Math 777 Graph Theory, Spring, 2006
Lecture Note 1
Planar graphs
Week 1 — Weak 2

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1 Planar graphs

Definition 1 *A* **drawing** *of a graph G is a function f defined on $V(G) \cup E(G)$ that assigns each vertex v a point $f(v)$ in the plane and assigns each edge a u, v-curve. A point in $f(e) \cap f(e')$ that is not a common endpoints is a* **crossing**.

Definition 2 *A graph is* planar *if it has a drawing without crossings. Such a drawing is a* planar embedding *of G. A* plane graph *is a particular planar embedding of a planar graph.*

A planar embedding of a graph cuts the plane into pieces. Each piece is called a *face* of the plane graph.

Definition 3 *The* dual graph *G^* of a plane graph G is a plane graph whose vertices correspond to the faces of G. The edge of G^* correspond to the edges of G as follows: if e is an edge of G with face X on one side and face Y on the other side, then the endpoints of the dual edge $e^* \in E(G^*)$ are the vertices x, y of G^* that represent the faces X, Y of G.*

Poropsition 1 *If $l(F_i)$ denotes the length of face F_i in a plane graph G, then $\sum_i l(F_i) = 2e(G)$.*

Theorem 1 (Euler's formula (1978)) *If a connected plane graph G has exactly n vertices, e edges, and f faces, then*

$$n - e + f = 2.$$

Proof: We use induction on the number of edges in G.

If $e(G) = n - 1$ and G is connected, then G is a tree. We have $f = 1$, $e = n - 1$. Thus $n - e + f = 2$ holds.

If $e(G) \geq n$ and G is connected, G contains a cycle C. Choose any edge e on C. Let $G' = G - e$. Then G' is connected and $e(G') \geq n - 1$. By inductive hypothesis, for G', we have

$$n' - e' + f' = 2.$$

Here $n' = n$, $e' = e - 1$, and $f' = f - 1$. Thus

$$n - e + f = 2.$$

Theorem 2 *If G is a simple graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If G is also triangle-free, then $e(G) \leq 2n(G) - 4$.*

Proof: It suffices to consider connected graphs; otherwise we could add edges. Delete any vertex with degree 1 if necessary.

Every face boundary in a simple graph contains at least three edges. Let $\{f_i\}$ be the list of face lengths. Then $2e = \sum_i f_i \geq 3f$. Hence $f \leq \frac{2}{3}e$. Substitute it into Euler's formula. We have

$$n - e + \frac{2}{3}e \geq 2.$$

Thus, $e \leq 3n - 6$.

When G is triangle-free, the faces have length at least 4. In this case $2e = \sum f_i \geq 4f$, and we obtain $e \leq 2n - 4$. $\square$

Example 1: K_5 is a non-planar graph since $e = 10 > 9 = 3n - 6$.

Example 2: $K_{3,3}$ is a non-planar graph since $e = 9 > 8 = 2n - 4$.

Poropsition 2 *If a graph G has subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is nonplanar.*

Theorem 3 (Kuratowski, 1930) *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Definition 4 *A* Kuratowski subgraph *of G is a subgraph of G that is a subdivision of K_5 or $K_{3,3}$. A* minimal nonplanar graph *is a nonplanar graph such that every proper subgraph is planar.*

Theorem 4 (Tutte, 1960) *If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then G has a convex embedding in the plane with no three vertices on a line.*

Proof: (Thomassen, 1980) We use induction on $n(G)$.

Basic step: $n(G) = 4$. The only 3 connected graph is K_4, which has such a planar embedding.

Inductive step: $n(G) \geq 5$. Let e be an edge such that $G \cdot e$ is 3-connnected. $G \cdot e$ has no Kuratowski subgraph. By inductive hypothesis, $G \cdot e$ has a convex embedding. Let z be the new vertex of $G \cdot e$ from edge xy. All faces contains z form a shape of a wheel. Let C be the boundary cycle. Let $x_1, x_2, \ldots, x_k$ be the neighbors of x in a cyclic oder on C. If all neighbors of y lie in the portion of C from x_i to x_{i+1}, then we obtain an embedding of G by putting x at z in H and putting y at a point close to z in the wedge formed by xx_i and xx_{i+1}.

If this does not occur, then one of the following case happens.

Case 1: y shares three neighbors u, v, w with x. In this case, the cycle C together with xy and edges from $\{x, y\}$ to $\{u, v, w\}$ form a subdivision of K_5.

Case 2: y has neighbors u, v that alternate on C with neighbors x_i, x_{i+1} of x. In this case, we get a subdivision of $K_{3,3}$.

Since G has no Kuratowski subgraph, the inductive step can always be carried out. $\square$.

Lemma 1 *If G has no Kruatowski subgraph, then $G \cdot e$ has no Kuratowski subgraph*

Proof: We prove the contrapositive: If $G \cdot e$ contains a Kuratowski subgraph, then so does G.

Let z be the vertex of $G \cdot e$ obtained by contracting $e = xy$. Let H be be a Kuratowski subgraph in $G \cdot e$. If $z \notin V(H)$, then $H \subset G$. If $z \in V(H)$ and $d_H(z) = 2$, then we obtain a Kuratowski subgraph of G form H by replacing z with x or y or with the edge xy.

If $d_H(z) = 3$, write $N_H(z) = A \cup B$, where the vertices in A are adjacent to x while the vertices in B are adjacency to y. If one of $|A|$ and $|B|$ equals to 3, we obtain a Kuratowski subgraph of G form H by replacing z with x or y. It suffices to consider the case $|A| = 2$ and $|B| = 1$. we obtain a Kuratowski subgraph of G form H by replacing z with the edge xy.

If $d_H(z) = 4$, then H is the subdivision of K_5. Write $d_H(z) = A \cup B$, where the vertices in A are adjacent to x while the vertices in B are adjacency to y. If one of $|A|$ or $|B|$ is at least 3, then we obtain a subdivision of K_5 from H by replacing z with x or y or with the edge xy. If $|A| = |B| = 2$, we obtain a subdivision of $K_{3,3}$ instead. $\square$

Lemma 2 (Thomassen, 1980) *Every 3-connected graph G with at least five vertices has an edge e such that $G \cdot e$ is 3-connected.*

Proof: We use contradiction. Suppose G has no edge whose contraction yields a 3-connected graph. For every edge xy, there is a mate z so that x, y, z is separating set. Choose the edge xy and their mate z so that $G - \{x, y, z\}$ has a component H with the largest order. Let H' be another component of $G - \{x, y, z\}$. Each x, y, z has a neighbor in each of H, H'. Let u be a neighbor of z in H', and let v be the mate of uz.

Since z, v is not a separating set of G, v can not be in H. Therefore $G_{V(H) \cup \{x,y\}} - v$ is contained in a component of $G - \{z, u, v\}$ that has more vertices than H, which contradicts the choice of x, y, z. $\square$

Proof of Theorem 3: We first prove the theorem for all 2-connected graphs. Let G be a 2-connected graphs containing no Kuratowski subgraph. We use induction on $n(G)$. It holds for any graphs with at most 4 vertices.

If G is 3-connected, then G has a convex planar drawing and we are done. Thus, G has a 2-separator $\{x, y\}$. If $xy \notin E(G)$, consider $G + xy$ instead. Notice that $G + xy$ is 2-connected and contains no Kuratowski subgraph. Without loss of generality, we can assume $xy \in E(G)$.

A xy-lobe of G is a connected component of $G - x - y$ together with edges to x, y and edge xy itself. Suppose G has x, y-lobes $G_1, G_2, \ldots, G_r$. By inductive hypothesis, each G_i admit a planar drawing. We can draw G_i so that xy is on the boundary of the outer face and other vertices in a thin slice of any specified xy-curve. The union of drawings of these G_i gives a planar drawing of G.

Now we prove the theorem for any connected graph G containing no Kuratowski subgraph. We use induction on $n(G)$. If G is 2-connected, we have done already. Otherwise, there exists acut vertex of G (say v). Let $G_1, G_2, \ldots, G_r$ be v-lobes of G. By inductive hypothesis, each G_i admits a planar drawing. We can assume v is in the boundary of the outer face. Deforming the drawing of G_i into areas bounded by thin angles of v. We get a planar drawing of G. $\square$

Definition 5 *A graph H is a* minor *of a graph G if a copy of H can be obtained from G by deleting and/or contracting edges of G.*

Here is another characterization of planar graphs.

Theorem 5 (Wagner 1937) *A graph is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

Definition 6 *The surface S_γ of* genus γ *is obtained by adding γ handles to a sphere.*

We can consider the similar problem: "which graphs can be embedded on S_γ without crossing?"

For $\gamma = 0$, S_γ is the sphere. The graph G embeddable on S_γ if and only if G is planar.

For $\gamma = 1$, S_γ is the torus. Similarly, we can consider which graphs can be embedded into a torus without crossings.

For example, K_5, $K_{3,3}$, K_7 are embeddable on torus.

On any surface, embeddability is preserved by deleting or contracting an edge. Thus, every surface has a list of "minor-minimal" obstructions to embeddability. Wagner's theorem states that the list for the sphere is $\{K_{3,3}, K_5\}$. Every nonplanar graph has one of these as a minor.

More than 800 minimal forbidden minors are known for the torus. For each surface, the list is finite. It follows by the following deep theorem.

Theorem 6 (Robertson-Seymour 1985) *In any infinite list of graphs, some graph is a minor of another.*

2 Four color theorem

History of Four Color Theorem:

- In 1878, Cayley announced the problem to London Mathematical Society.
- In 1879, Kempe published a "solution".
- In 1890, Heawood proved the Five Color Theorem.

- From 1976 to 1983, Appel, Haken, and Koch proved the Four Color Theorem.
- In 1997, Robertson, Sanders, Seymour, and Thomas simplify the proof.

Theorem 7 (Heawood, 1890) *Every planar graph is 5-colorable.*

Proof: We use induction on $n(G)$.

Initial step: $n(G) \leq 4$. All graphs with at most 4 vertices are 5-colorable.

Inductive step: $n(G) \geq 5$. Since $e(G) \leq 3n(G) - 6$, there is a vertex v of degree at most 5. By inductive hypothesis, $G - v$ is 5-colorable. Let $f\colon V(G - v) \to [5]$ be a proper 5-coloring of $G - v$. If $deg(v) \leq 4$ or f uses only at most 4 colors for neighbors of v, then we can extend f to obtain a proper 5-coloring of G.

Without loss of generality, we can assume v has 5 neighbors v_1, v_2, v_3, v_4, v_5 in clockwise order around v and $f(v_i)$'s are distinct. After renaming the colors, we assume $f(v_i) = i$, for $i = 1, 2, 3, 4, 5$. Let $G_{i,j}$ denote the subgraph $G - v$ induced by the vertices of colors i and j. Switching the two colors on any component of $G_{i,j}$ yields another proper 5-coloring of $G - v$. If the component of $G_{i,j}$ containing v_i does not contain v_j, then we can switch the colors on it to remove color i from $N(v)$. Now giving v to color i yields a proper 5-coloring of G. It suffices to show such $G_{i,j}$ exists.

Otherwise, for each i, j, an alternative i, j-path connects v_i and v_j. Let $P_{1,3}$ and $P_{2,4}$ be such two paths. $P_{1,3}$ and v forms a closed cycle, which cuts the plane into two connected regions. Note that v_2 and v_4 in the different regions. $P_{2,4}$ and $P_{1,3}$ must intersect at some vertex u because of the planarity. Then $f(u) \in \{1, 3\} \cap \{2, 4\} = \emptyset$. Contradiction. $\square$

We state the well-known Four Color theorem as follows. The proof requires substantial computing hours and is omitted in the class.

Theorem 8 (Appel, Haken, and Koch) *Every planar graph is 4-colorable.*

3 Crossing numbers

Definition 7 *The crossing number $\nu(G)$ of a graph G is the minimum number of crossings in a drawing of G in a plane.*

Trivial fact: Let G be a n-vertex graph with m edges. If k is the maximum number of edges in a planar subgraph of G, then $\nu(G) \geq m - k$.

In particular, if G is a simple graph with at least 3 vertices, then

$$\nu(G) \geq e(G) - 3n + 6.$$

If G is also triangle-free, then we have

$$\nu(G) \geq e(G) - 2n + 4.$$

Theorem 9 *Zarankiewicz(1954)*

$$\nu(K_{m,n}) \leq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor.$$

This bound is conjectured to be optimal by Guy.
Kleitman[1970] proved it for $\min\{m, n\} \leq 6$.

Theorem 10 (R. Guy,1972)

$$\frac{1}{80} n^4 + O(n^3) \leq \nu(K_n) \leq \frac{1}{64} n^4 + O(n^3).$$

Proof: The upper bound is obtained by a drawing of K_n on a can. It suffices to define a drawing of K_{2k}. (For odd n, draw K_n as a subgraph of K_{n+1}.) Place k vertices on the top rim of the can and k vertices on the bottom rim, drawing chords on the top and bottom for these k-cliques. The edges from top to bottom are drawn to wind around the can as little as possible in the same direction. We have

$$\nu(K_n) \leq 2\binom{k}{4} + k\binom{k}{3} = \frac{1}{64} n^4 + O(n^3).$$

The lower bound uses Kleitman's result that

$$\nu(K_{6,n-6}) = 6 \lfloor \frac{n-6}{2} \rfloor \lfloor \frac{n-7}{2} \rfloor.$$

Since K_n contains $\binom{n}{6}$ copies of $K_{6,n-6}$ and each pairs of crossed edges can be in at most $4\binom{n-4}{4}$ copies of $K_{6,n-6}$. Thus,

$$\begin{aligned} \nu(K_n) &\geq \frac{\binom{n}{6}\nu(K_{6,n-6})}{4\binom{n-4}{4}} \\ &= \frac{\binom{n}{6} 6 \lfloor \frac{n-6}{2} \rfloor \lfloor \frac{n-7}{2} \rfloor}{4\binom{n-4}{4}} \\ &= \frac{1}{80} n^4 + O(n^3). \end{aligned}$$

Theorem 11 (Ajtai-Chvátal-Newborn-Szemeédi 1982, Leighto 1983) *Let G be a simple graph. If $e(G) \geq 4n(G)$, then*

$$\nu(G) \geq \frac{1}{64} \frac{e(G)^3}{n(G)^2}.$$

Proof: For a simple graph G with at least three vertices, we have

$$\nu(G) \geq e(G) - 3n(G) + 6.$$

For any simple graph G, we have

$$\nu(G) \geq e(G) - 3n(G).$$

Consider a random induced subgraph G_p. A vertex of G is selected in G_p with probability p independently. For G_p, we have

$$\nu(G_p) \geq e(G_p) - 3n(G_p).$$

By taking the expectation, we have

$$\begin{aligned} \mathrm{E}(\nu(G_p)) &= p^4\nu(G) \\ \mathrm{E}(e(G_p)) &= p^2 e(G) \\ \mathrm{E}(n(G_p) &= pn(G) \end{aligned}$$

We have

$$p^4\nu(G) \geq p^2 e(G) - pn(G).$$

Choose $p = \frac{4n(G)}{e(G)} \leq 1$. We obtained

$$\nu(G) \geq \frac{1}{64}\frac{e(G)^3}{n(G)^2}.$$

Remark: Choose $p = \frac{9n(G)}{2e(G)}$. We conclude that $\nu(G) \geq \frac{4}{243}\frac{e(G)^3}{n(G)^2}$ for all simple graph G with $e(G) \geq 4.5n(G)$. The best know bound is due to Pach and Tóth.

Theorem 12 (Pach, Tóth, 1997) *Let G be a simple graph. If $e(G) \geq 7.5n(G)$, then*

$$\nu(G) \geq \frac{1}{33.75}\frac{e(G)^3}{n(G)^2}.$$

Remark: They show that for any simple graph G with at least 3 vertices

$$\nu(G) \geq 5e(G) - 25v(G) + 50.$$

Lower bound: Consider $G = sK_r$. We have $n = sr$, $m = s\binom{r}{2}$, and $\nu(G) \leq s\frac{1}{64}r^4$. Therefore, $\nu(G) \leq \frac{1}{8}\frac{m^3}{n^2}$.

Theorem 13 (Szemerédi-Trotter theorem) *Given m points and n lines in the Euclidean plane, the number of incidences between them is at most*

$$cm^{2/3}n^{2/3} + m + n.$$

Proof: WLOG, we assume that every line and every point is involved in at least one incidence, and that $n \geq m$, by duality.

Theorem 14 (Spencer-Szemerédi-Trotter, 1984) *There are at most $4n^{4/3}$ pairs of points at distance 1 among a set of n points in the plane.*

Proof by Székely, 1997 By moving points or pairs of points without reducing the number of pairs at distance 1 among a set of n points in the plane. We can assume that for each point there is at least two points from distance 1.

Let P be an optimal n-point configuration, with q unit distance pairs. We define a graph G from P. The vertex set is the set of points. The edge set are arcs partitioned by these points. G is a loopless graph with $2q$ edges. G may have edge of multiplicity 2 but no larger multiplicity. We delete one copy of each duplicated edge to obtain a simple graph G' with at least q edges. We can assume $q \geq 4n$. Since every pairs of cycles cross at most twice. Thus,

$$2\binom{n}{2} \geq \nu(G) \geq \frac{q^3}{64n^2}.$$

We have $q \leq 4n^3 4$. $\square$

This lecture is based on the chapter 7 of our textbook [1]. This note is only for your convenience.

References

[1] D. B. West, Introduction to Graph Theory, second edition, 2001.