On families of subsets with a forbidden subposet

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Joined work with Jerry Griggs

Poset

A poset is a set S together with a partial ordering $\preceq$ on it.

- Reflexive

$$a \preceq a$$

- Anti-symmetric

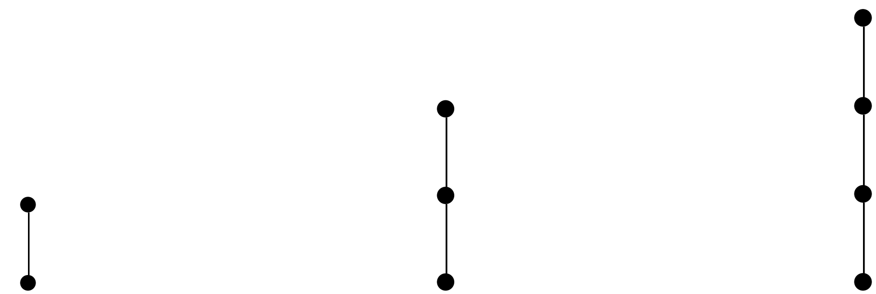
$$a \preceq b \text{ and } b \preceq a \text{ implies } a = b$$

- Transitive

$$a \preceq b \text{ and } b \preceq c \text{ implies } a \preceq c$$

Posets and Hasse diagrams

- Chains $P_n = ([n], \leq)$



P_2 $\quad$ P_3 $\quad$ P_4

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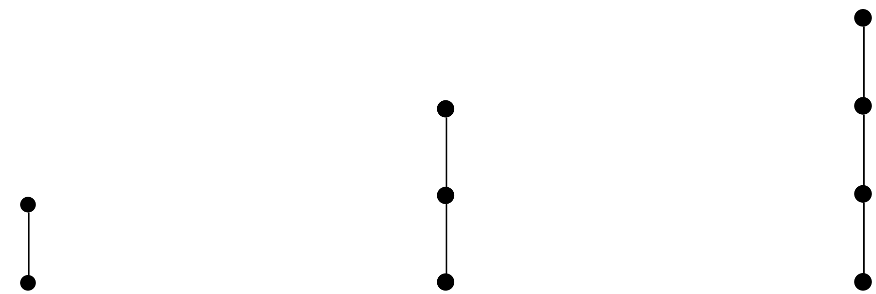

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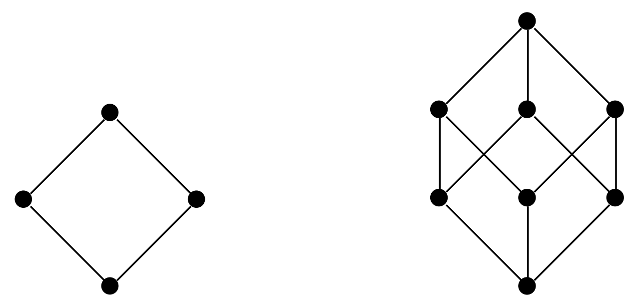
- Boolean lattice $Q_n = (2^{[n]}, \subseteq)$



Q_1 $\quad$ Q_2 $\quad$ Q_3

Sperner's theorem

Theorem (Sperner 1928)
Let $\mathcal{F}$ be a family of subsets of $[n] = \{1, 2, \ldots, n\}$. Suppose

"For any $A, B \in \mathcal{F}$, neither $A \subset B$ nor $B \subset A$."

Then

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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YBLM inequality:
Under the same condition, we have

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1.$$

Yamomoto
Bollobas
Lubell
Meshalkin

Subposets

A poset $(S, \preceq)$ contains a subposet $(S', \preceq')$ if there exists an injection $f\colon S' \to S$ which preserves partial orderings.

$$f(a') \preceq f(b') \text{ if } a' \preceq' b'$$

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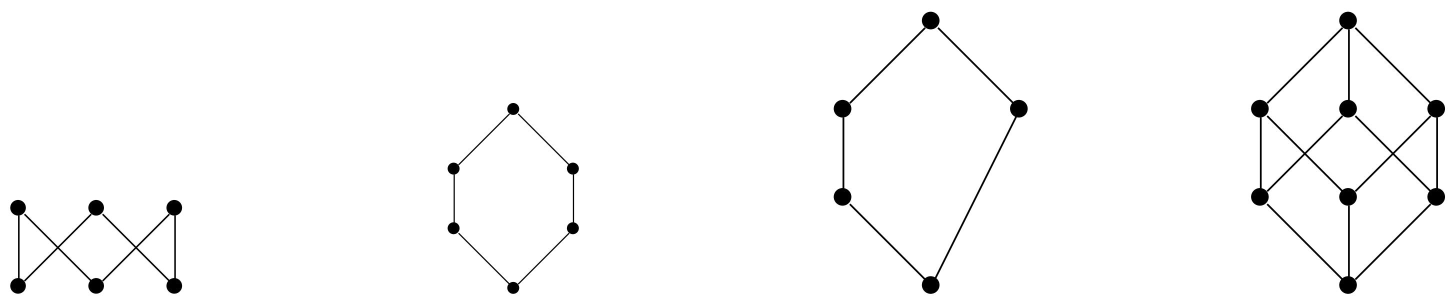
Subposets of Q_3:



C_6 $\quad$ C'_6 $\quad$ N_5 $\quad$ Q_3

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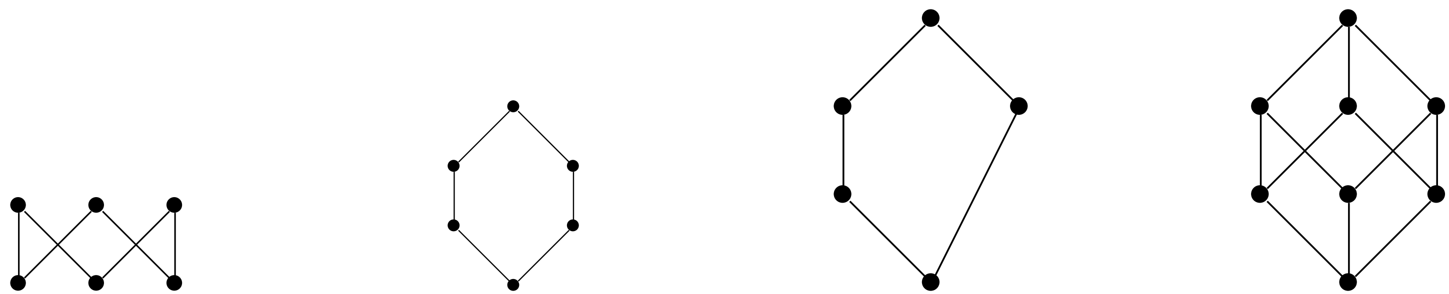

C_6 $\quad$ C_6' $\quad$ N_5 $\quad$ Q_3

We say $(S, \preceq)$ is $(S', \preceq')$-free if no such injection f exists.

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Erdős(1938):

$$\begin{aligned} \mathrm{La}(n, P_r) &= \sum_{i=\lfloor \frac{n-r+1}{2} \rfloor}^{\lfloor \frac{n+r-1}{2} \rfloor} \binom{n}{i} \\ &= (r - 1 + o_n(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}. \end{aligned}$$

Trivial bounds of $\mathrm{La}(n, H)$

- If $H' \subset H$, then

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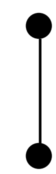
$$d(H) - 1 - o_n(1) \leq \frac{\mathrm{La}(n, H)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq |H| - 1.$$

Here depth $d(H)$ is the order of the longest chain in H.

Known results

- P_2 **Sperner (1928):**



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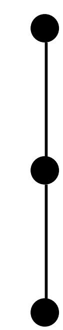

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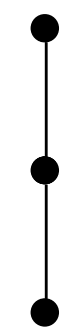

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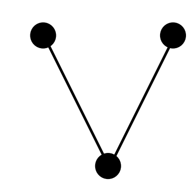
- V_2 **Tarján (1975)**



$$\mathrm{La}(n, V_2) = (1 + O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

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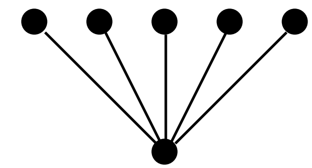

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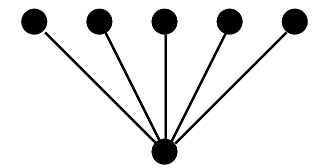

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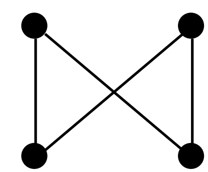

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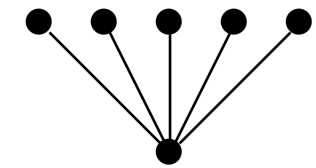

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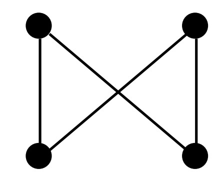

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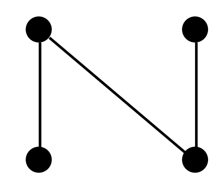
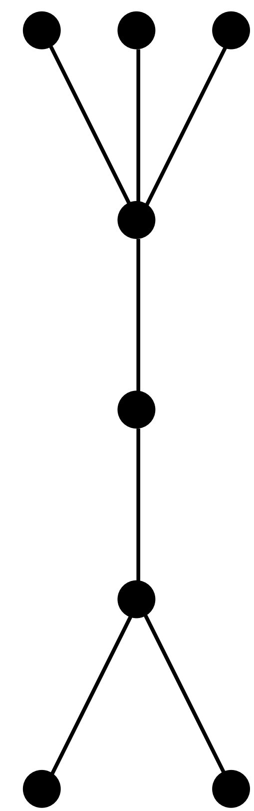
- **“N” Griggs, Katona (2006)**



$$\mathrm{La}(n, \text{“N”}) = (1 + O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Result on $P_k(s,t)$

$P_k(s,t)$: the "blow-up" of chain P_k by duplicating top element $s-1$ times and bottom element $t-1$ times.



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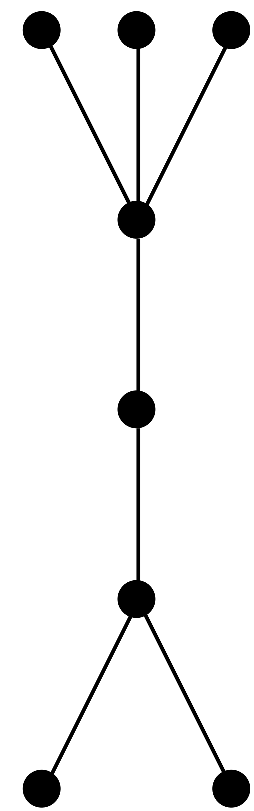

Theorem (Griggs, Lu 2007) *For* $s, t \geq 1$ *and* $k \geq 3$,

$$\mathrm{La}(n, P_k(s,t)) = (k - 1 + O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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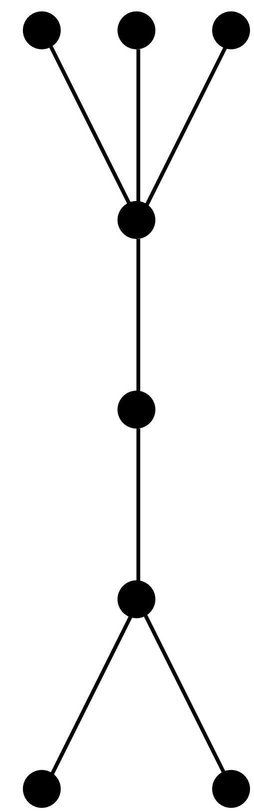

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Remark: The hidden constant in $O()$ is a function of s, t, and k.

Result on poset with depth 2

Theorem (Griggs, Lu 2007) *For any poset H of depth 2,*

$$(1 - O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \mathrm{La}(n, H) \leq (2 + O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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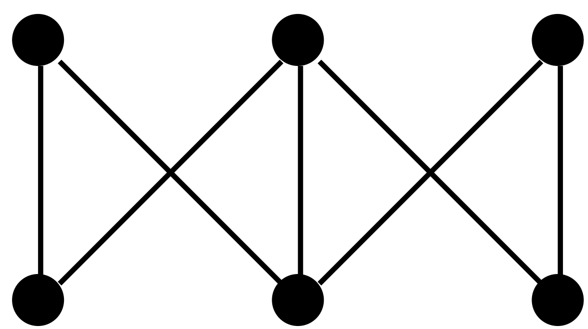
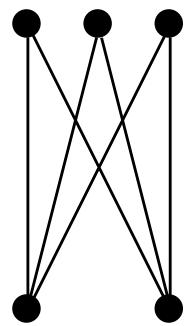
Moreover, if H contains the butterfly C_4, then

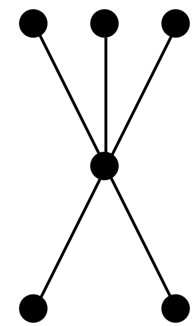
$$\mathrm{La}(n, H) = (2 + O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$



Proof of the upper bound:

Since H has depth 2. Assume H has s elements in upper level and t elements in lower level. H is a subposet of $P_3(s,t)$.





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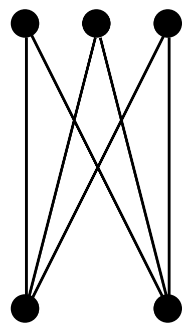
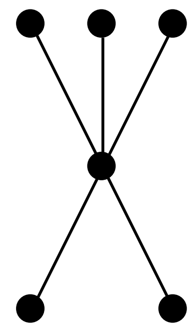




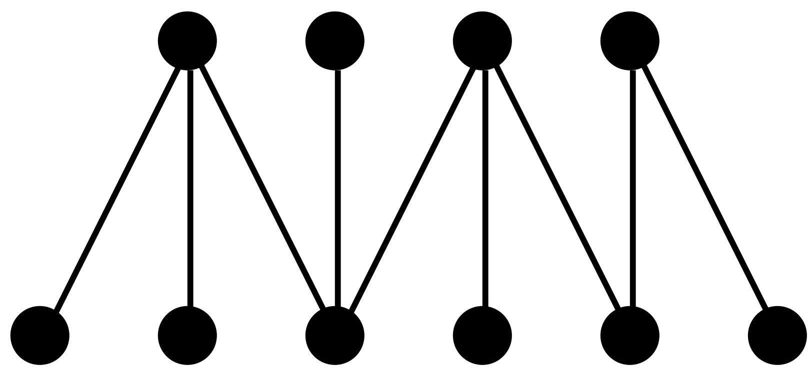
We have

$$\mathrm{La}(n,H) \le \mathrm{La}(n,P_3(s,t)) \le (2+O(\frac{1}{n}))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Result on up-down tree

T is an up-down tree if

- T is a poset of depth 2.
- T is a tree as a graph.



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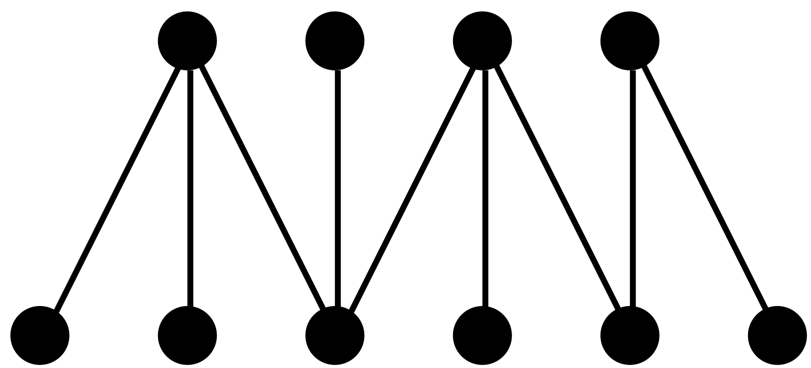
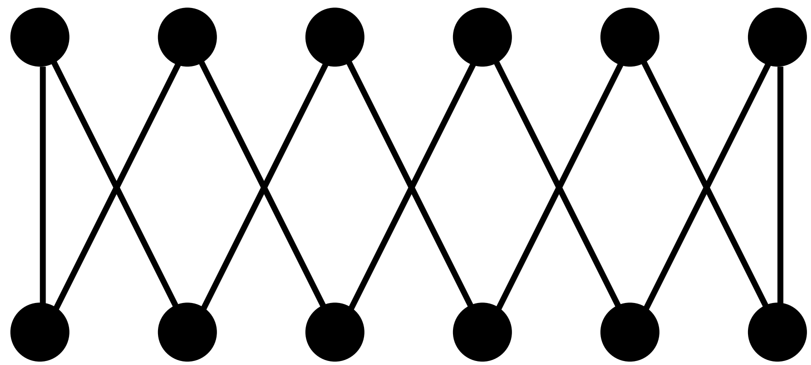


Theorem (Griggs, Lu 2007) *For any up-down tree T,*

$$\mathrm{La}(n, T) = \left(1 + O\left(\frac{1}{n}\right)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Result on cycles

Let C_{2k} be the poset of depth 2 on $2k$ elements which is also an even cycle as a graph.



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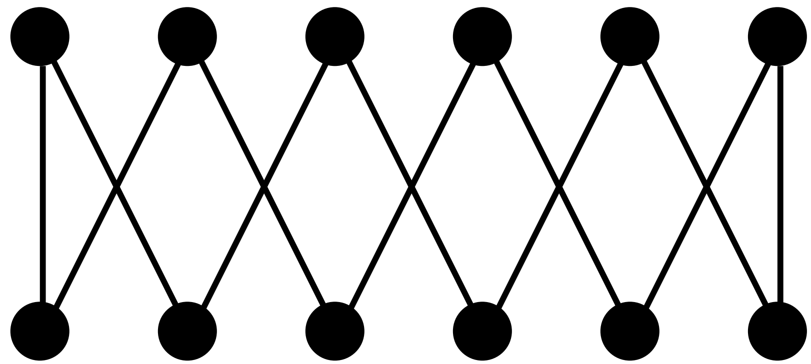

Theorem (Griggs, Lu 2007) *For* $k \geq 3$,

$$\mathrm{La}(n, C_{4k}) = (1 + o_n(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Next...

We will prove a special case of our theorem here:

$$\mathrm{La}(n, P_3(s,t)) \leq \left(2 + \frac{C}{n} + O(n^{-3/2}\sqrt{\ln n})\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

where $C = 6(s + t - 2)$.

A probabilistic lemma

Lemma: Let X be a random variable taking values of non-negative integers. For any integers $k > r$, if $\mathrm{E}(X) > k-1$, then

$$\mathrm{E}\binom{X}{k} \geq \mathrm{E}\binom{X}{r}\frac{r!}{k!}\prod_{i=r}^{k-1}(\mathrm{E}(X)-i).$$

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Special case $r = k-1$,

$$\mathrm{E}\binom{X}{k} \geq \mathrm{E}\binom{X}{k-1}\frac{1}{k}(\mathrm{E}(X)-k+1).$$

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- $\mathrm{E}\binom{X}{3} = \sum_{B \in \mathcal{F}} \frac{1}{\binom{n}{|B|}} \sum_{\substack{A \in \mathcal{F} \\ A \subset B}} \frac{1}{\binom{|B|}{|A|}} \sum_{\substack{C \in \mathcal{F} \\ B \subset C}} \frac{1}{\binom{n-|B|}{n-|C|}}$.

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Reduction

A fact on binomial coefficients:

$$\sum_{|i-\frac{n}{2}|>2\sqrt{n\ln n}} \binom{n}{i} = O\left(n^{-3/2}\binom{n}{\lfloor \frac{n}{2} \rfloor}\right).$$

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Reduction

A fact on binomial coefficients:

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WLOG, we can assume $\mathcal{F}$ only contains k-sets

with $k \in (\frac{n}{2} - 2\sqrt{n\ln n}, \frac{n}{2} + 2\sqrt{n\ln n})$.

An upper bound of $\mathrm{E}\binom{X}{3}$

For any $B \in \mathcal{F}$, one of the followings must occur:

- At most $s-1$'s A's satisfy $A \in \mathcal{F}$ and $A \subset B$.

$$\sum_{\substack{A\in\mathcal{F}\\ A\subset B}} \frac{1}{\binom{|B|}{|A|}} \le (s-1)\frac{1}{\frac{n}{2}-2\sqrt{n\ln n}}.$$

- At most $t-1$'s C's satisfy $C \in \mathcal{F}$ and $B \subset C$.

$$\sum_{\substack{C\in\mathcal{F}\\ B\subset C}} \frac{1}{\binom{n-|B|}{n-|C|}} \le (t-1)\frac{1}{\frac{n}{2}-2\sqrt{n\ln n}}.$$

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Put together

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The proof is finished since $|\mathcal{F}| \le \mathrm{E}(X)\binom{n}{\lfloor\frac{n}{2}\rfloor}$. $\square$

Open problems

Let $\pi(H) = \lim_{n\to\infty} \frac{\mathrm{La}(n,H)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ if it exists.

For poset H of depth 2, we proved

$$1 \leq \pi(H) \leq 2.$$

$$\pi(H) = \begin{cases} 1 & \text{if } H \text{ is a tree;} \\ 2 & \text{if } H \text{ contains } C_4. \end{cases}$$

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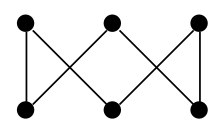

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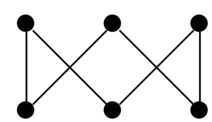

- Determine $\pi(Q_2)$.