Graph theory was founded by the great Swiss mathematician Leonhard Euler (1707-1783) after he solved the Königsberg Bridge problem: Is it possible to cross seven bridges exactly once?

Using graph terminology, Euler gave a complete solution for any similar problem with any number of bridges connecting any number of landmasses! We will show and prove his result later.

**Definition 1** A graph $G$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its endpoints.

**Example 1** For the problem of the Königsberg Bridges, one can define the graph as follows. Vertices are landmasses while edges are bridges. We have

\[
V(G) = \{w, x, y, z\}
\]

\[
E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}
\]

\[
e_1 \leftrightarrow \{w, x\}
\]

\[
e_2 \leftrightarrow \{w, x\}
\]

\[
e_3 \leftrightarrow \{w, z\}
\]

\[
e_4 \leftrightarrow \{w, z\}
\]

\[
e_5 \leftrightarrow \{w, y\}
\]

\[
e_6 \leftrightarrow \{x, y\}
\]

\[
e_7 \leftrightarrow \{y, z\}
\]

**Definition 2** A loop is an edge whose endpoints are the same. Multiple edges are edges that have the same pair of endpoints. A simple graph is a graph without loops or multiple edges.
For a simple graph, an edge $e$ is uniquely represented by its endpoints $u$ and $v$. In this case, we write $e = uv$ (or $e = vu$), and we say $u$ and $v$ are adjacent. $u$ is a neighbor of $v$, and vice versa. An edge $e$ is incident to $u$ (and $v$).

A simple graph $G$ on $n$ vertices can have at most $\left(\begin{array}{c}n \\ 2 \end{array}\right)$ edges. The simple graph with $n$ vertices and $\left(\begin{array}{c}n \\ 2 \end{array}\right)$ edges is called the complete graph, denoted by $K_n$. The simple graph $G$ on $n$ vertices with 0 edge is called the empty graph. The graph with 0 vertices and 0 edges is called the null graph.

![Figure 3: Complete graphs $K_3$, $K_4$, $K_5$, and $K_6$.](image)

A cycle on $n$ vertices, written $C_n$, is a graph with vertex set $V(G) = [n] = \{1, 2, 3, \ldots, n\}$ and edge set $E(G) = \{12, 23, 34, \ldots, (n-1)n, n1\}$.

![Figure 4: Cycles $C_3$, $C_4$, $C_5$, and $C_6$.](image)

A path on $n$ vertices, written $P_n$, is a graph with vertex set $V(G) = [n] = \{1, 2, 3, \ldots, n\}$ and edge set $E(G) = \{12, 23, 34, \ldots, (n-1)n\}$.

**Definition 3** The complement $\bar{G}$ of a simple graph $G$ is the simple graph with vertex set $V(\bar{G}) = V(G)$ and edge set $E(\bar{G})$ defined by $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. A clique in a graph is a set of pairwise adjacent vertices. An independent set in a graph is a set of pairwise non-adjacent vertices.

**Definition 4** A graph $G$ is bipartite if $V(G)$ is the union of two disjoint independent sets called partite sets of $G$.

**Definition 5** A graph is $k$-partite if $V(G)$ can be expressed as the union of $k$ independent sets.

**Definition 6** The chromatic number of a graph $G$, written $\chi(G)$, is the minimum number $k$ such that $G$ is $k$-partite.
A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say $G$ contains $H$. A induced subgraph of a graph $G$ is a subgraph $H$ satisfying

$$E(H) = \{uv \mid u, v \in V(H), uv \in E(G)\}.$$ 

**Example 2** Let $G_1$ be the simple graph defined by (see Figure 5)

$$V(G_1) = \{1, 2, 3, 4, 5\}$$
$$E(G_1) = \{12, 23, 13, 24, 34, 45\}.$$ 

![Figure 5: A simple graph $G_1$.](image)

{1, 2, 3} is a clique. {1, 4} is an independent set. $G_1$ is not a bipartite graph. The chromatic number is $\chi(G_1) = 3$.

![Figure 6: A subgraph of $G_1$.](image)  
![Figure 7: A induced subgraph of $G_1$.](image)
An isomorphism from a simple graph $G$ to a simple graph $H$ is a bijection $f : V(G) \to V(H)$ satisfying

$$uv \in E(G) \text{ iff } f(u)f(v) \in E(H).$$

We say $G$ is isomorphic to $H$, denoted $G \cong H$.

The isomorphism relation is an equivalence relation. I.e., it is

1. reflexive: $A \cong A$.
2. symmetric: if $A \cong B$, then $B \cong A$.
3. transitive: if $A \cong B$ and $B \cong C$, then $A \cong C$.

An isomorphism class of graphs is an equivalence class of graphs under the isomorphism relation. It is also known as an “unlabeled graph”.

**Definition 7** The adjacency matrix of a graph $G$, written $A(G)$, is the $n$-by-$n$ matrix in which entry $a_{ij}$ is the number of edges with endpoints $\{v_i, v_j\}$ in $G$. Here $V(G) = v_1, v_2, \ldots, v_n$ is the vertex set of $G$.

For a simple graph $G$, we have

$$A(G) = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

$A(G)$ is a symmetric 0-1 matrix with 0s on the diagonal. For a vertex $v$ of the graph $G$, the degree $d_v$ is the number of edges which are incident to $v$. If $v = v_i$, then $d_{v_i}$ is the $i$-th row/column sum of $A(G)$.

**Example 3** For graph $G_1$ (Figure 5), the adjacency matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

**Definition 8** A walk (on a graph $G$) is a list $v_0, e_1, v_1, e_1, \ldots, e_k, v_k$, satisfying $e_i = v_{i-1} v_i$ is an edge for all $i = 1, 2, \ldots, k$. $k$ is called the length of the walk. A walk is a walk with no repeated vertices.

A path is a walk with no repeated vertices.

A trail is a walk with no repeated vertices.

A closed walk is a walk with the same endpoints, i.e., $v_0 = v_k$.

A cycle is a closed walk with no repeated vertices except for the endpoints.

**Lemma 1** Every $u, v$-walk contains a $u, v$-path.
Proof: We prove it by induction on the length \( k \) of the walk.

When \( k = 1 \), a \( u, v \)-walk is a \( u, v \)-path.

We assume that every \( u, v \)-walk with length at most \( k \) contains a \( u, v \)-path. Now let us consider a \( u, v \)-walk \( u = v_0, e_1, v_1, e_1, \ldots, e_{k+1}, v_{k+1} = v \) of length \( k + 1 \). If the walk has no repeated vertices, it is a \( u, v \)-path by definition. Otherwise, say \( v_i = v_j \) for \( 0 \leq i < j \leq k + 1 \). By deleting all vertices and edges between \( v_i \) and \( v_j \), we get a new walk: \( v_0, \ldots, v_i = v_j, \ldots, v_{k+1} \). This walk has length at most \( k \). By the inductive hypothesis, it contains a \( u, v \)-path. \( \Box \).

Definition 9 A graph \( G \) is connected if it has a \( u, v \)-path for any \( u, v \in V(G) \).

For any \( u, v \), we define a connected relation \( u \sim v \) over \( V(G) \) if there is a \( u, v \)-path in \( G \). The connected relation is an equivalence relation. The equivalence classes for this relation are called connected components of \( G \).

A component is trivial if it contains only one vertex. A trivial component is also called an isolated vertex.

Lemma 2 Every closed odd walk contains an odd cycle.

Proof: We prove it by induction on the length \( k \) of the closed walk.

When \( k = 1 \), the walk of length 1 is a loop. Thus, it is an odd cycle.

We assume that every odd walk with length at most \( k = 2r - 1 \) contains a \( u, v \)-path. Now let us consider a closed walk \( u = v_0, e_1, v_1, e_1, \ldots, e_{2r+1}, v_{2r+1} = v \) of length \( 2r + 1 \). If the walk has no repeated vertices, it is an odd cycle by definition. Otherwise, say \( v_i = v_j \) for \( 0 \leq i < j \leq 2r + 1 \). The walk can be split into two closed walks:

\[
v_0, \ldots, v_i = v_j, \ldots, v_{k+1} \]
\[
v_i, \ldots, e_{i+1}, \ldots, v_j.
\]

One of them must have odd length of at most \( 2r - 1 \). By the inductive hypothesis, it contains an odd cycle.

The proof is finished. \( \Box \).

Theorem 1 (König 1936) A graph is bipartite if and only if it has no odd cycle.

Proof: It is sufficient to prove this for any connected graph.

Necessity: Let \( G \) be bipartite graph. Every walk of \( G \) alternates between the two sets of a bipartition. The lengths of any closed walks are even. Therefore, it has no odd cycle.

Sufficiency: Let \( G \) be a graph with no odd cycle. We will construct a bipartition as follows. Choose any vertex \( u \). We define a partition \( V(G) = X \cup Y \) as follows.

\[
X = \{ v \in V(G) \mid \text{there is a } u, v \text{-path of odd length} \}.
\]
\[
Y = \{ v \in V(G) \mid \text{there is a } u, v \text{-path of even length} \}.
\]
Since $G$ is connected, we have $X \cup Y = V(G)$. We will show $X \cap Y = \emptyset$. Otherwise, let $u \in X \cap Y$. There are two $u, v$-paths. One path has even length while the other one has odd length. Put them together. We have an odd closed walk. By lemma, $G$ has an odd cycle. Contradiction. Therefore, $V(G) = X \cup Y$ is a partition.

Next, we will show there are no edges with both ends in $X$ or $Y$. Otherwise, suppose that $vw$ is such an edge. The $u, v$-path, the edge $vw$, and the $w, u$-path together form an odd closed walk. By previous lemma, $G$ contains an odd cycle. Contradiction.

Hence, $G$ is a bipartite graph. □

A complete bipartite graph $K_{s,t}$ has a vertex set partition $V = X \cup Y$ with $|X| = s$ and $|Y| = t$ and an edge set $E(G) = xy \mid x \in X, y \in Y$.

**Theorem 2** Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ be the eigenvalues of the adjacency matrix of a simple graph $G$. The following statements are equivalent.

1. $G$ is a bipartite graph.
2. For all $1 \leq i \leq n$, $\lambda_{n+1-i} = -\lambda_i$.

**Proof:** Suppose $G$ is a bipartite graph. Then there exists a vertex set partition $V = X \cup Y$. Reorder vertices so that the vertices in $X$ are before the vertices in $Y$. The adjacency matrix $A$ has the following shape:

$$A = \begin{pmatrix} 0 & B \\ B' & 0 \end{pmatrix}.$$

Let $\alpha = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}$ be an eigenvector corresponding to an eigenvalue $\lambda$. Since $A\alpha = \lambda \alpha$, we have

$$B\gamma = \lambda \beta$$
$$B'\beta = \lambda \alpha.$$

Let $\tilde{\alpha} = \begin{pmatrix} \beta \\ -\gamma \end{pmatrix}$ Then we have

$$A\tilde{\alpha} = \begin{pmatrix} 0 & B \\ B' & 0 \end{pmatrix} \begin{pmatrix} \beta \\ -\gamma \end{pmatrix}$$
$$= \begin{pmatrix} -B\gamma \\ B'\beta \end{pmatrix}$$
$$= \begin{pmatrix} -\lambda \beta \\ \lambda \gamma \end{pmatrix}$$
$$= -\lambda \tilde{\alpha}.$$

This shows $-\lambda$ is also an eigenvalue of $A$. The eigenvalues are symmetric about 0.
Suppose $\lambda_i = -\lambda_{n+1-i}$ for all $i$. We have

$$\text{tr}(A^{2k+1}) = \sum_{i=1}^{n} \lambda_i^{2k+1} = 0.$$  

The number of odd closed walks is 0. $G$ has no odd cycle. Thus, $G$ is bipartite by König’s theorem.