}{2} \rfloor - 2$. The right option has value

   $$2C(1,n+1) = 2n.$$

   We have

   $$C(2,n) = \{\lfloor \frac{n+1}{2} \rfloor - 2 \mid 2n\} = \lfloor \frac{n+1}{2} \rfloor - 1.$$

   The inductive proof is finished.

2. Two players are playing the cut cake game. The current game position consists of three rectangles: 6×4, 3×3, 3×5. What is the game value? If it is Left's turn now, what is his best move?

   Solution: Note that $C(6,4) = 0$, $C(3,3) = 0$, and $C(3,5) = 1$. The game value is

   $$0 + 0 + 1 = 1.$$

   If it is Left's turn, his best move is to restore the game value to 0 by cut the 3×5 cake to two cakes of size 3×3 and 3×2. It is easy to check that all other moves result in a negative value. So this move the his best move.

3. Find the game value of the rectangle 399×400 in the Maundy Cake game.

   Solution: Write

   $$399 \to 133 \to 19 \to 1$$
   $$400 \to 200 \to 100 \to 50 \to 25 \to 5 \to 1.$$

   By the theorem we proved in the lecture, this Maundy Cake game has the value

   $$25 + 5 + 1 = 31.$$

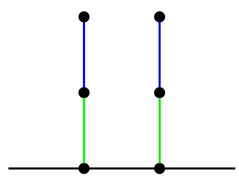
4. What are the meanings of $G \geq 0$ and $G \,|\!\triangleright 0$?

   Solution: $G \geq 0$ means that Left has a winning strategy as the second player.
   $G|\!\triangleright 0$ means that Left has a winning strategy as the first player.

5. For each case, find a pair of two fuzzy games G and H so that

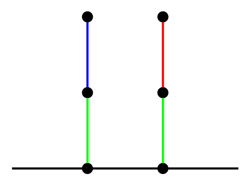
   - $G + H > 0$.
   - $G + H = 0$.
   - $G + H < 0$.
   - $G + H || 0$.

   Solution: Consider the following Hackenbush games:



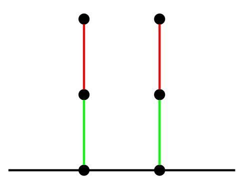
G H

$G + H > 0$



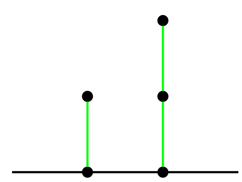
G H

$G + H = 0$



G H

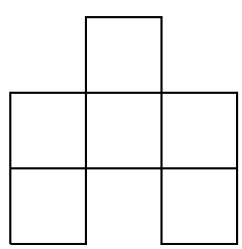
$G + H < 0$



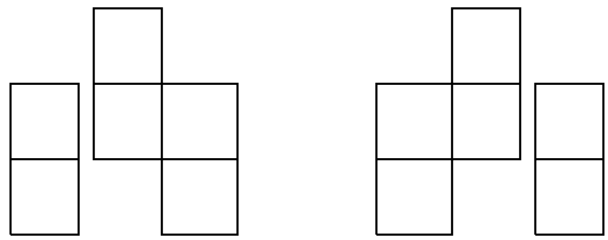
G H

$G + H || 0$

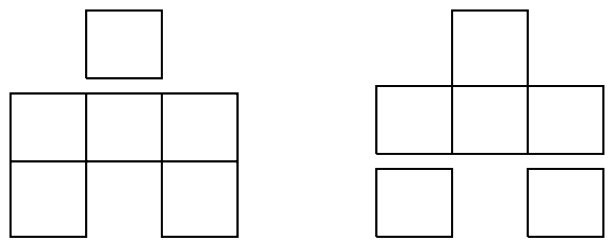
6. Two players are playing the Cut Cake game over an non-rectangle cake. What's the game value of the following cake?



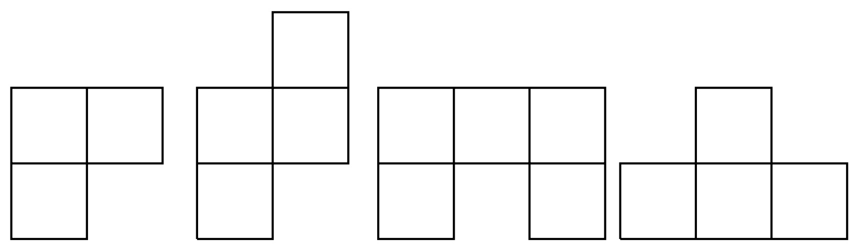
   Solution: There are two options of Left, and they are symmetric.



Below are two options of Right.



We calculate the following game values:



$\{-1 \mid 1\} = 0$ $\{-2 \mid 0\} = -1$ $\{-1 \mid 2\} = 0$ $\{0 \mid 2\} = 1$

Therefore, the game value of the orignal board is

$$\{(-1) + (-1) \mid 0 + 0, 0 + 1\} = \{-2 \mid 0, 1\} = -1.$$