Solutions for selected problems from the end of chapters

Chapter 13, problem 5 Find the equations of the tangent line and the normal line at $t = 0$ for the curve:

$$x = 2t^3 - 4t + 7, \quad y = t + \ln(t + 1).$$

Solution: We have

$$\frac{dx}{dt} = 6t^2 - 4, \quad \frac{dy}{dt} = 1 + \frac{1}{t+1}.$$

At $t = 0$, the nomal vector is $\langle -4, 2\rangle$. The point at $t = 0$ is $(7, 0)$. The equation of the tangent line is

$$-4(x - 7) + 2(y - 0) = 0.$$

After simiplifying it, we have

$$2x - y - 14 = 0.$$

The normal line is given by

$$\frac{x - 7}{-4} = \frac{y - 0}{2}.$$

I.e.,

$$x + 2y - 7 = 0.$$

Chapter 13, problem 7 Find the lenght of the curve

$$\begin{aligned} x &= \cos t + t \sin t \\ y &= \sin t - t \cos t \end{aligned}$$

from 0 to 2π.

Solution: We have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \\ &= \int_0^{2\pi} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt \\ &= \int_0^{2\pi} t dt \\ &= \frac{t^2}{2} |_0^{2\pi} \end{aligned}$$

$2\pi^2$.

Chapter 13, problem 9 Let $\mathbf{a} = \langle 2, -5\rangle$, $\mathbf{b} = \langle 1, 1\rangle$, and $\mathbf{c} = \langle -6, 0\rangle$. Find each of the following

(a) $3\mathbf{a} - 2\mathbf{b}$
Solution: We have
$$3\mathbf{a} - 2\mathbf{b} = \langle 4, -17\rangle.$$

(b) $\mathbf{a} \cdot \mathbf{b}$
Solution: We have
$$\mathbf{a} \cdot \mathbf{b} = -3.$$

(c) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$
Solution: We have
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = -15.$$

(d) $(4\mathbf{a} + 5\mathbf{b}) \cdot 3\mathbf{c}$
Solution: We have
$$(4\mathbf{a} + 5\mathbf{b}) \cdot 3\mathbf{c} = -234.$$

(e) $|\mathbf{c}|c \cdot 3\mathbf{b}$
Solution: We have
$$|\mathbf{c}|c \cdot 3\mathbf{b} = -36.$$

(f) $\mathbf{c} \cdot \mathbf{c} - |\mathbf{c}|$
Solution: We have
$$\mathbf{c} \cdot \mathbf{c} - |\mathbf{c}| = 30.$$

Chapter 13, problem 17 Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ for each of the following:

(a) $\mathbf{r}(t) = (\ln t)\mathbf{i} - 3t^2\mathbf{j}$
Solution: We have
$$\begin{aligned}\mathbf{r}'(t) &= \frac{1}{t}\mathbf{i} - 6t\mathbf{j}\\ \mathbf{r}''(t) &= -\frac{1}{t^2}\mathbf{i} - 6\mathbf{j}\end{aligned}$$

(b) $\mathbf{r}(t) = \sin t\mathbf{i} + \cos 2t\mathbf{j}$
Solution: We have
$$\begin{aligned}\mathbf{r}'(t) &= \cos t\mathbf{i} - 2\sin 2t\mathbf{j}\\ \mathbf{r}''(t) &= -\sin t\mathbf{i} - 4\cos 2t\mathbf{j}\end{aligned}$$

(c) $\mathbf{r}(t) = \tan t\mathbf{i} - t^4\mathbf{j}$
Solution: We have
$$\begin{aligned}\mathbf{r}'(t) &= \sec^2 t\mathbf{i} - 4t^3\mathbf{j}\\ \mathbf{r}''(t) &= 2\sec^2 t\tan t\mathbf{i} - 12t^2\mathbf{j}\end{aligned}$$

Chapter 14, problem 4 Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, and $\mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Find each of the following:

(a) $\mathbf{a} \times \mathbf{b}$.
Solution: We have

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \left\langle \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}, - \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} \right\rangle \\ &= \langle -5, -5, 5 \rangle \end{aligned}$$

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$.
Solution: We have

$$\mathbf{b} + \mathbf{c} = \langle 0\mathbf{i} + 5\mathbf{j} + \mathbf{k} \rangle.$$

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \left\langle \begin{vmatrix} -1 & 1 \\ 5 & 1 \end{vmatrix}, - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 0 & 5 \end{vmatrix} \right\rangle \\ &= \langle -6, -2, 10 \rangle \end{aligned}$$

(c) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
Solution: We have

$$\begin{aligned} \mathbf{b} \times \mathbf{c} &= \left\langle \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}, - \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} \right\rangle \\ &= \langle -7, 1, -5 \rangle \end{aligned}$$

Thus,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 2 \cdot (-7) + (-1) \cdot 1 + 1 \cdot (-5) = -20.$$

(a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.
Solution: We have

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \left\langle \begin{vmatrix} -1 & 1 \\ 1 & -5 \end{vmatrix}, - \begin{vmatrix} 2 & 1 \\ -7 & -5 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ -7 & 1 \end{vmatrix} \right\rangle \\ &= \langle 4, 3, -5 \rangle \end{aligned}$$

Chapter 14, Problem 9 Find the value of C if the plane $x + 5y + Cz + 6 = 0$ is perpendicular to the plane $4x - y + z - 17 = 0$.

Solution: The dot product of two normal vectors are zero.

$$\langle 1, 5, C \rangle \cdot \langle 4, -1, 1 \rangle = 0$$

Solve for C, we have

$$C = 1.$$

Chapter 14, Problem 11 Find the parametric equations for the line throuth $(-2, 1, 5)$ and $(6, 2, -3)$.

Solution: Calculate the direction vector by taking the differene of two points:

$$\mathbf{v} = \langle 8, 1, -8 \rangle$$

Picking any point, together with the direction vector, we have

$$\begin{cases} x &= 8t - 2 \\ y &= t + 1 \\ z &= -8t + 5. \end{cases}$$

Chapter 14, Problem 19 Find the length of the curve $\mathbf{r}'(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + e^t\mathbf{k}$, $1 \le t \le 5$.

Solution: We have

$$\begin{aligned} L &= \int_1^5 \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} dt \\ &= \int_1^5 \sqrt{(e^t(\sin t + \cos t))^2 + (e^t(\cos t - \sin t))^2 + (e^t)^2} dt \\ &= \int_1^5 e^t \sqrt{3} dt \\ &= \sqrt{3} e^t |_1^5 \\ &= \sqrt{3}(e^5 - e). \end{aligned}$$

Chapter 14, Problem 21 If $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ is the position vector for a moving particle at time t, find the tangential and normal components, a_T and a_N, of the acceleration vector at $t = 1$.

Solution: We have

$$\begin{aligned} \mathbf{r}'(t) &= \langle 1, 2t, 3t^2 \rangle. \\ \mathbf{r}''(t) &= \langle 0, 2, 6t \rangle. \\ \mathbf{r}'(1) &= \langle 1, 2, 3 \rangle. \\ \mathbf{r}''(1) &= \langle 0, 2, 6 \rangle. \\ a_T &= \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|} \\ &= \frac{1 \cdot 0 + 2 \cdot 2 + 3 \cdot 6}{\sqrt{1^2 + 2^2 + 3^2}} \\ &= \frac{22}{\sqrt{14}}. \\ a_N &= \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left|\left\langle \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix}, - \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \right\rangle\right|}{\sqrt{1^2+2^2+3^2}} \\
&= \frac{|\langle 6, -6, 2\rangle|}{\sqrt{14}} \\
&= \frac{\sqrt{6^2+(-6)^2+2^2}}{\sqrt{14}} \\
&= \frac{2\sqrt{19}}{\sqrt{14}}.
\end{aligned}
$$

Chapter 15, Problem 7 If $F(x,y) = 5x^3y^6 - xy^7$. find $\frac{\partial^3 F(x,y)}{\partial x \partial y^2}$.

Solution: We have

$$
\begin{aligned}
F_x &= 15x^2y^6 - y^7 \\
F_{xy} &= 90x^2y^5 - 7y^6 \\
FF_{xyy} &= 450x^2y^4 - 42y^5.
\end{aligned}
$$

Chapter 15, Problem 11 Does $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ exist? Explain.

Solution: The limit does NOT exist. Taking the limit along the line $y=0$, we get 1. Taking the limit along the line $x=0$, we get -1. They are not the same value.

Chapter 15, Problem 15 Find the slope of the tangent line to the curve of intersection of the vertical plane $x - \sqrt{3}y + 2\sqrt{3} - 1 = 0$ and the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$.

Solution: This is the same question asking the directional derivative $\nabla_{\mathbf{u}} f(1,2)$ along the the unit vector of the direction $\langle 1, -\sqrt{3}\rangle$.

Since the length is 2, the unit vector is

$$\mathbf{u} = \langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle.$$

We have

$$\nabla f(1,2) = \langle 2x, 2y\rangle = \langle 2, 4\rangle.$$

Thus,

$$\nabla_{\mathbf{u}} f(1,2) = 2 \cdot \frac{1}{2} + 4 \cdot (-\frac{\sqrt{3}}{2}) = 1 - 2\sqrt{3}.$$

Remark: The answer in the book is wrong.

Chapter 15, Problem 23 Find the gradient vector of $F(x,y,z) = 9x^2+4y^2+9z^2-34$ at the point $P(1,2,-1)$. Write the equation of the tangent plane to the surface $F(x,y,z) = 0$ at the given point.

Solution: We have

$$\nabla F = \langle 18x, 8y, 18z \rangle.$$

At the point $P(1, 2, -1)$, we have

$$\nabla F(1, 2, -1) = \langle 18, 16, -18 \rangle.$$

The equation of the tangent plane is

$$18(x-1) + 16(y-2) - 18(z+1) = 0.$$

Simiplify it.

$$9x + 8y - 9z - 34 = 0.$$

Chapter 15, Problem 25 If $f(x, y, z) = xy^2/(1+z^2)$, use differentials to estimate $f(1.01, 1.98, 2.03)$.

Solution: We will use the following linear approximation formula:

$$f(\mathbf{p} + \mathbf{h}) \approx f(\mathbf{p}) + \nabla(\mathbf{p}) \cdot \mathbf{h}$$

Here $\mathbf{p} = (1, 2, 2)$ and $\mathbf{h} = (0.01, -0.02, 0.03)$. We have

$$\begin{aligned} \nabla f(\mathbf{p}) &= \langle \frac{y^2}{1+z^2}, \frac{2xy}{1+z^2}, -\frac{2xy^2z}{(1+z^2)^2} \rangle(\mathbf{p}) \\ &= \langle \frac{2^2}{1+2^2}, \frac{2 \cdot 1 \cdot 2}{1+2^2}, -\frac{2 \cdot 1 \cdot 2^2 \cdot 2}{(1+2^2)^2} \rangle \\ &= \langle \frac{4}{5}, \frac{4}{5}, -\frac{16}{25} \rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} f(\mathbf{p} + \mathbf{h}) &\approx f(\mathbf{p}) + \nabla(\mathbf{p}) \cdot \mathbf{h} \\ &= \frac{1 \cdot 2^2}{1+2^2} + \frac{4}{5} \cdot 0.01 + \frac{4}{5} \cdot (-0.02) + (-\frac{16}{25}) \cdot 0.03 \\ &= 0.7728 \end{aligned}$$

Chapter 15, Problem 27 A rectangular box whose edges are parallel to the coordinate axes in inscribed in the ellipsoid $36x^2 + 4y^2 + 9z^2 = 36$. What is the greatest possible volume for such a box?

Solution: Let (x, y, z) be the cornner of the rectangular box in the first octant. We have $x, y, z > 0$. The volume of the rectangular box is

$$V = 8xyz.$$

We will apply Langrange's method.

$$\begin{aligned} 8yz &= 72x\lambda && (1) \\ 8xz &= 8y\lambda && (2) \\ 8xy &= 18z\lambda && (3) \\ 36x^2 + 4y^2 + 9z^2 &= 36 && (4) \end{aligned}$$

Take product of equation (1), (2), and (3). We have

$$xyz = \frac{81}{4}\lambda^3.$$

Together with equation (1), we can cancel yz. We get

$$x^2 = \frac{9}{4}\lambda^2.$$

Since x, y, z are positive, λ must be positive. We have

$$x = \frac{3}{2}\lambda.$$

Similarly, we have

$$y = \frac{9}{2}\lambda, \quad z = 3\lambda.$$

Substitute them into equation (4). We can solve for λ. We get $\lambda = \frac{2}{9}\sqrt{3}$. Therefore, we have

$$x = \frac{1}{3}\sqrt{3}, \quad y = \sqrt{3}, \quad z = \frac{2}{3}\sqrt{3}.$$

The maximum volume is

$$V = 8xyz = \frac{16}{3}\sqrt{3}.$$

Chapter 16, Problem 1 Evaluate the integral $\int_0^1 \int_x^{\sqrt{x}} xydydx$.

Solution:

$$\begin{aligned}
\int_0^1 \int_x^{\sqrt{x}} xydydx &= \int_0^1 \frac{1}{2}xy^2|_{y=x}^{y=\sqrt{x}} dx \\
&= \int_0^1 \frac{1}{2}(x^2 - x^3)dx \\
&= \frac{1}{6}x^3 - \frac{1}{8}x^4|_{x=0}^{x=1} \\
&= \frac{1}{6} - \frac{1}{8} \\
&= \frac{1}{24}.
\end{aligned}$$

Chapter 16, Problem 5 Rewrite the iterated integral with the indicated order of integration. $\int_0^1 \int_x^1 f(x, y)dydx; dxdy$

Solution: The iterated integrals can be written as a double integral over a triangle region:

$$\{(x, y)|x \le y \le 1, \text{ and } 0 \le x \le 1\}$$

The same triangle can be also described as

$$\{(x, y)|0 \le x \le y, \text{ and } 0 \le y \le 1\}$$

Hence, we have

$$\int_0^1 \int_x^1 f(x, y)dydx = \int_0^1 \int_0^y f(x, y)dxdy.$$

Chapter 16, Problem 7 Rewrite the iterated integral with the indicated order of integration. $\int_0^1 \int_0^{(1-x)/2} \int_0^{1-x-2y} f(x, yx, z)dzdydx; dxdzdy$

Solution: The iterated integrals can be written as a triple integral over a tetrahedron:

$$\{(x, y, z)|0 \le z \le 1 - x - 2y, 0 \le y \le (1 - x)/2, \text{ and } 0 \le x \le 1\}$$

The same tetrahedron can be also described as

$$\{(x, y)|x + 2y + z \le 1 \text{ and } x, y, z \ge 0\}$$

Hence, we have

$$\int_0^1 \int_0^{(1-x)/2} \int_0^{1-x-2y} f(x, yx, z)dzdydx = \int_0^{\frac{1}{2}} \int_0^{1-2y} \int_0^{1-2y-z} f(x, yx, z)dxdzdy$$

Chapter 16, Problem 11 Evaluate $\iiint_S z^2dV$, where S is the region bounded by $x^2 + z = 1$ and $y^2 + z = 1$ and the xy-plane.

Proof: We have

$$\begin{aligned}
\iiint_S z^2dV &= \iint_D \int_0^{\min\{1-x^2,1-y^2\}} z^2dzdA \\
&= \iint_D \frac{1}{3}(\min\{1 - x^2, 1 - y^2\})^3dA \\
&= 8\iint_T \frac{1}{3}(1 - x^2)^3dA \\
&= 8\int_0^1 \int_0^x \frac{1}{3}(1 - x^2)^3dydx \\
&= 8\int_0^1 \frac{1}{3}(1 - x^2)^3xdx \\
&= -\frac{1}{3}(1 - x^2)^4|_{x=0}^{x=1} \\
&= \frac{1}{3}.
\end{aligned}$$

Here D is the square $\{(x, y) : -1 \le x, y \le 1\}$, and T is the triangle $\{(x, y) : 0 \le y \le x, 0 \le x \le 1\}$.

Remark: The answer in the book is wrong.

Chapter 16, Problem 13 Find the center of mass of the rectangular lemina bounded by $x = 1$, $x = 3$, $y = 0$ and $y = 2$ if the density is $\delta(x, y) = xy^2$.

Solution: First we calculate the mass m, moments m_x and m_y. We have

$$\begin{aligned} m &= \iint_D \delta(x,y)dA \\ &= \int_1^3 \int_0^2 xy^2 dydx \\ &= \int_1^3 x\frac{y^3}{3}|_{y=0}^{y=2} dx \\ &= \int_1^3 x\frac{8}{3}dx \\ &= \frac{4}{3}x^2|_{x=1}^{x=3} \\ &= \frac{32}{3}. \end{aligned}$$

$$\begin{aligned} m_x &= \iint_D x\delta(x,y)dA \\ &= \int_1^3 \int_0^2 x^2y^2 dydx \\ &= \int_1^3 x^2\frac{y^3}{3}|_{y=0}^{y=2} dx \\ &= \int_1^3 x^2\frac{8}{3}dx \\ &= \frac{8}{9}x^3|_{x=1}^{x=3} \\ &= \frac{224}{9}. \end{aligned}$$

$$\begin{aligned} m_y &= \iint_D y\delta(x,y)dA \\ &= \int_1^3 \int_0^2 xy^3 dydx \\ &= \int_1^3 x\frac{y^4}{4}|_{y=0}^{y=2} dx \\ &= \int_1^3 4x dx \\ &= 2x^2|_{x=1}^{x=3} \\ &= 16. \end{aligned}$$

We have

$$\begin{aligned} \bar{x} &= \frac{m_x}{m} = \frac{13}{6}. \\ \bar{y} &= \frac{m_y}{m} = \frac{3}{2}. \end{aligned}$$

The center of the mass is $(\frac{13}{6}, \frac{3}{2})$.

Chapter 17, Problem 2 Find div$\mathbf{F}$, curl$\mathbf{F}$, grad(div$\mathbf{F}$), and div(curl$\mathbf{F}$) if $\mathbf{F}(x,y,z) = 2xyz\mathbf{i} - 3y^2\mathbf{j} + 2y^2z\mathbf{k}$.

Solution: We have

$$\begin{aligned} \text{div}\mathbf{F} &= 2yz - 6y + 2y^2 \\ \text{curl}\mathbf{F} &= \langle 4yz, 2xy, -2xz\rangle \\ \text{grad}(\text{div}\mathbf{F}) &= \langle 0, 2z - 6 + 4y, 2y\rangle \\ \text{div}(\text{curl}\mathbf{F}) &= 0 + 2x - 2x = 0 \end{aligned}$$

Chapter 17, Problem 5 Evaluate

(a) $\int_C (1-y^2)ds$; C is the quarter circle from $(0,-1)$ to $(1,0)$, centered at the origin.

Solution: C has the following parametric equation: $x = \cos t, y = \sin t$, $-\pi/2 \leq t \leq 0$. We have

$$\begin{aligned} \int_C (1-y^2)ds &= \int_{-\pi/2}^{0} (1-\sin^2 t)\sqrt{(-\sin t)^2 + \cos^t} dt \\ &= \int_{-\pi/2}^{0} \cos^2 t dt \\ &= \int_{-\pi/2}^{0} \frac{1}{2}(1+\cos 2t)dt \\ &= \frac{t}{2} + \frac{1}{4}\sin 2t|_{t=-\pi/2}^{t=0} \\ &= \frac{\pi}{4}. \end{aligned}$$

(b) $\int_C xydx + z\cos xdy + zdz$; C is the curve $x = t$, $y = \cos t$, $z = \sin t$, $0 \leq t \leq \pi/2$.

Solution:

$$\begin{aligned} \int_C xydx + z\cos xdy + zdz &= \int_0^{\pi/2} t\cos tdt + \sin t\cos td(\cos t) + \sin td(\sin t) \\ &= \int_0^{\pi/2} t\cos t - \sin^2 t\cos t + \sin t\cos tdt \end{aligned}$$

$$\begin{aligned}
&= (t\sin t + \cos t - \frac{1}{3}\sin^3 t - \frac{1}{4}\cos 2t)|_{t=0}^{t=\pi/2} \\
&= (\frac{\pi}{2} - \frac{1}{3} + \frac{1}{4}) - (1 - \frac{1}{4}) \\
&= \frac{\pi}{2} - \frac{5}{6}.
\end{aligned}$$

Chapter 17, Problem 6 Show that $\int_C y^2dx + 2xydy$ is independent of path, and use this to claculate the integral on any path from $(0,0)$ to $(1,2)$.

Solution: It is enough to find f so that

$$\nabla f = \langle y^2, 2xy\rangle.$$

We choose $f = xy^2$. We have

$$\begin{aligned}
\int_C y^2dx + 2xydy &= f(1,2) - f(0,0) \\
&= 4.
\end{aligned}$$

Chapter 17, Problem 8 Evaluate

$$\int_{(0,0,0)}^{(1,1,4)} (yz - e^{-x})dx + (xz + e^y)dy + xydz.$$

Solution: Let $f(x,y,z) = xyz + e^{-x} + e^y$. Then we have

$$\nabla f = \langle yz - e^{-x}, xy + e^y, xy\rangle.$$

The line integral is independent of path. Therefore we have,

$$\begin{aligned}
\int_{(0,0,0)}^{(1,1,4)} (yz - e^{-x})dx + (xz + e^y)dy + xydz &= f(1,1,4) - f(0,0,0) \\
&= (4 + e^{-1} + e^1) - (0 + e^0 + e^0) \\
&= 2 + e^{-1} + e.
\end{aligned}$$