Math 778S Spectral Graph Theory
Handout #3: Eigenvalues of Adjacency Matrix

The Cartesian product (denoted by \( G \square H \)) of two simple graphs \( G \) and \( H \) has the vertex-set \( V(G) \times V(H) \). For any \( u, v \in V(G) \) and \( x, y \in V(H) \), \((u, x)\) is adjacent to \((v, y)\) if either “\( u = v \) and \( xy \in E(H) \)” or “\( uv \in E(G) \) and \( x = y \)”.

**Lemma 1** Suppose \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of the adjacency matrix of a graph \( G \) and \( \mu_1, \ldots, \mu_m \) are eigenvalues of the adjacency matrix of a graph \( H \). Then the eigenvalues of the adjacency matrix of the Cartesian product \( G \square H \) are \( \lambda_i + \mu_j \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).

**Proof:** Let \( A \) (or \( B \)) be the adjacency matrix of \( G \) (or \( H \)) respectively. For any eigenvalue \( \lambda \) of \( A \) and any eigenvalue \( \mu \) of \( B \), we would like to show \( \lambda + \mu \) is an eigenvalue of \( G \square H \). Let \( \alpha \) be the eigenvector of \( A \) corresponding to \( \lambda \) and \( \beta \) be the eigenvector of \( B \) corresponding to \( \mu \). We have

\[
A \alpha = \lambda \alpha \quad (1)
\]
\[
B \beta = \mu \beta \quad (2)
\]

Equivalently, for any \( u \in V(G) \),
\[
\sum_{v \sim u} \alpha_v = \lambda \alpha_u;
\]
for any \( x \in V(H) \),
\[
\sum_{y \sim x} \beta_y = \mu \beta_x.
\]

Let \( \alpha \otimes \beta \) be the \( n \times m \) column vector defined by entries
\[
(\alpha \otimes \beta)_{u,x} = \alpha_u \beta_x.
\]
Let \( C \) be the adjacency matrix of \( G \square H \). We would like to show \( \alpha \otimes \beta \) is an eigenvector of \( C \). We have, for any \((u, x) \in V(G \square H)\),

\[
\sum_{(v,y) \sim (u,x)} (\alpha \otimes \beta)_{v,y} = \sum_{(v,y) \sim (u,x)} \alpha_v \beta_y \\
= \sum_{(u,y) \sim (u,x)} \alpha_u \beta_y + \sum_{(v,x) \sim (u,x)} \alpha_v \beta_x \\
= \sum_{y \sim x} \alpha_u \beta_y + \sum_{v \sim u} \alpha_v \beta_x \\
= \alpha_u \sum_{y \sim x} \beta_y + \beta_x \sum_{v \sim u} \alpha_v \\
= \alpha_u \beta_x + \beta_x \lambda \alpha_u \\
= (\lambda + \mu)(\alpha \otimes \beta)_{u,x}.
\]
This is equivalent to
\[ C(\alpha \times \beta) = (\lambda + \mu)(\alpha \times \beta). \]

Thus, \( \lambda + \mu \) is an eigenvalue of \( G \square H \).

For \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), \( \lambda_i + \mu_j \) are eigenvalues of \( G \square H \). Since \( G \square H \) has \( nm \) vertices, these eigenvalues (with multiplicity) are all eigenvalues of \( G \square H \). \( \square \)

**Remark:** The adjacency matrix of \( G \square H \) can be written as \( A \otimes I_m + I_n \otimes B \). Here \( \otimes \) is tensor product of matrices.

**Hypercube** \( Q_n \): The vertices of \( Q_n \) are points in \( n \)-dimensional space over the field of two elements \( F_2 = \{0, 1\} \). Two points are adjacent in \( Q_n \) if and only if they differ by exactly one coordinate.

We have \( Q_1 = P_2 \), \( Q_2 = C_4 \), and \( Q_3 \) is the cube in 3-dimensional space. We have \( Q_{n+1} = Q_1 \square Q_n \). The eigenvalues of \( Q_n \) can be determined from the eigenvalues of \( Q_1 \) and the above lemma.

\( Q_1 = P_2 \) has eigenvalues \( \pm 1 \). \( Q_n \) has eigenvalues \( n - 2i \) with multiplicity \( \binom{n}{i} \) for \( 0 \leq i \leq n \).

**Regular graphs:** The degree of a vertex \( v \) in \( G \) is the number of edges incident to \( v \). If all degrees are equal to \( d \), then \( G \) is called a \( d \)-regular graph. Let \( \mathbf{1} \) be the column vector of all entries equal to 1. If \( G \) is a regular graph, then \( A\mathbf{1} = d\mathbf{1} \). Hence, \( \mathbf{1} \) is an eigenvector for the eigenvalue \( d \).

**Eigenvalues of** \( K_n \): Let \( J = \mathbf{1}'\mathbf{1} \) be the \( n \times n \)-matrix with all entries 1. Since \( J \) is a rank 1 matrix, \( J \) has eigenvalues 0 with multiplicity \( n - 1 \). It is easy to see that the nonzero eigenvalue of \( J \) is \( n \). The complete graph \( K_n \) has the adjacency matrix \( J - I \). Thus, \( K_n \) has an eigenvalue \( n - 1 \) of multiplicity 1 and \( -1 \) of multiplicity \( n - 1 \).

**Eigenvalues of** \( C_n \): Let \( Q = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \).

\( Q \) can be viewed as the adjacency matrix of the directed cycle. We have \( A = Q + Q' \). Note that \( Q^n = I \). Let \( \lambda \) be the eigenvalue of \( Q \). We have \( \lambda^n = 1 \). The eigenvalues of \( Q \) are precisely \( n \)-th root of 1:

\[ \rho^k = \cos\left(\frac{2k\pi}{n}\right) + \sqrt{-1}\sin\left(\frac{2k\pi}{n}\right), \quad \text{for } 0 \leq k \leq n - 1. \]

Note \( Q' = Q^{n-1} \). Thus, \( A = Q + Q' \) has eigenvalues

\[ \rho^k + \rho^{k(n-1)} = 2\Re(\rho^k) = 2\cos\left(\frac{2k\pi}{n}\right) \]

for \( k = 0, 1, 2, \ldots, n - 1 \).
Let \( \mu_1 \geq \mu_2 \geq \ldots \mu_n \) be the eigenvalues of the adjacency matrix of a graph \( G \). We refer \( \mu_1 = \mu_{\text{max}} \) and \( \mu_n = \mu_{\text{min}} \). We have

\[
\mu_{\text{max}} = \sup_{\|x\|=1} x'Ax \\
\mu_{\text{min}} = \inf_{\|x\|=1} x'Ax
\]

Suppose \( f(x) = x'Ax \) reaches the maximum at \( \alpha \) on the unit sphere. Then all coordinates of \( \alpha \) are non-negative.

**Lemma 2** If \( H \) is a subgraph of \( G \), then we have

\[
\mu_{\text{max}}(G) \geq \mu_{\text{max}}(H).
\]

**Proof:** Without loss of generality, we assume \( V(H) = V(G) \). (Otherwise, we add some isolated vertices to \( H \). It doesn’t change the maximum eigenvalue of \( H \).)

Let \( \alpha \) be the eigenvector \( A_H \) corresponding to \( \mu_{\text{max}}(H) \). We have

\[
\mu_{\text{max}}(H) = \alpha' A_H \alpha \\
= 2 \sum_{ij \in E(H)} \alpha_i \alpha_j \\
\leq 2 \sum_{ij \in E(G)} \alpha_i \alpha_j \\
= \alpha' A_G \alpha \\
\leq \sup_{\|x\|=1} x' A_G x \\
= \mu_{\text{max}}(G).
\]

Let \( \delta \) be the minimum degree and \( \Delta \) be the maximum degree of \( G \). We have the following bound on \( \mu_{\text{max}} \).

**Lemma 3** For every graph \( G \), we have

\[
\delta(G) \leq \mu_{\text{max}}(G) \leq \Delta(G).
\]

**Proof:** Let \( \alpha \) be an eigenvector for eigenvalue \( \mu = \mu_{\text{max}}(G) \). Since \( \alpha \neq 0 \), we can assume \( \alpha \) has at least one positive coordinate. (If all coordinates are non-positive, we consider \(-\alpha\) instead.)

Let \( \alpha_k = \max_i \alpha_i \) be the largest coordinate of \( \alpha \). Since \( A\alpha = \mu \alpha \), we have

\[
\mu \alpha_k = (A\alpha)_k = \sum_{i \sim k} \alpha_i \leq \Delta \alpha_k.
\]

Thus, \( \mu \leq \Delta \).
Now we show $\mu_{\text{max}}(G) \geq \delta(G)$.

$$\mu_{\text{max}} = \sup_{\|x\|=1} x' A_G x \geq \frac{1}{\sqrt{n}} 1' A_G \frac{1}{\sqrt{n}} 1 \geq \frac{1}{n} \sum_{i \sim j} a_{ij} = \frac{2|E(G)|}{n} \geq \delta(G).$$

□

A $k$-coloring of a graph $G$ is a map $c : V(G) \to [k] = \{1, 2, \ldots, k\}$. A $k$-coloring is said to be proper if the end vertices of any edge in $G$ receive different colors. I.e.,

$$c(u) \neq c(v) \text{ for any } u \sim v.$$ In this case, we say $G$ is $k$-colorable.

The chromatic number denoted by $\chi(G)$ is the minimum integer $k$ such that $G$ is $k$-colorable. For example, $\chi(K_n) = n$. $\chi(G) = 2$ if and only if $G$ is a nonempty bipartite graph.

There is a simple bound on $\chi(G)$.

**Theorem 1** For every $G$, $\chi(G) \leq 1 + \Delta(G)$.

**Proof:** Given any order $v_1, v_2, \ldots, v_n$, we color vertices one by one using $\Delta + 1$ colors. At time $i$, we assume $v_1, \ldots, v_{i-1}$ has been colored properly. Note that $v_i$ has at most $\Delta$ neighbors in $v_1, \ldots, v_{i-1}$. We can pickup a distinct color for $v_i$ other than those neighbors received. The resulted coloring is a proper coloring.

□.

**Theorem 2** (Wilf 1967) For every $G$, $\chi(G) \leq 1 + \lambda_{\text{max}}(G)$.

**Proof:** In the proof of the previous lemma, the graph $G$ is $k$-colorable if $v_i$ has at most $k - 1$ neighbors in the induced subgraph on $v_1, v_2, \ldots, v_i$ for all $i = 1, 2, \ldots, n$.

Since the order of the vertices can be arbitrary, we choose $v_n$ to be the vertex having the minimum degree. For $i = n, n - 1, \ldots, 1$, let $v_i$ be the vertex having minimum degree in the induced subgraph $G_i$ on $v_1, v_2, \ldots, v_i$. Note

$$\delta(G_i) \leq \mu_{\text{max}}(G_i) \leq \mu_{\text{max}}(G).$$

Thus, under this order, the previous greedy algorithm results a proper $k$-coloring for any $k \leq 1 + \mu_{\text{max}}(G)$. □
Remark: Brook’s theorem states that if $G$ is a simple connected graph other than the complete graph and odd cycles then

$$\chi(G) \leq \Delta(G).$$

It is unknown whether similar result can be proved using $\mu_{\text{max}}(G)$ instead.

Assume $\mu_1 > \mu_2 > \ldots > \mu_k$ are distinct eigenvalues of $A$. The $\phi(x) = \prod_{i=1}^{k}(x - \mu_k)$ is called the minimal polynomial of $A$. We have

$$\phi(A) = 0.$$ 

Any polynomial $f(x)$ with $f(A) = 0$ is divisible by $\phi(x)$.

For any pair of vertices $u, v$, the distance $d(u, v)$ is the shortest length of any $uv$-path. The diameter of graph $G$ is the maximum distance among all pairs of vertices which belongs to the same connected component.

**Theorem 3** The diameter of a graph is less than its number of distinct eigenvalues.

**Proof:** Without loss of generality, we can assume $G$ is connected. Let $k$ be the number of distinct eigenvalues. The minimum polynomial $\phi(x)$ has degree $k$. Since $\phi(A) = 0$, $A^k$ can be expressed as a linear combination of $I, A, \ldots, A^{k-1}$. Suppose the diameter of $G$ is greater than or equal to $k$. There exists a pair of vertices $u$ and $v$ satisfying $d(u, v) = k$. We have $(A^k)_{uv} \geq 1$ and $(A^i)_{uv} = 0$ for $i = 0, 1, 2, \ldots, A^{k-1}$. This is a contradiction to the fact $A^k$ is a linear combination of $I, A, \ldots, A^{k-1}$. □

This result is tight for the hypercube $Q_n$.  

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