Math 777 Homework 3 Solution

1. Use the first moment method (i.e. using the expected value) to show

$$R(4,n) \geq c\left(\frac{n}{\log n}\right)^{3/2}$$

for some positive constant c.

Solution: Consider random graph $G(N,p)$. The expected number of K_4 is

$$\binom{N}{4}p^6.$$

The expected number of independent set of size n is

$$\binom{N}{n}(1-p)^{\binom{n}{2}}.$$

Thus, $R(4,n) \geq N$ if

$$\binom{N}{4}p^6 + \binom{N}{n}(1-p)^{\binom{n}{2}} < 1. \tag{1}$$

Choose $N = c\left(\frac{n}{\log n}\right)^{3/2}$ and $p = \frac{\ln n}{n}$. We have

$$\begin{aligned}
\binom{N}{n}(1-p)^{\binom{n}{2}} &< \frac{N^n}{n!}e^{-pn(n-1)/2} \\
&< \left(\frac{eN}{n}e^{-p(n-1)/2}\right)^n \\
&= \left(\frac{ec\frac{n^{3/2}}{\ln^{3/2} n}}{n}e^{-\frac{1}{2}\ln n + \frac{1}{2n}\ln n}\right)^n \\
&= \left(\frac{ec}{\ln^{3/2} n}e^{\frac{\ln n}{2n}}\right)^n \\
&= o(1)
\end{aligned}$$

and

$$\begin{aligned}
\binom{N}{4}p^6 &< \frac{N^4p^6}{24} \\
&= \frac{c^4}{24}.
\end{aligned}$$

Choose $c < \sqrt[4]{24}$. Equation (2) is satisfied. $\square$

2. Use the deletion method to show

$$R(4,n) \geq c\left(\frac{n}{\log n}\right)^2$$

for some positive constant c.

Solution: Consider random graph $G(N,p)$. Suppose that a graph has at most $N/2$ bad configurations. We can destroy all bad configurations by deleting one vertex from each bad configuration. The remaining graph has at least $\frac{N}{2}$ vertices and has no bad configuration. Deletion method implies $R(4,n) \geq N/2$ if

$$\binom{N}{4}p^6 + \binom{N}{n}(1-p)^{\binom{n}{2}} < \frac{N}{2}. \tag{2}$$

Choose $N = c\left(\frac{n}{\log n}\right)^{5/2}$ and $p = \frac{2\ln n}{n}$. We have

$$\begin{aligned}
\binom{N}{n}(1-p)^{\binom{n}{2}} &< \frac{N^n}{n!}e^{-pn(n-1)/2} \\
&< \left(\frac{eN}{n}e^{-p(n-1)/2}\right)^n \\
&= \left(\frac{ec\frac{n^2}{\ln^2 n}}{n}e^{-\ln n + \frac{1}{n}\ln n}\right)^n \\
&= \left(\frac{ec}{\ln^2 n}e^{\frac{2\ln n}{n}}\right)^n \\
&= o(1);
\end{aligned}$$

and

$$\begin{aligned}
\binom{N}{4}p^6 &< \frac{N^3p^6}{12}\frac{N}{2} \\
&= \frac{2c^3}{3}\frac{N}{2}.
\end{aligned}$$

Choose $c < \sqrt[3]{1.5}$. Equation (2) is satisfied. □

3. Use Lovász local lemma to show

$$R(4,n) \geq c\left(\frac{n}{\log n}\right)^{5/2}$$

for some positive constant c.

Solution: Consider random graph $G(N,p)$. For any 4-set S, the probability that S forms a clique is p^6. For any n-set T, the probability that T is an independent set is $(1-p)^{\binom{n}{2}}$. For any 4-set S, there is at most

$6\binom{N}{2} < 3n^2$ other 4-set S' with $|S \cap S'| \geq 2$; there is at most $6\binom{N}{n-2}$ n-set T' with $|S \cap T'| \geq 2$. For any n-set T, there is at most $\binom{n}{2}\binom{N}{2} < N^2n^2/4$ other 4-set S' with $|T \cap S'| \geq 2$; there is at most $\binom{n}{2}\binom{N}{n-2}$ n-set T' with $|T \cap T'| \geq 2$. By Lovász local lemma, $R(4,n) \geq N$ if there is an $x, y \in (0,1)$ satisfying

$$\begin{cases} p^6 < x(1-x)^{3n^2}(1-y)^{6\binom{N}{n-2}} \\ (1-P)^{\binom{n}{2}} < y(1-x)^{N^2n^2/4}(1-y)^{\binom{n}{2}\binom{N}{n-2}} \end{cases}$$

Choose $y = \frac{1}{\binom{n}{2}\binom{N}{n-2}}$. The above equation can be simplified as follows.

$$\begin{cases} p^6 < x(1-x)^{3n^2}(1-o(1)) \\ (1-p)^{\binom{n}{2}} < (1-x)^{N^2n^2/4}\frac{1}{e\binom{n}{2}\binom{N}{n-2}} \end{cases}$$

Choose $x = (1+o(1))p^6$ and $p = (1+o(1))\frac{\ln n}{n}$ so that the first equation is satisfied. Take the nature logrithm at the both sides of the second equation. It suffices to require

$$\binom{n}{2}p > N^2n^2/4x + \ln(e\binom{n}{2}\binom{N}{n-2}).$$

Since $e\binom{n}{2}\binom{N}{n-2} < \binom{N}{n} < (\frac{Ne}{n})^n$, we have

$$\binom{n}{2}p > N^2n^2/4p^6 + n\ln N.$$

Choose $N = c\left(\frac{n}{\ln n}\right)^{2.5}$ and devide both sides by $n \log n$. We have

$$\frac{1}{2} > \frac{c^2}{4} - 2.5 - o(1).$$

It is suffice to choose $c < 2\sqrt{3}$. □

4. Let ϵ be a small positive constant and $p = n^{\epsilon-1}$. Prove that almost surely the chromatic number of random graph $G(n,p)$ is at least

$$(1-o(1))\frac{np}{2\ln(np)}.$$

Proof: Since $\chi(G) \geq \frac{n}{\alpha(G)}$, it suffices to prove almost surely

$$\alpha(G) \leq (1+o(1))\frac{2\ln(np)}{p}.$$

For any k-set S, the probability that S is an independent set is $(1-p)^{\binom{k}{2}}$. Thus, we have

$$\begin{aligned}\Pr(\alpha(G) > k) &\leq \sum_{|S|=k} (1-p)^{\binom{k}{2}} \\ &= \binom{n}{k}(1-p)^{\binom{k}{2}} \\ &< (\frac{ne}{k})^k e^{-p\binom{k}{2}} \\ &= \left(\frac{ne}{k} e^{-p(k-1)}\right)^k.\end{aligned}$$

Choose $k = (1+o(1))\frac{2\ln(np)}{p}$. We have

$$\frac{ne}{k} e^{-p(k-1)} \approx \frac{e}{2\ln(np)}.$$

Since $np = n^{\epsilon} \to \infty$, the above quantity goes to 0. Thus, $\Pr(\alpha(G) > k) = o(1)$. □

5. Consider a random walk on the plane. At $t = 0$, a chip is at the origin. Each time a chip can move one step in a random chosen direction independently. Prove that with probability at least $1 - \frac{1}{n}$, the chip at time n is within the distance of $O(\sqrt{n \log n})$ from the origin.

   Proof: Let (X_i, Y_i) be the vector of i-th step movement. At time $t = n$, the chip is at position $(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i)$. Both x-coordinate and y-coordinate of the chip is a sum of n independent random variables. Let $X = \sum_{i=1}^n X_i$ and $Y = \sum_{i=1}^n Y_i$. By theorem, we have

   $$\Pr(|X - \mathrm{E}(X)| > \lambda) \leq 2e^{-\frac{\lambda^2}{-2(\mathrm{Var}(X)+M\lambda/3)}}.$$

   For any i, $\mathrm{E}(X_i) = \mathrm{E}(Y_i) = 0$, $|X_i| \leq 1$, and $\mathrm{Var}(X_i) = \mathrm{Var}(Y_i) \leq 1$. We have $\mathrm{E}(X) = \mathrm{E}(Y) = 0$ and

   $$\mathrm{Var}(X) = \sum_{i=1}^n \mathrm{Var}(X_i) \leq n.$$

   Choose $\lambda = 2\sqrt{n \log n}$. We have

   $$\begin{aligned}\Pr(|X| > 2\sqrt{n\log n}) &\leq 2e^{-\frac{\lambda^2}{-2(\mathrm{Var}(X)+M\lambda/3)}} \\ &\leq 2e^{-\frac{\lambda^2}{-2(n+\lambda/3)}} \\ &= 2e^{-2\ln n + o(1)} \\ &< \frac{1}{2n}\end{aligned}$$

   Similary, we have

   $$\Pr(|Y| > 2\sqrt{n\log n}) < \frac{1}{2n}.$$

With probability at least $1 - \frac{1}{n}$, we have

$$|X| \leq 2\sqrt{n \log n} \quad \text{and} \quad \Pr(|Y| \leq 2\sqrt{n \log n}).$$

Thus, with probability at least $1 - \frac{1}{n}$,

$$\sqrt{X^2 + Y^2} \leq 2\sqrt{2n \log n}.$$

□