Math 776 Graph Theory, Spring 2006
Lecture Note 2: Combinatorial probabilistic methods
Week 3–Week 5

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1 First moment method

Definition 1 *A discrete probability space is a finite or countable set Ω together with nonnegative weights on the elements that sum to 1. An event is a subset of A. The probability $P(A)$ of an event A is the sum of weights of the elements of A. Events A and B are independent if $P(A \cap B) = P(A)P(B)$.*

A random variable is a function $X : \Omega \to \mathbb{R}$. The expected value of X is

$$\mathrm{E}(X) = \sum_{a \in \Omega} X(a) w_a.$$

Here w_a is the weight of an element a in Ω.

Poropsition 1 *If $X = \sum X_i$, then $\mathrm{E}(X) = \sum \mathrm{E}(X_i)$.*

Let $R(s,t)$ denote the least number N such that if each edge of K_N is colored in either red or blue then there is a red clique K_s or a blue clique K_t.

Examples: $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(4,4) = 18$, $R(4,5) = 9$ $43 \leq R(5,5) \leq 49$,

The following theorem is often referred as the the birth of combinatorial probabilistic methods.

Theorem 1 (Erdős, 1947)

$$R(n,n) > \frac{1+o(1)}{e\sqrt{2}} n 2^{n/2}.$$

Proof: Color edges of K_N in two colors randomly and independently. For any set $S \subset V(K_N)$ of order n, the probability that S forms a monochromatic clique is

$$2^{1-\binom{n}{2}}.$$

Let X be the number of monochromatic cliques of order n. Then we have

$$\mathrm{E}(X) = \binom{N}{n} 2^{1-\binom{n}{2}}. \tag{1}$$

If $\mathrm{E}(X) < 1$, then there exist some colorings with $X = 0$. In another word, $R(n,n) > N$.

We will choose a maximum N so that $\mathrm{E}(X) < 1$. Stirling formula gives an approximation to $n!$.

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Note that

$$\begin{aligned} \binom{N}{n} 2^{1-\binom{n}{2}} &< \frac{N^n}{n!} 2^{1-\binom{n}{2}} \\ &\approx \frac{2}{\sqrt{2\pi n}} \left(\frac{Ne}{n 2^{(n-1)/2}}\right)^n. \end{aligned}$$

Choose $N = \lfloor \frac{n}{e} 2^{(n-1)/2} \rfloor$. Then, $\mathrm{E}(X) < 1$. □

Theorem 2

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Proof: Let $N = R(s-1,t) + R(s,t-1)$. For any edge-coloring of K_N, we pick any vertex v. A vertex u is called a red (or blue) neighbor of v if edge uv is a red (or blue) edge.

Since the degree of v is $N-1$. By pigeonhole principle, v has either $R(s-1,t)$ red neighbors or $R(s,t-1)$ blue neighbors.

For the first case, we consider the induced subgraphs on the set $R(v)$ of v's red neighbors. By the definition of $R(s-1,t)$, $R(v)$ contains either a red $s-1$-clique or a blue t-clique. Any red $s-1$-clique together with v forms a red s-clique. The argument is similar for the second case. Thus, $R(s,t) \geq N$. □

Corollary 1 *For all $s, t \geq 2$,*

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

Proof: We use induction on $s+t$. If $s = t = 2$, $R(2,2) = 2 = \binom{2}{1}$.

If one of s, t is 2, it is true since $R(s,2) = \binom{s}{2}$ and $R(2,t) = \binom{t}{2}$. Now we assume $s, t \geq 3$. We have

$$\begin{aligned} R(s,t) &\leq R(s-1,t) + R(s,t-1) \quad \text{by Theorem 2} \\ &\leq \binom{s-1+t-2}{s-2} + \binom{s+t-1-2}{s-1} \quad \text{by inductive hypothesis} \\ &= \binom{s+t-2}{s-1}. \end{aligned}$$

The inductive step is finished. □

Lemma 1 (Markov's Inequality) *If X takes only nonnegative values, then*

$$\Pr(X \geq t) \leq \frac{1}{t}\mathrm{E}(X).$$

In particular, if X is integer-valued, then $\mathrm{E}(X) \to 0$ *implies* $\Pr(X = 0) \to 1$.

Proof: We have

$$\begin{aligned}
\mathrm{E}(X) &= \sum_i x_i \Pr(X = x_i) \\
&= \sum_{0 \le x_i < t} x_i \Pr(X = x_i) + \sum_{x_i \ge t} x_i \Pr(X = x_i) \\
&\ge \sum_{x_i \ge t} x_i \Pr(X = x_i) \\
&\ge \sum_{x_i \ge t} t \Pr(X = x_i) \\
&= t \Pr(X \ge t).
\end{aligned}$$

Thus,

$$\Pr(X \ge t) \le \frac{1}{t}\mathrm{E}(X).$$

If X is integer-valued, we have

$$\Pr(X = 0) = 1 - \Pr(X \ge 1) \ge 1 - \mathrm{E}(X) \to 1.$$

The proof of this lemma is finished. □

Theorem 3 (Caro 1979, Wei 1981) *For any simple graph G, the independent number satisfies*

$$\alpha(G) \ge \sum_{v \in V(G)} \frac{1}{d_G(v) + 1}.$$

Proof: Number vertices of G from 1 to n in an arbitrary way. We get an directed graph D by orientating each edge from the smaller vertex to the larger vertex. Let S be the set of vertices whose indegree is 0 in D. Then S forms an independent set of G. For each v, $v \in S$ if v is smaller than its neighbors. The probability of $v \in S$ is exactly $\frac{1}{1+d_v}$. By linearity, we have

$$\mathrm{E}(|S|) = \sum_v \Pr(v \in S) = \sum_v \frac{1}{1 + d_v}.$$

In particular, there is an order of vertices so that the resulting set S is at least the expected value of $|S|$. Hence, $\alpha(G) \ge \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$. □

2 Deletion method

A random graph is a collection of graphs together with a probability distribution over it. Here are two classical Erdős-Rényi models of random graphs:
A uniform random graph model $G_{n,m}$: Given n and $m = m(n)$, let each graph with vertex set $[n]$ and m edges occur with probability $\binom{N}{m}^{-1}$, where $N = \binom{n}{2}$.

A edge-independent random graph model $G(n,p)$**:** Given n and $p = p(n)$, generate graph with vertex set $[n]$ by letting each pair be an edge with probability p.

Deletion Method: when a randomly generated object is close to having a desired property, a slight alternation may produce it.

Theorem 4 (Spencer)

$$R(n,n) \geq \frac{1+o(1)}{en} 2^{-n}.$$

Proof:

Color edges of K_N in two colors randomly and independently. For any set $S \subset V(K_N)$ of order n, the probability that S forms a monochromatic clique is

$$2^{1-\binom{n}{2}}.$$

Let X be the number of monochromatic cliques of order n. Then we have

$$\mathrm{E}(X) = \binom{N}{n} 2^{1-\binom{n}{2}}.$$

For any $\epsilon > 0$, if $\mathrm{E}(X) < \epsilon N$, then there exist some colorings with $X < \epsilon N$. We can destroy all monochromatic cliques of order n by delete one vertex from each mono-chromatic clique. We delete at most ϵN vertices. In another word, $R(n,n) > (1-\epsilon)N$.

We will choose a maximum N so that $\mathrm{E}(X) < \epsilon N$. Choose $N = \lfloor (1-\epsilon)\frac{n}{e} 2^{n/2} \rfloor$. We can show $\mathrm{E}(X) < \epsilon N$ is satisfied for n large enough. Because ϵ is arbitrary, we conclude that

$$R(n,n) \geq (1 - o(1)) \frac{n}{e} 2^{n/2}.$$

The proof of this lemma is finished. □

Theorem 5 (Erdős, 1959) *Given $m \geq 3$ and $g \geq 3$, there exists a graph with girth at least g and chromatic number at least m.*

Proof: Consider a random graph G on n vertices. For any pair of vertices, an edge is added with probability p independently. Here we choose $p = n^{t-1}$ with a positive constant $t < \frac{1}{g}$.

Let X_i be the number of cycles of length i and $X = \sum_{i=3}^{g-1} X_i$. We have

$$\begin{aligned} \mathrm{E}(X) &= \sum_{i=3}^{g-1} \mathrm{E}(X_i) \\ &= \sum_{i=3}^{g-1} P(n,i) p^i/(2i) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=3}^{g-1} (np)^i/(2i) \\ &= \sum_{i=3}^{g-1} n^{ti}/(2i) \\ &= O(n^{tg}). \end{aligned}$$

Since $tg < 1$, $\mathrm{E}(X) = o(n)$. For n sufficient large, we have $\mathrm{E}(X) < \frac{n}{4}$.

By Markov's inequality, we have

$$\Pr(X \geq n/2) \leq \frac{2}{n}\mathrm{E}(X) \leq \frac{1}{2}.$$

For the independent number of G, we have

$$\Pr(\alpha(G) \geq r) \leq \binom{n}{r}(1-p)^{\binom{n}{2}} < n^r e^{-pr(r-1)/2}.$$

Choose $r = \lceil \frac{3}{p} \ln n \rceil$. We have $\Pr(X \geq n/2) \to 0$ and $\Pr(\alpha(G) \geq r) \to 0$.

With positive probability, G has at most $\frac{n}{2}$ cycles with length at most $g-1$ and with independent number at most r. For each small cycle, delete one vertex from it. Let G' be the remaining graph. Then we have

$$\begin{aligned} n(G') &\geq \frac{n}{2} \\ \alpha(G') &\leq r. \end{aligned}$$

The graph G' has girth at least g. We also have

$$\chi(G') \geq \frac{n(G')}{\alpha(G')} \geq \frac{n^t}{3 \ln n}.$$

Since $\frac{n^t}{3 \ln n} \to \infty$, it is large than any given number m for n sufficient large. □

3 Lovász local lemma

Definition 2 *Given events A and B, the conditional probability of A given B is defined as*

$$\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)}.$$

If A and B are independent then, $\Pr(A|B) = \Pr(A)$.

$$\Pr(A|BC) = \frac{\Pr(AB|C)}{\Pr(B|C)}.$$

$$\Pr(A_1 A_2 \cdots A_n) = \prod_{i=1}^{n} \Pr(A_i | A_{i+1} \cdots A_n).$$

In the scenario of showing the existence of certain good event, Lovász local lemma is a very powerful tool. Here is the symmetric version.

Lemma 2 (Lovász local lemma) *Let $A_1, A_2, \ldots, A_n$ be events satisfying*

1. *For $1 \le i \le n$, each event A_i is mutually independent of all but at most d ($d \ge 1$.) other events.*
2. *For $1 \le i \le n$, $\Pr(A_i) \le p$.*
3. *$4dp < 1$.*

Then

$$\Pr(\cap_{i=1}^{n} \bar{A}_i) > 0.$$

Proof: We show by induction on s that if $|S| \le s$, then for any $i \notin S$

$$\Pr(A_i | \cap_{j \in S} \bar{A}_j) \le 2p.$$

For $S = \emptyset$ this is true by assumption. Renumber for convenience so that $i = n$, $S = \{1, \ldots, s\}$ and A_n is mutually independent of events $\{A_x\}_{x \ge s}$. We have

$$\Pr(A_n | \bar{A}_1 \cdots \bar{A}_s) = \frac{\Pr(A_n \bar{A}_1 \cdots \bar{A}_d | \bar{A}_{d+1} \cdots \bar{A}_s)}{\Pr(\bar{A}_1 \cdots \bar{A}_d | \bar{A}_{d+1} \cdots \bar{A}_s)}.$$

The numerator can be bounded as follows

$$\begin{aligned} \Pr(A_n \bar{A}_1 \cdots \bar{A}_d | \bar{A}_{d+1} \cdots \bar{A}_s) &\le \Pr(A_n | \bar{A}_{d+1} \cdots \bar{A}_s) \\ &\le \Pr(A_n) \\ &\le p. \end{aligned}$$

We bound the denominator

$$\begin{aligned} \Pr(\bar{A}_1 \cdots \bar{A}_d | \bar{A}_{d+1} \cdots \bar{A}_s) &\ge 1 - \sum_{i=1}^{d} \Pr(A_i | \bar{A}_{d+1} \cdots \bar{A}_s) \\ &\ge 1 - \sum_{i=1}^{d} 2p \\ &= 1 - 2dp \\ &\ge \frac{1}{2}. \end{aligned}$$

Hence we have the quotient

$$\Pr(A_n | \bar{A}_1 \cdots \bar{A}_s) \le 2p.$$

The induction is finished. Finally,

$$\Pr(\bar{A}_1 \cdots \bar{A}_n) = \prod_{i=1}^{n} \Pr(\bar{A}_i | \bar{A}_1 \cdots \bar{A}_{i-1}) \ge \prod_{i=1}^{n} (1 - 2p) > 0.$$

The proof of this lemma is finished.

Theorem 6

$$R(n,n) \geq \frac{\sqrt{2}+o(1)}{en}2^{-n}.$$

Proof:

Color edges of K_N in two colors randomly and independently. For any set $S \subset V(K_N)$ of order n, let A_S be the bad event that S forms a monochromatic clique.

$$Pr(A_S) = 2^{1-\binom{n}{2}}.$$

A_S and A_T are independent if $|S \cap T| \leq 1$. Let $d = \sum_{k=2}^{n-1} \binom{n}{k}\binom{N-n-k}{n-k} < \frac{\binom{n}{2}N^{n-2}}{(n-2)!}$. By Lovász local lemma, $R(n,n) \geq N$ if

$$4\frac{\binom{n}{2}N^{n-2}}{(n-2)!}2^{1-\binom{n}{2}} < 1.$$

We will choose a maximum N satisfying above equation. A similar estimation show $N = \lfloor (1-o(1))\frac{\sqrt{2}n}{e}2^{n/2}\rfloor$.The proof of this lemma is finished. □

Definition 3 *A graph G on vertices $[n]$ is called a* dependency graph *for events $A_1, \cdots, A_n$ if for all i A_i is mutually independent of all A_j with $\{i,j\} \notin G$.*

Lemma 3 (Lovász local lemma) *(General case). Let $A_1, \ldots, A_n$ be events with dependency graph G, Assume there exist $x_1, \ldots, x_n \in [0,1)$ with*

$$\Pr(A_i) < x_i \prod_{ij \in E(G)} (1-x_j)$$

for all i. Then

$$\Pr(\cap_{i=1}^n A_i) < \prod_{i=1}^n (1-x_i) > 0.$$

Proof: We show by induction on s that if $|S| \leq s$, then for any $i \notin S$

$$\Pr(A_i | \cap_{j\in S} \bar{A}_j) \leq x_i.$$

For $S = \emptyset$, $\Pr(A_i) < x_i \prod_{ij\in E(G)}(1-x_j) < x_i$.

Renumber for convenience so that $i = n$, $S = \{1, \ldots, s\}$ and among $x \in S$, $nx \in E(G)$ for $x = 1, 2, \ldots, d$. We have

$$\Pr(A_n|\bar{A}_1 \cdots \bar{A}_s) = \frac{\Pr(A_n\bar{A}_1 \cdots \bar{A}_d|\bar{A}_{d+1}\cdots\bar{A}_s)}{\Pr(\bar{A}_1 \cdots \bar{A}_d|\bar{A}_{d+1}\cdots\bar{A}_s)}.$$

The numerator can be bounded as follows

$$\begin{aligned}\Pr(A_n\bar{A}_1 \cdots \bar{A}_d|\bar{A}_{d+1}\cdots\bar{A}_s) &\leq \Pr(A_n|\bar{A}_{d+1}\cdots\bar{A}_s)\\ &\leq \Pr(A_n)\end{aligned}$$

We bound the denominator

$$\begin{aligned}\Pr(\bar{A}_1 \cdots \bar{A}_d | \bar{A}_{d+1} \cdots \bar{A}_s) &\geq \prod_{i=1}^{d} \Pr(\bar{A}_i | \bar{A}_{i+1} \cdots \bar{A}_s) \\ &\geq \sum_{i=1}^{d} (1 - x_i).\end{aligned}$$

Hence we have the quotient

$$\Pr(A_n | \bar{A}_1 \cdots \bar{A}_s) \leq \frac{\Pr(A_n)}{\prod_{i=1}^{d}(1 - x_i)} < x_i.$$

The induction is finished. Finally,

$$\Pr(\bar{A}_1 \cdots \bar{A}_n) = \prod_{i=1}^{n} \Pr(\bar{A}_i | \bar{A}_1 \cdots \bar{A}_{i-1}) \geq \prod_{i=1}^{n} (1 - x_i).$$

The proof of this lemma is finished.

Corollary 2 *In the symmetric version of Lovász local lemma, the condition* $4dp < 1$ *can be replaced by* $(d + 1)ep < 1$.

Proof: We will apply Lemma 3. By symmetry, we choose all $x_i = x$, for some x. It is enough to show that for each event A_i, there is an x so that

$$\Pr(A_i) \leq x(1 - x)^d.$$

In another word, we need a sufficient condition that $p = x(1-x)^d$ has a positive solution $x < 1$.

Note that $f(x) = x(1 - x)^d$ reaches the maximum at $x = \frac{1}{d+1}$. Since $f(\frac{1}{d+1}) > \frac{1}{(d+1)e}$, $p = f(x)$ has a solution if $p \leq \frac{1}{(d+1)e}$. □

4 Comparison of three methods

We will use Ramsey number $R(3, n)$ to compare three methods —first moment method, deletion method, and Lovász loca lemma.

First we use first moment method. Consider that a random graph $G(N, p)$. The bad structures are triangles and independent set of order n.

The expected number of triangles is

$$\binom{N}{3} p^3.$$

The expected number of independent set of size n is

$$\binom{N}{n} (1 - p)^{\binom{n}{2}}.$$

Thus $R(3,n) > N$, if

$$\binom{N}{3}p^3 + \binom{N}{n}(1-p)^{\binom{n}{2}} < 1. \tag{2}$$

Note that

$$\binom{N}{n}(1-p)^{\binom{n}{2}} < \frac{N^n}{n!}e^{-p\binom{n}{2}} < \left(\frac{eN}{ne^{p(n-1)/2}}\right)^n.$$

In this case, the first moment method does not give any substantially better bound than the trivial bound n.

Deletion method: If the expected number of bad structures is less than $\frac{N}{2}$, we can destroy all bad structures by deleting one vertex from each bad structure. The remaining subgraph has at least $\frac{N}{2}$ vertices and has no bad structures. In particular, $R(3,n) > N/2$ if

$$\binom{N}{3}p^3 + \binom{N}{n}(1-p)^{\binom{n}{2}} < \frac{N}{2}. \tag{3}$$

Choose $N = \frac{n^{1.5}}{\log^{1.5} n}$ and $p = N^{-2/3} = \frac{\ln n}{n}$. We have

$$\begin{aligned}
\binom{N}{3}p^3 + \binom{N}{n}(1-p)^{\binom{n}{2}} &< \frac{N}{6} + \frac{N^n}{n!}e^{-pn(n-1)/2} \\
&< \frac{N}{6} + e^{n(\ln N - \log n + 1 - p(n-1)/2)} \\
&< \frac{N}{6} + e^{n(-\ln \ln^{1.5} n + 1 + \ln n/(2n))} \\
&< \frac{N}{2}
\end{aligned}$$

Thus, $R(3,n) \geq \frac{n^{1.5}}{\log^{1.5} n}$.

Lovász local lemma: We consider the random graph $G(N,P)$. For any 3-set S, let A_S be "S is a triangle". For any n-set T, let B_T be "T is an independent set." Then

$$\begin{aligned}
\Pr(A_S) &= p^3 \\
\Pr(B_T) &= (1-p)^{\binom{n}{2}} \approx e^{-pn^2/2}.
\end{aligned}$$

Let S, S' be adjacent in the dependency graph if they have a common edge; the same for S, T or T, T'. Each S is adjacent to $3(n-3) \approx 3n$ of other S' and to less than $3\binom{N}{n-2}$ of T. Each T is adjacent to $\binom{n}{2}N < n^2N/2$ of S and to at most $\binom{N}{n}$ of other T'. Lovász local lemma takes the following form:

If there exist p, x, y with

$$\begin{aligned}
p^3 &< x(1-x)^{3N}(1-y)^{3\binom{N}{n-2}} \\
e^{-pn^2/2} &< y(1-x)^{n^2N/2}(1-y)^{\binom{N}{n}},
\end{aligned}$$

then $R(3,n) > N$.

Choose $y = \frac{1}{\binom{N}{n}+1}$ to maximize $y(1-y)^{\binom{N}{n}}$. We can simplify the system as follows

$$
\begin{aligned}
p^3 &< (1+o(1))x(1-x)^{3N} \quad (4)\\
e^{-pn^2/2} &< (1-x)^{n^2N/2}\frac{1}{e\binom{N}{n}}. \quad (5)
\end{aligned}
$$

Choose $x = (1+\epsilon)p^3$ so that equation 4 is satisfied. Take logarithm of equation 5. We have

$$-pn^2/2 < -(1+\epsilon)p^3n^2N/2 - 1 - n\ln(eN/n).$$

Choose $p = \frac{(2+2\epsilon)\ln n}{n}$ and $N = \frac{2\epsilon n^2}{(1+\epsilon)^4\ln^2 n}$. The above inequality is satisfied. Choose $\epsilon = \frac{1}{3}$ to maximize N. We have

$$R(3,n) \geq (\frac{27}{128} - o(1))\frac{n^2}{\ln^2 n}.$$

The best lower bound is due to Kim:

$$R(3,n) \geq c_1\frac{n^2}{\ln n}.$$

It matches the best known upper bound up to a constant factor.

This lecture is partly based on [1] and [2] This note is only for your convenience.

References

[1] D. B. West, *Introduction to Graph Theory*, second edition, Prentice-Hall, 2001.

[2] J. Spencer, *Ten Lectures on the Probabilistic Method* SIAM, Philadephia, 1987.