On a problem of Erdős and Lovász on Coloring Non-Uniform Hypergraphs

Lincoln Lu

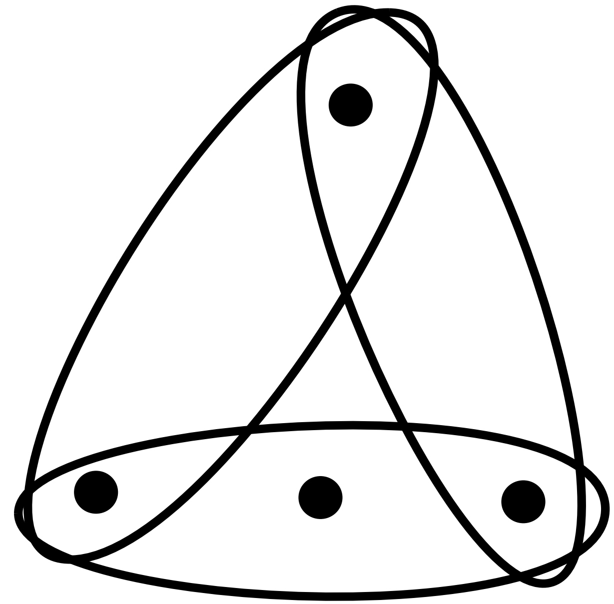
lu@math.sc.edu.

University of South Carolina

Hypergraphs

Hypergraph H:

- $V(H)$: the set of vertices.
- $E(H)$: the set of edges.



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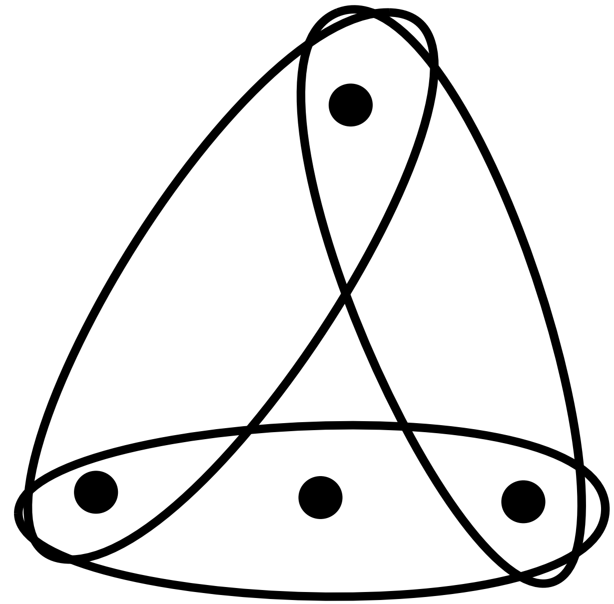

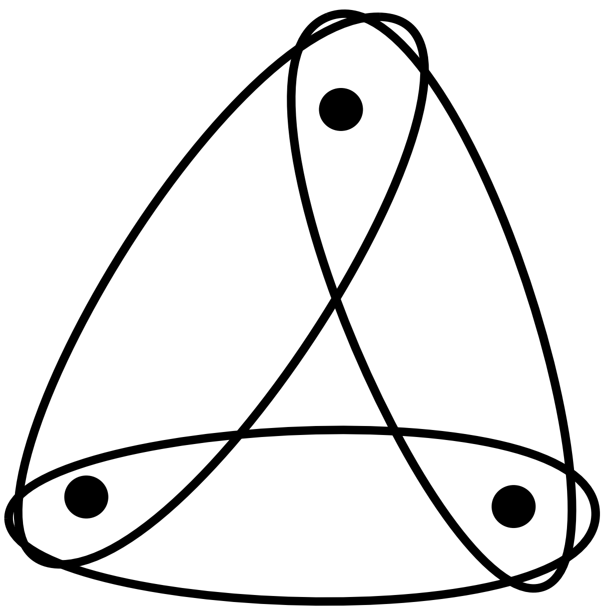

H is r-uniform if $|F| = r$ for every edge F of H.

Property B

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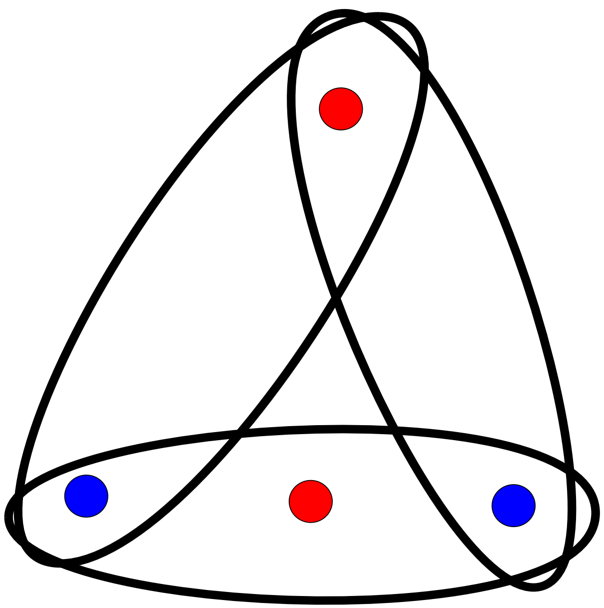

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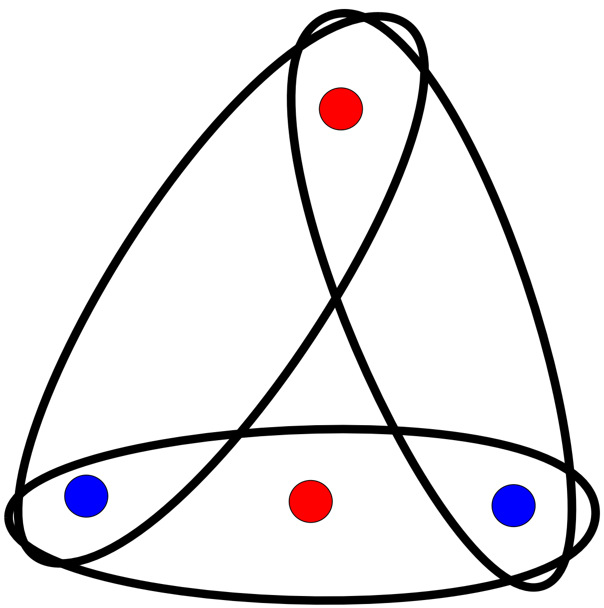

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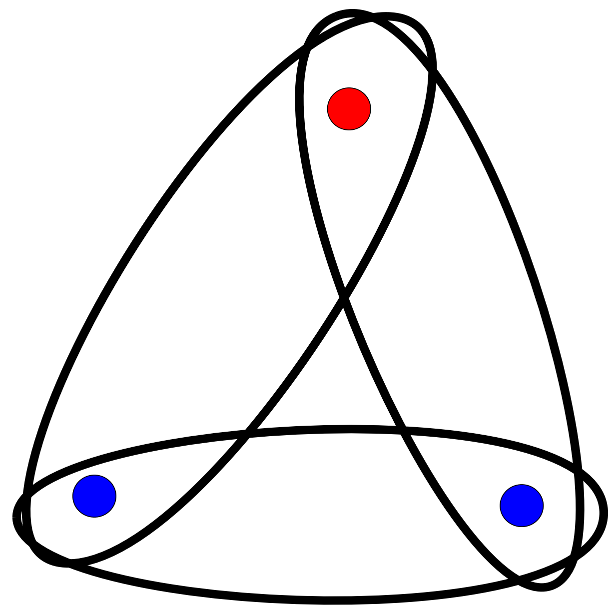

Without Property B

History

Property B is first introduced by Miller in 1937.

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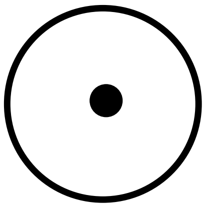
Erdős (1963) asked:

"What is the minimum edge number $m_2(r)$ of a r-uniform hypergraph not having property B?"

Edge cardinality matters!

- $m_2(1) = 1$:



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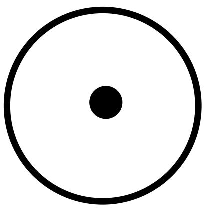
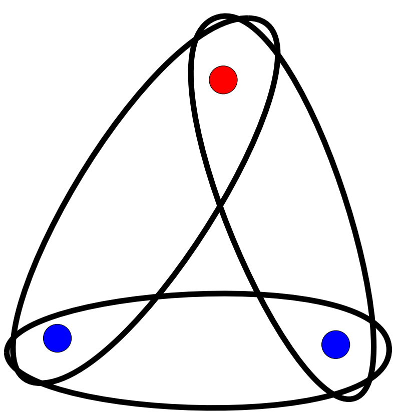


- $m_2(2) = 3$:



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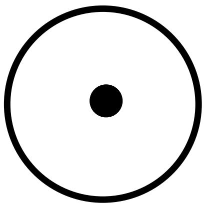

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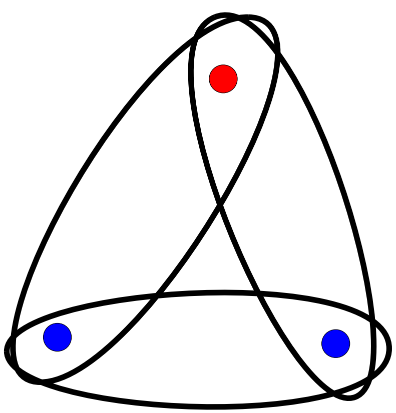
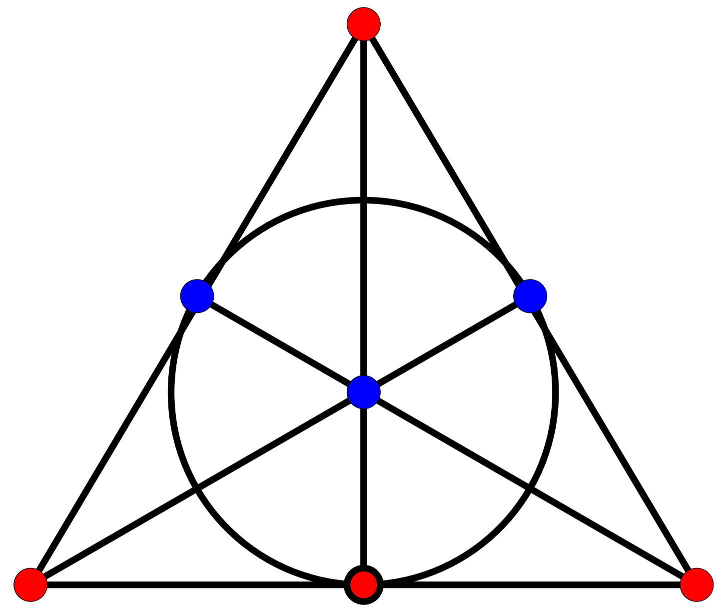


- $m_2(3) = 7$:



Fano plane

Erdős and Lovász (1975)

Perhaps $r2^r$ is the correct order of magnitude of $m_2(r)$; it seems likely that

$$\frac{m_2(r)}{2^r} \to \infty.$$

A stronger conjecture would be: Let $E_{k=1}^{m}$ be a 3-chromatic (not necessarily uniform) hypergraph. Let

$$f(r) = \min \sum_{k=1}^{m} \frac{1}{2^{|E_k|}},$$

where the minimum is extended over all hypergraphs with $\min |E_k| = r$. We conjecture that $f(r) \to \infty$ as $r \to \infty$.

Previous results

- Erdős (1963)

$$2^{r-1} \le m_2(r) \le (1+\epsilon)\frac{2\ln 2}{4} r^2 2^r.$$

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$$m_2(r) > r^{\frac{1}{3}-o(1)} 2^r.$$

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$$m_2(r) > r^{\frac{1}{3}-o(1)} 2^r.$$

- Radhakrishnan and Srinivasan (2000)

$$m_2(r) > (\frac{\sqrt{2}}{2} - o(1))\frac{\sqrt{r}}{\sqrt{\ln r}} 2^r.$$

Non-uniform hypergraphs

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The function $\log^*(x)$ grows very slowly since it is the inverse function of

$$n \to 2^{\overbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}^{n}}.$$

b

Our result

Theorem (Lu) For any $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$, for all $r > r_0$, we have

$$f(r) \geq (\frac{1}{16} - \epsilon)\frac{\ln r}{\ln \ln r}.$$

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An obvious upper bound:

$$f(r) \leq \frac{m_2(r)}{2^r} \leq (1+\epsilon)\frac{2\ln 2}{4}r^2.$$

Recoloring method

Theorem (Beck 1978) Any r-hypergraph H with at most $r^{1/3-o(1)}2^r$ edges has Property B.

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Spencer's Proof:

- Randomly and independently color each vertex red and blue with probability $\frac{1}{2}$.
- With small probability p, independently flip the color of vertices lying in monochromatic edges.

Type I: a red edge survives.

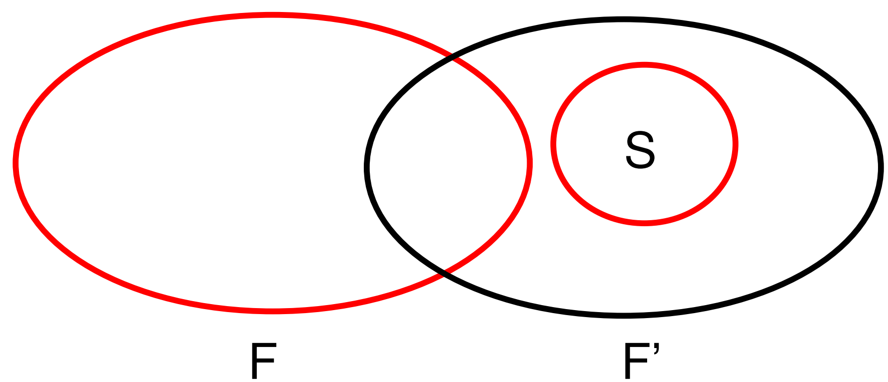
Let $h = |E(H)|2^{-r}$ be the expected number of red edges.

The probability of this event is

$$|E(H)|2^{-r}(1-p)^r \leq he^{-rp}.$$

Type II: a blue edge is created.

$$\begin{aligned}
&\sum_{i\geq 1}\sum_{|F\cap F'|=i} 2^{-2r+i}\sum_{s\geq 0}\binom{r-i}{s}p^{i+s} \\
&= 2^{-2r}\sum_{i\geq 1}(2p)^i\sum_{|F\cap F'|=i}(1+p)^{r-i} \\
&\leq 2^{-2r}(1+p)^r\frac{2p}{1+p}|E(H)|^2 \\
&\leq 2ph^2e^{pr}.
\end{aligned}$$



Put together

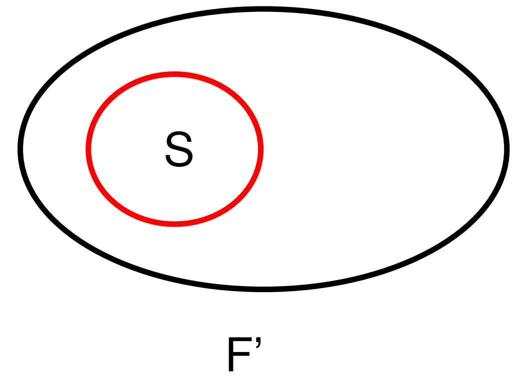
H has Property B if

$$2he^{-rp} + 4ph^2e^{pr} < 1.$$

Choose $h = r^{(1-\epsilon)/3}$ and $p = \frac{(1+\epsilon)\ln h}{r}$. Done!

The difficulty

A critical case:



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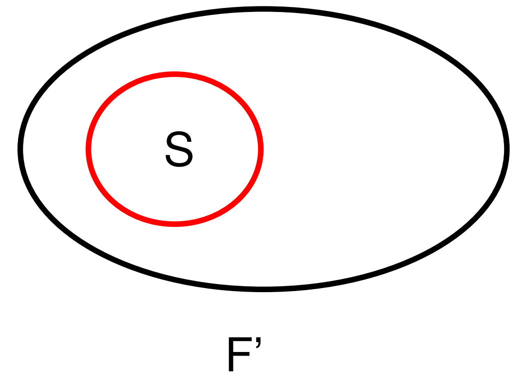

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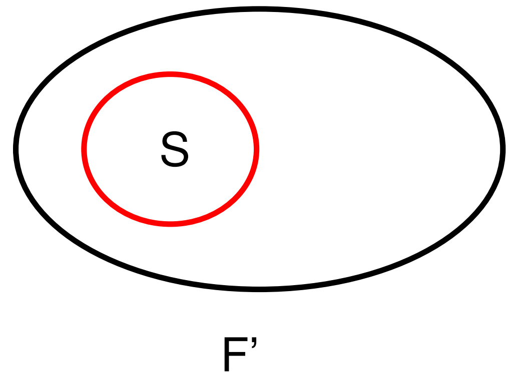

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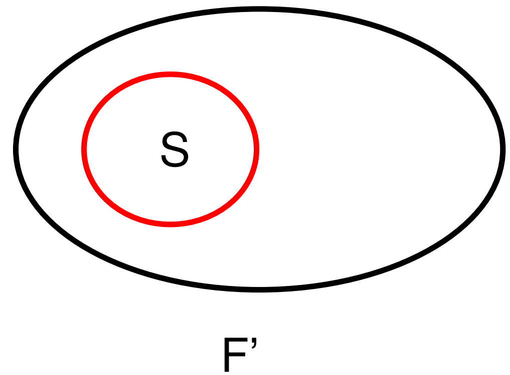

- S is red while $F' \setminus S$ is blue.
- For any $v \in S$, there exists an red edge F containing v.
- The size of F' is unbounded.
- There are too many choices of S.

Our approach

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- Simplifying “twin-hypergraph” into an irreducible core.
- Recoloring vertices in large edges first.

Twin-hypergraphs

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twin-hypergraph (H_1, H_2) is said *to have Property B* if there exists a red-blue vertex-coloring satisfying

- H_1 has no red edge.
- H_2 has no blue edge.

A theorem on twin-hypergraph.

Theorem (Lu) Suppose a twin-hypergraph $H = (H_1, H_2)$ with minimum edge-cardinality r satisfies

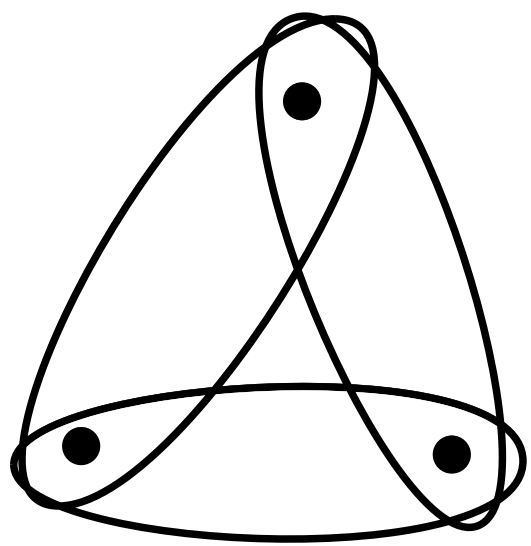
$$\sum_{F \in E(H_i)} \frac{1}{2^{|F|}} \leq \left(\frac{1}{16} - o(1)\right) \frac{\ln r}{\ln \ln r}$$

for $i = 1, 2$. Then H has property B.

Irreducibility

A twin-hypergraph $H = (H_1, H_2)$ is called *irreducible* if

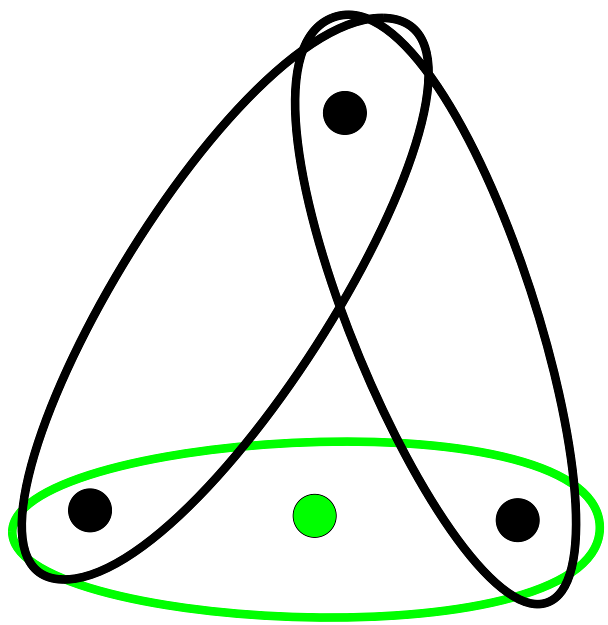
1. $\forall F_1 \in E(H_1)$ and $v \in F_1$, $\exists F_2 \in E(H_2)$ such that $F_1 \cap F_2 = \{v\}$.
2. $\forall F_2 \in E(H_2)$ and $v \in F_2$, $\exists F_1 \in E(H_1)$ such that $F_1 \cap F_2 = \{v\}$.



Reducibility

A twin-hypergraph $H = (H_1, H_2)$ is called *reducible* if there is an evidence (F, v) satisfying

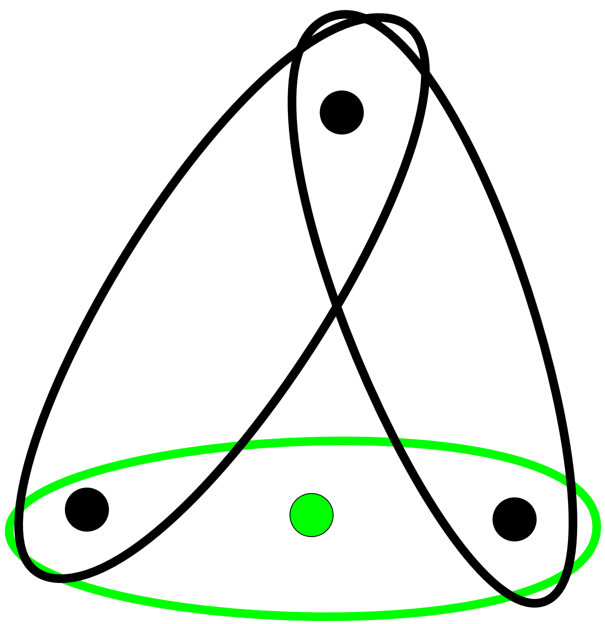
1. $v \in F$, and $F \in E(H_i)$ for $i = 1$ or 2.
2. $\forall F' \in E(H_{3-i})$, if $v \in F'$ then $|F \cap F'| \geq 2$.



Reducing twin-hypergraphs

If H is reducible, there is an evidence (F, v). Repeatedly removing F from H until an irreducible twin-hypergraph is reached.

$$H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(s)}.$$



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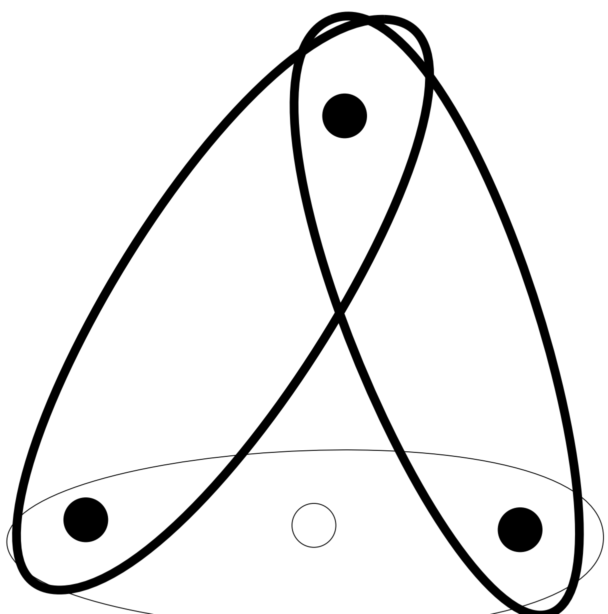

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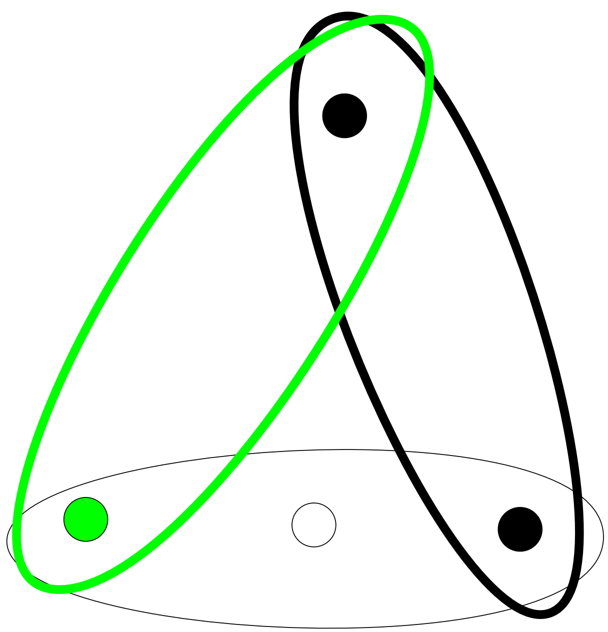

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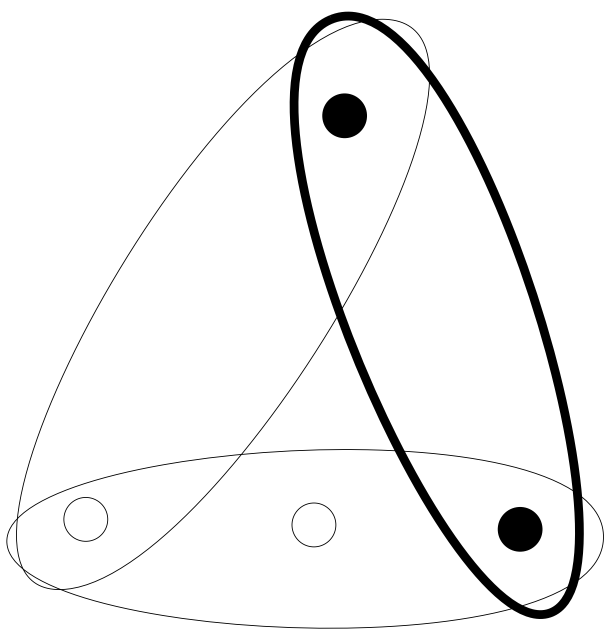

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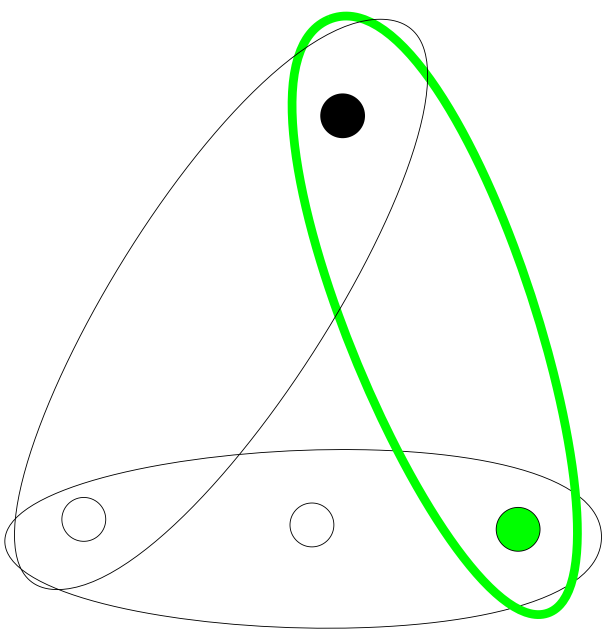

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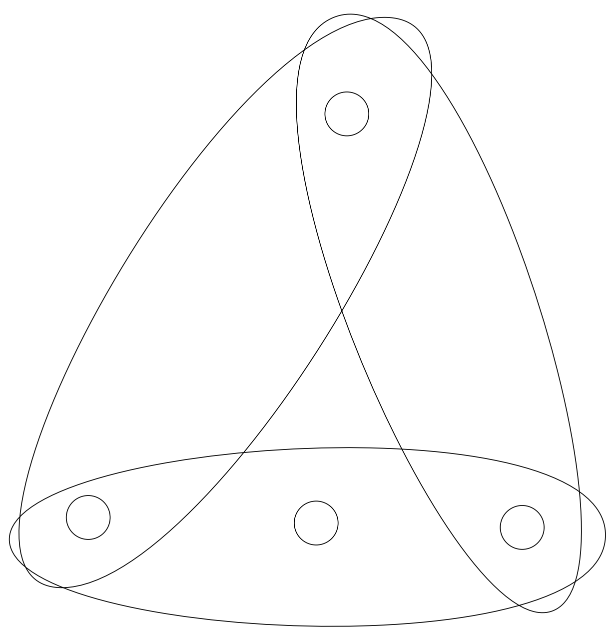

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Lemma on irreducible core

Lemma 3 A twin-hypergraph H has Property B if and only if its irreducible core has Property B.

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Proof: It suffices to add a removed edge F back.

- If F is not monochromatic, do nothing.
- Otherwise, flip the color of v. For any F' containing v, F' contains another vertex of F. Thus, F' is not monochromatic.

Randomized algorithm

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- Red edges with higher rank are destroyed first.
- Reduce it to the irreducible core whenever possible.
- Abort the program if a red edge survives or a blue edge is created.

Sketch of the proof

The probability of success is at least

$$1 - \frac{2}{M} - 2he^{-q} - \frac{2he^{8Mhq}}{Mr}.$$

Choose $M = 2(1+\epsilon)$, $q = \ln\ln r$, and $h = \frac{1-\epsilon}{16}\frac{\ln r}{\ln\ln r}$.
The above probability is

$$\frac{\epsilon}{1+\epsilon} - \frac{2h}{\ln r} - \frac{2h}{Mr^{\epsilon^2}} > 0$$

for sufficiently large r.

Therefore, H has Property B.

k colors

Theorem (Lu) Let $H_1, H_2, \ldots, H_k$ be hypergraphs over a common vertex set V with minimum edge cardinality r satisfying

$$\sum_{F \in E(H_i)} \frac{1}{k^{|F|}} \leq \left(\frac{k-1}{4k^2} - o(1)\right) \frac{\ln r}{\ln \ln r}.$$

Then, there exists a k-coloring of V such that H_i contains no monochromatic edge in i-th color for all $1 \leq i \leq k$.

Open Problems

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- Find a better upper bound for $f(r)$ using non-uniform hypergraph.
- Prove of disprove Erdős-Lovász's stronger conjecture $m_2(r) = \Theta(r2^r)$.