Math 778S Spectral Graph Theory
Handout #2: Basic graph theory

Graph theory was founded by the great Swiss mathematician Leonhard Euler (1707-1783) after he solved the Königsberg Bridge problem: Is it possible to cross seven bridges exactly once?

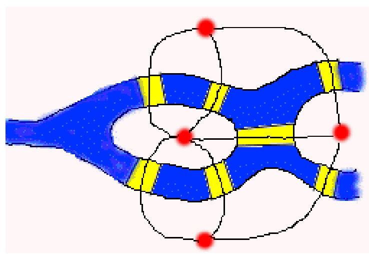

Figure 1: The map of Königsberg

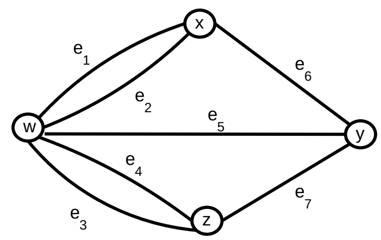

Figure 2: The graph associated with the Königsberg Bridges

Using graph terminology, Euler gave a complete solution for any similar problem with any number of bridges connecting any number of landmasses! We will show and prove his result later.

Definition 1 *A* graph G *is a triple consisting of a vertex set* $V(G)$*, an edge set* $E(G)$*, and a relation that associates with each edge two vertices called its endpoints.*

Example 1 *For the problem of the Königsberg Bridges, one can define the graph as follows. Vertices are landmasses while edges are bridges. We have*

$$\begin{aligned}
V(G) &= \{w, x, y, x\} \\
E(G) &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \\
e_1 &\leftrightarrow \{w, x\} \\
e_2 &\leftrightarrow \{w, x\} \\
e_3 &\leftrightarrow \{w, z\} \\
e_4 &\leftrightarrow \{w, z\} \\
e_5 &\leftrightarrow \{w, y\} \\
e_6 &\leftrightarrow \{x, y\} \\
e_7 &\leftrightarrow \{y, z\}
\end{aligned}$$

Definition 2 *A* loop *is an edge whose endpoints are the same.* Multiple *edges are edges that have the same pair of endpoints. A* simple *graph is a graph without loops or multiple edges.*

For a simple graph, an edge e is uniquely represented by its endpoints u and v. In this case, we write $e = uv$ (or $e = vu$), and we say u and v are *adjacent*. u is a *neighbor* of v, and vice versa. An edge e is incident to u (and v).

A simple graph G on n vertices can have at most $\binom{n}{2}$ edges. The simple graph with n vertices and $\binom{n}{2}$ edges is called the *complete* graph, denoted by K_n. The simple graph G on n vertices with 0 edge is called the *empty* graph. The graph with 0 vertices and 0 edges is called the *null* graph.

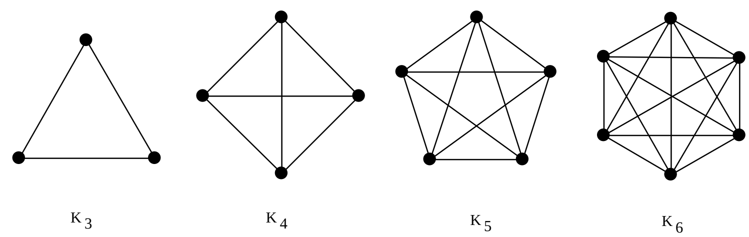

Figure 3: Complete graphs K_3, K_4, K_5, and K_6.

A cycle on n vertices, written C_n, is a graph with vertex set $V(G) = [n] = \{1, 2, 3, \ldots, n\}$ and edge set $E(G) = \{12, 23, 34, \ldots, (n-1)n, n1\}$.

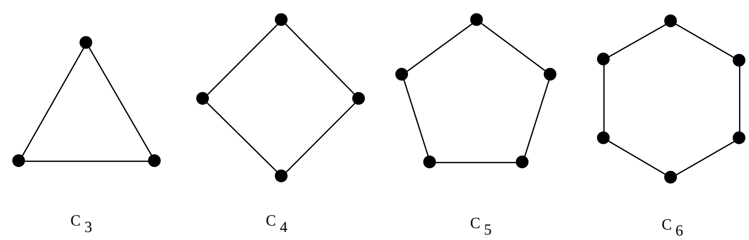

Figure 4: Cycles C_3, C_4, C_5, and C_6.

A path on n vertices, written P_n, is a graph with vertex set $V(G) = [n] = \{1, 2, 3, \ldots, n\}$ and edge set $E(G) = \{12, 23, 34, \ldots, (n-1)n\}$.

Definition 3 *The* complement $\bar{G}$ *of a simple graph* G *is the simple graph with vertex set* $V(\bar{G}) = V(G)$ *and edge set* $E(\bar{G})$ *defined by* $uv \in E(\bar{G})$ *if and only if* $uv \notin E(G)$. *A* clique *in a graph is a set of pairwise adjacent vertices. An* independent *set in a graph is a set of pairwise non-adjacent vertices.*

Definition 4 *A graph* G *is* bipartite *if* $V(G)$ *is the union of two disjoint independent sets called partite sets of* G.

Definition 5 *A graph is* k*-partite if* $V(G)$ *can be expressed as the union of* k *independent sets.*

Definition 6 *The* chromatic *number of a graph* G, *written* $\chi(G)$, *is the minimum number* k *such that* G *is* k*-partite.*

A *subgraph* of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We says G contains H. A *induced subgraph* of a graph G is a subgraph H satisfying

$$E(H) = \{uv \mid u, v \in V(H), uv \in E(G)\}.$$

Example 2 *Let G_1 be the simple graph defined by (see Figure 5)*

$$\begin{aligned} V(G_1) &= \{1, 2, 3, 4, 5\} \\ E(G_1) &= \{12, 23, 13, 24, 34, 45\}. \end{aligned}$$

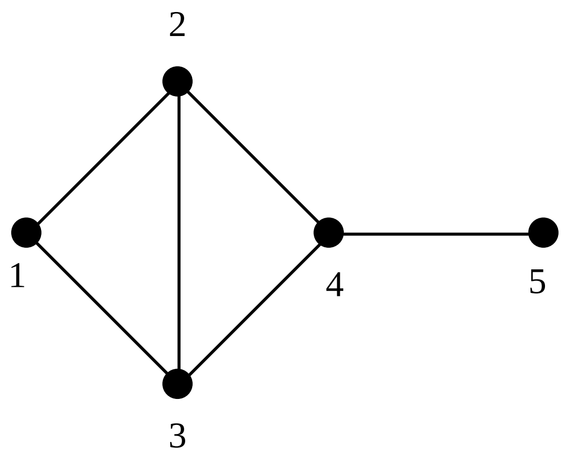

Figure 5: A simple graph G_1.

$\{1, 2, 3\}$ is a clique. $\{1, 4\}$ is an independent set. G_1 is not a bipartite graph. The chromatic number is $\chi(G_1) = 3$.

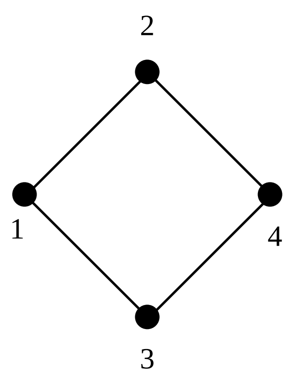

Figure 6: A subgraph of G_1.

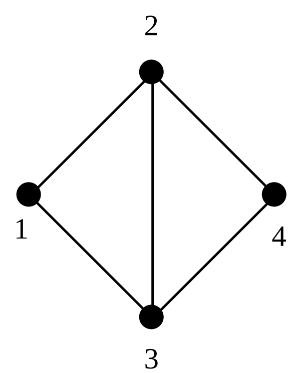

Figure 7: A induced subgraph of G_1.

An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ satisfying

$$uv \in E(G) \text{ iff } f(u)f(v) \in E(H).$$

We say G is isomorphic to H, denoted $G \cong H$.

The isomorphism relation is an equivalence relation. I.e., it is

1. reflexive: $A \cong A$.
2. symmetric: if $A \cong B$, then $B \cong A$.
3. transitive: if $A \cong B$ and $B \cong C$, then $A \cong C$.

An isomorphism class of graphs is an equivalence class of graphs under the isomorphism relation. It is also known as an"unlabeled graph".

Definition 7 *The* adjacency matrix *of a graph G, written $A(G)$, is the n-by-n matrix in which entry a_{ij} is the number of edges with endpoints $\{v_i, v_j\}$ in G. Here $V(G) = v_1, v_2, \ldots, v_n$ is the vertex set of G.*

For a simple graph G, we have

$$A(G) = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

$A(G)$ is a symmetric 0-1 matrix with 0s on the diagonal. For a vertex v of the graph G, the degree d_v is the number of edges which are incident to v. If $v = v_i$, then d_{v_i} is the i-th row/column sum of $A(G)$.

Example 3 *For graph G_1 (Figure 5), the adjacency matrix is given by*

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Definition 8 *A* walk *(on a graph G) is a list $v_0, e_1, v_1, e_1, \ldots, e_k, v_k$, satisfying $e_i = v_{i-1}v_i$ is an edge for all $i = 1, 2, \ldots, k$. k is called the length of the walk.*

A u, v-walk *is a walk with $v_0 = u$ and $v_k = v$.*

A trail *is a walk with no repeated edge.*

A path *is a walk with no repeated vertices.*

A closed walk *is a walk with the same endpoints, i.e., $v_0 = v_k$.*

A cycle *is a closed walk with no repeated vertices except for the endpoints.*

Lemma 1 *Every u, v-walk contains a u, v-path.*

Proof: We prove it by induction on the length k of the walk.

When $k = 1$, a u, v-walk is a u, v-path.

We assume that every u, v-walk with length at most k contains a u, v-path. Now let us consider a u, v-walk $u = v_0, e_1, v_1, e_1, \ldots, e_{k+1}, v_{k+1} = v$ of length $k + 1$. If the walk has no repeated vertices, it is a u, v-path by definition. Otherwise, say $v_i = v_j$ for $0 \leq i < j \leq k + 1$. By deleting all vertices and edges between v_i and v_j, we get a new walk: $v_0, \ldots, v_i = v_j, \ldots, v_{k+1}$. This walk has length at most k. By the inductive hypothesis, it contains a u, v-path. $\square$.

Definition 9 *A graph G is* connected *if it has a u, v-path for any $u, v \in V(G)$.*

For any u, v, we define a connected relation $u \sim v$ over $V(G)$ if there is a u, v-path in G. The connected relation is an equivalence relation. The equivalence classes for this relation are called *connected components* of G.

A component is *trivial* if it contains only one vertex. A trivial component is also called an *isolated* vertex.

Lemma 2 *Every closed odd walk contains an odd cycle.*

Proof: We prove it by induction on the length k of the closed walk.

When $k = 1$, the walk of length 1 is a loop. Thus, it is an odd cycle.

We assume that every odd walk with length at most $k = 2r-1$ contains a u, v-path. Now let us consider a closed walk $u = v_0, e_1, v_1, e_1, \ldots, e_{2r+1}, v_{2r+1} = v$ of length $2r + 1$. If the walk has no repeated vertices, it is an odd cycle by definition. Otherwise, say $v_i = v_j$ for $0 \leq i < j \leq 2r + 1$. The walk can be split into two closed walks:

$$v_0, \ldots, v_i = v_j, \ldots, v_{k+1}$$

$$v_i, \ldots, e_{i+1}, \ldots, v_j.$$

One of them must have odd length of at most $2r-1$. By the inductive hypothesis, it contains an odd cycle.

The proof is finished. $\square$.

Theorem 1 (König 1936) *A graph is bipartite if and only if it has no odd cycle.*

Proof: It is sufficient to prove this for any connected graph.

Necessity: Let G be bipartite graph. Every walk of G alternates between the two sets of a bipartition. The lengths of any closed walks are even. Therefore, it has no odd cycle.

Sufficiency: Let G be a graph with no odd cycle. We will construct a bipartition as follows. Choose any vertex u. We define a partition $V(G) = X \cup Y$ as follows.

$$X = \{v \in V(G) | \text{ there is a } u, v\text{-path of odd length}\}.$$

$$Y = \{v \in V(G) | \text{ there is a } u, v\text{-path of even length}\}.$$

Since G is connected, we have $X \cup Y = V(G)$. We will show $X \cap Y = \emptyset$. Otherwise, let $u \in X \cap Y$. There are two u, v-paths. One path has even length while the other one has odd length. Put them together. We have an odd closed walk. By lemma, G has an odd cycle. Contradiction. Therefore, $V(G) = X \cup Y$ is a partition.

Next, we will show there are no edges with both ends in X or Y. Otherwise, suppose that vw is such an edge. The u, v-path, the edge vw, and the w, u-path together form an odd closed walk. By previous lemma, G contains an odd cycle. Contradiction.

Hence, G is a bipartite graph. □

A complete bipartite graph $K_{s,t}$ has a vertex set partition $V = X \cup Y$ with $|X| = s$ and $|Y| = t$ and an edge set $E(G) = xy \mid x \in X, y \in Y$.

Theorem 2 *Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ be the eigenvalues of the adjacency matrix of a simple graph G. The the following statements are equivalent.*

1. *G is a bipartite graph.*
2. *For all $1 \leq i \leq n$, $\lambda_{n+1-i} = -\lambda_i$.*

Proof: Suppose G is a bipartite graph. Then there exists a vertex set partition $V = X \cup Y$. Reorder vertices so that the vertices in X are before the vertices in Y. The adjacency matrix A has the following shape:

$$A = \begin{pmatrix} 0 & B \\ B' & 0 \end{pmatrix}.$$

Let $\alpha = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ be an eigenvector corresponding to an eigenvalue λ. Since $A\alpha = \lambda\alpha$, we have

$$\begin{aligned} B\gamma &= \lambda\beta \\ B'\beta &= \lambda\alpha. \end{aligned}$$

Let $\tilde{\alpha} = \begin{pmatrix} \beta \\ -\gamma \end{pmatrix}$ Then we have

$$\begin{aligned} A\tilde{\alpha} &= \begin{pmatrix} 0 & B \\ B' & 0 \end{pmatrix} \begin{pmatrix} \beta \\ -\gamma \end{pmatrix} \\ &= \begin{pmatrix} -B\gamma \\ B'\beta \end{pmatrix} \\ &= \begin{pmatrix} -\lambda\beta \\ \lambda\gamma \end{pmatrix} \\ &= -\lambda\tilde{\alpha}. \end{aligned}$$

This shows $-\lambda$ is also an eigenvalue of A. The eigenvalues are symmetric about 0.

Suppose $\lambda_i = -\lambda_{n+1-i}$ for all i. We have

$$\operatorname{tr}(A^{2k+1}) = \sum_{i=1}^{n} \lambda_i^{2k+1} = 0.$$

The number of odd closed walks is 0. G has no odd cycle. Thus, G is bipartite by Konig's theorem.