A weighted graph $G$ has a weight function $w: E(G) \to [0, +\infty)$. It can be extended to $w: V(G) \times V(G) \to \mathbb{R}$ so that for any $uv \notin E(G)$, then $w(u, v) = 0$. The adjacency matrix of a weighted graph is simply the weight matrix. I.e. $A = W$.

The degree $d_u$ of a vertex $u$ is $d_u = \sum_v w_{uv}$. Let $T$ be the diagonal matrix of degrees. The Laplacian matrix is defined as $L = I - T^{-\frac{1}{2}}AT^{-\frac{1}{2}}$.

Let $g$ be an eigenvector of $L$ for an eigenvalue $\lambda$. Then $T^{-\frac{1}{2}}g$ is an eigenvector of $T^{-1}A$ for the eigenvalue $1 - \lambda$. I.e., for any $u \in V(G)$,

$$\sum_v \frac{1}{d_u} w_{uv}f_v = (1 - \lambda)f_u.$$ 

Suppose $G$ and $H$ are two weighted graphs. A projection is an onto map $\pi: V(G) \to V(H)$ satisfying

$$\sum_{x \in \pi^{-1}(u)} \sum_{y \in \pi^{-1}(v)} w_{xy} = w_{uv} \quad \text{for any } u, v \in V(H).$$

A projection $\pi: G \to H$ is called regular, if for any $x_1, x_2 \in V(G)$ and any $v \in V(H)$ the sum

$$\sum_{y: \pi(y) = v} w_{x_1 y} = \sum_{y: \pi(y) = v} w_{x_2 y} \quad \text{if } \pi(x_1) = \pi(x_2).$$

**Lemma 1** If the graph projection $\pi: G \to H$ is regular, then any Laplacian eigenvalue of $H$ is a Laplacian eigenvalue of $G$.

**Proof:** Suppose $\lambda$ is an eigenvalue of $H$. There exists a harmonic function $f: V(H) \to \mathbb{R}$ satisfying for any $u \in V(H)$,

$$\sum_{v \in V(H)} \frac{1}{d_u} w_{uv}^Hf_v = (1 - \lambda)f_u.$$ 

Now we extend $f$ to $\tilde{f}: V(G) \to \mathbb{R}$:

$$\tilde{f}_x = f_{\pi(x)}.$$
We would like to show $\tilde{f}$ is the harmonic function $\mathcal{L}_G$. We have, for any $x \in V(G)$, letting $u = \pi(x)$, we have

$$\sum_{y \in V(G)} \frac{1}{d_x} w_{xy} \tilde{f}_y = \sum_{v \in V(H)} \sum_{y : \pi(y) = v} \frac{1}{d_x} w_{xy}^G f_v$$

$$= \sum_{v \in V(H)} \frac{\sum_{y : \pi(y) = v} w_{xy}^G}{d_x} f_v$$

$$= \sum_{v \in V(H)} w_{uv}^H d_u f_v$$

$$= (1 - \lambda) f_u$$

$$= (1 - \lambda) \tilde{f}_x.$$

**Example 1:** Let $G = K_3$. Identifying two vertices together, we get a projection from $G$ to $H$. Here

$$A_H = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}.$$

$H$ has degree 2 and 4. We have the Laplacian matrix

$$\mathcal{L}_H = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{pmatrix}.$$

Note

$$\det(\lambda I - \mathcal{L}_H) = (\lambda - 1)(\lambda - \frac{1}{2}) - \frac{1}{2} = \lambda(\lambda - \frac{3}{2}).$$

$H$ has Laplacian eigenvalues 0 and $\frac{3}{2}$. Note $G$ has Laplacian eigenvalues $0, \frac{3}{2}, \frac{3}{2}$. The set of all Laplacian eigenvalues of $H$ is a subset of all Laplacian eigenvalues of $G$.

**Example 2:** Let $P_{k+1}(2)$ be the weighed graph by assigning each edge of $P_{k+1}$ a weight 2. $P_{k+1}(2)$ and $P_{k+1}$ have the same set of Laplacian eigenvalues.

There is a regular projection from $C_{2k} \rightarrow P_{k+1}(2)$. We conclude that Laplacian eigenvalues of $P_{k+1}$ are Laplacian eigenvalues of $C_{2k}$.

**Example 3:** If the projection is not regular, then the lemma may not hold. For example, there is a projection from $2P_2 \rightarrow P_3$ by identifying two vertices into one vertex. The Laplacian eigenvalues of $P_3$ is $0, 1, 2$ while the Laplacian eigenvalues of $2P_2$ is $0, 0, 2, 2$.

**Lemma 2** If $\pi : G \rightarrow H$ is a projection, then

$$\lambda_G \leq \lambda_H.$$

**Proof:** Let $f$ be the harmonic function for $\lambda_H$. For any $u \in V(H)$, we have

1. $\sum_{u \in V(H)} d_u^H f_u = 0.$
2. $\lambda_H = \frac{\sum_{u,v} (f_u - f_v)^2 w_{uv}^H}{\sum_u d_u^H f_u}$.

Now we extend $f$ to $\tilde{f}: V(G) \to R$:

$$\tilde{f}_x = f_{\pi(x)}.$$ 

In general $\tilde{f}$ is not a harmonic function of $G$. However, we have

$$\sum_{x \in V(G)} d_x^G \tilde{f}_x = \sum_{u \in V(H)} d_u^H f_u = 0.$$

We have

$$\frac{\lambda_G}{\lambda_H} \leq \frac{\sum_{x,y} (\tilde{f}_x - \tilde{f}_y)^2 w_{xy}^G}{\sum_x d_x^G \tilde{f}_x}$$

$$= \frac{\sum_{u,v} (f_u - f_v)^2 w_{uv}^H}{\sum_u d_u^H f_u}$$

$$= \lambda_H.$$  

$\square$