On a problem of Erdős and Lovász on Coloring Non-Uniform Hypergraphs

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Hypergraphs

Hypergraph $H$:

- $V(H)$: the set of vertices.
- $E(H)$: the set of edges.
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$H$ is $r$-uniform if $|F| = r$ for every edge $F$ of $H$. 
Property B

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With Property B
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With Property B

Without Property B
History

Property B is first introduced by Miller in 1937.

Bernstein (1908) proved: Suppose an infinite hypergraph $H$ has countable edges and each edge has infinite vertices. Then $H$ has Property B.
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Bernstein (1908) proved: Suppose an infinite hypergraph $H$ has countable edges and each edge has infinite vertices. Then $H$ has Property B.

Erdős (1963) asked:
“What is the minimum edge number $m_2(r)$ of a $r$-uniform hypergraph not having property $B$?”
Edge cardinality matters!

$m_2(1) = 1$: 

[Diagram of a single vertex and edge]
Edge cardinality matters!

- $m_2(1) = 1$: 

- $m_2(2) = 3$: 

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  - ![Diagram for $m_2(1) = 1$]

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- $m_2(3) = 7$:
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Fano plane
Perhaps \( r2^r \) is the correct order of magnitude of \( m_2(r) \); it seems likely that

\[
\frac{m_2(r)}{2^r} \to \infty.
\]

A stronger conjecture would be: Let \( E_{k=1}^m \) be a 3-chromatic (not necessarily uniform) hypergraph. Let

\[
f(r) = \min \sum_{k=1}^m \frac{1}{2|E_k|},
\]

where the minimum is extended over all hypergraphs with \( \min |E_k| = r \). We conjecture that \( f(r) \to \infty \) as \( r \to \infty \).
Previous results

Erdős (1963)

\[ 2^{r-1} \leq m_2(r) \leq (1 + \epsilon) \frac{2 \ln 2}{4} r^2 2^r. \]
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- Radhakrishnan and Srinivasan (2000)
  \[ m_2(r) > (\frac{\sqrt{2}}{2} - o(1)) \frac{\sqrt{r}}{\sqrt{\ln r}} 2^r. \]
Non-uniform hypergraphs

Beck (1978) proved

\[ f(r) \geq \frac{\log^*(r) - 100}{7}. \]
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The function \( \log^*(x) \) grows very slowly since it is the inverse function of

\[ n \rightarrow 2^2^\cdot^2 \cdot^\ldots^\cdot^2 \]
Our result

**Theorem (Lu)** For any $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$, for all $r > r_0$, we have

$$f(r) \geq \left( \frac{1}{16} - \epsilon \right) \frac{\ln r}{\ln \ln r}.$$
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An obvious upper bound:

$$f(r) \leq \frac{m_2(r)}{2r} \leq (1 + \epsilon) \frac{2 \ln 2}{4} r^2.$$
Recoloring method

Theorem (Beck 1978) Any $r$-hypergraph $H$ with at most $r^{1/3-o(1)}2^r$ edges has Property B.
Recoloring method

Theorem (Beck 1978) Any \( r \)-hypergraph \( H \) with at most \( r^{1/3-o(1)} 2^r \) edges has Property B.

Spencer’s Proof:

- Randomly and independently color each vertex red and blue with probability \( \frac{1}{2} \).
- With small probability \( p \), independently flip the color of vertices lying in monochromatic edges.
Type I: a red edge survives.

Let $h = |E(H)|2^{-r}$ be the expected number of red edges.

The probability of this event is

$$|E(H)|2^{-r}(1 - p)^r \leq he^{-rp}.$$
Type II: a blue edge is created.

\[
\sum_{i \geq 1} \sum_{|F \cap F'| = i} 2^{-2r+i} \sum_{s \geq 0} \left( \begin{array}{c} r - i \\ s \end{array} \right) p^{i+s} \\
= 2^{-2r} \sum_{i \geq 1} (2p)^i \sum_{|F \cap F'| = i} (1 + p)^{r-i} \\
\leq 2^{-2r} (1 + p)^r \frac{2p}{1 + p} |E(H)|^2 \\
\leq 2ph^2 e^{pr}.
\]
Put together

$H$ has Property B if

$$2he^{-rp} + 4ph^2e^{pr} < 1.$$ 

Choose $h = r^{(1-\epsilon)/3}$ and $p = \frac{(1+\epsilon)\ln h}{r}$. Done!
The difficulty

A critical case:

- $S$ is red while $F' \setminus S$ is blue.
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- For any $v \in S$, there exists an red edge $F$ containing $v$. 
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- The size of $F'$ is unbounded.
A critical case:

- $S$ is red while $F' \setminus S$ is blue.
- For any $v \in S$, there exists an red edge $F$ containing $v$.
- The size of $F'$ is unbounded.
- There are too many choices of $S$. 
Our approach

- Extending hypergraphs to “twin-hypergraphs”.
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- Extending hypergraphs to “twin-hypergraphs”.
- Simplifying “twin-hypergraph” into an irreducible core.
- Recoloring vertices in large edges first.
A twin-hypergraph is a pair of hypergraphs \((H_1, H_2)\) with the same vertex set \(V(H_1) = V(H_2)\).
Twin-hypergraphs

A *twin-hypergraph* is a pair of hypergraphs $\langle H_1, H_2 \rangle$ with the same vertex set $V(H_1) = V(H_2)$. The twin-hypergraph $\langle H_1, H_2 \rangle$ is said to have *Property B* if there exists a red-blue vertex-coloring satisfying

- $H_1$ has no red edge.
- $H_2$ has no blue edge.
Theorem (Lu) Suppose a twin-hypergraph $H = (H_1, H_2)$ with minimum edge-cardinality $r$ satisfies

$$\sum_{F \in E(H_i)} \frac{1}{2|F|} \leq \left(\frac{1}{16} - o(1)\right) \frac{\ln r}{\ln \ln r}$$

for $i = 1, 2$. Then $H$ has property $B$. 

Irreducibility

A twin-hypergraph \( H = (H_1, H_2) \) is called \textit{irreducible} if

1. \( \forall F_1 \in E(H_1) \text{ and } v \in F_1, \exists F_2 \in E(H_2) \text{ such that } F_1 \cap F_2 = \{v\} \).

2. \( \forall F_2 \in E(H_2) \text{ and } v \in F_2, \exists F_1 \in E(H_1) \text{ such that } F_1 \cap F_2 = \{v\} \).
A twin-hypergraph $H = (H_1, H_2)$ is called *reducible* if there is an evidence $(F, v)$ satisfying

1. $v \in F$, and $F \in E(H_i)$ for $i = 1$ or $2$.
2. $\forall F' \in E(H_{3-i})$, if $v \in F'$ then $|F \cap F'| \geq 2$. 

![Diagram of a twin-hypergraph showing reducibility conditions](image)
Reducing twin-hypergraphs

If $H$ is reducible, there is an evidence $(F, v)$. Repeatedly removing $F$ from $H$ until an irreducible twin-hypergraph is reached.

$$H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(s)}.$$ 

$H^{(s)}$ is called the irreducible core.
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Lemma 3 A twin-hypergraph $H$ has Property B if and only if its irreducible core has Property B.
Lemma on irreducible core

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Proof: It suffices to add a removed edge $F$ back.

- If $F$ is not monochromatic, do nothing.
- Otherwise, flip the color of $v$. For any $F'$ containing $v$, $F'$ contains another vertex of $F$. Thus, $F'$ is not monochromatic.
Randomized algorithm

- Randomly color each vertex red or blue with probability \( \frac{1}{2} \).
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- Randomly color each vertex red or blue with probability $\frac{1}{2}$.

- An edge $F$ has rank $i$ if $r2^{i-1} \leq |F| < r2^i$. For each $v$ lying in edges of rank $i$, flip the color of $v$ with probability $\frac{q}{r2^{i-1}}$ independently.
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- Red edges with higher rank are destroyed first.
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- Reduce it to the irreducible core whenever possible.
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- Red edges with higher rank are destroyed first.

- Reduce it to the irreducible core whenever possible.

- Abort the program if a red edge survives or a blue edge is created.
Sketch of the proof

The probability of success is at least

\[
1 - \frac{2}{M} - 2he^{-q} - \frac{2he^{8Mhq}}{Mr}.
\]

Choose \( M = 2(1 + \epsilon) \), \( q = \ln \ln r \), and \( h = \frac{1 - \epsilon}{16} \frac{\ln r}{\ln \ln r} \).

The above probability is

\[
\frac{\epsilon}{1 + \epsilon} - \frac{2h}{\ln r} - \frac{2h}{Mr^{\epsilon^2}} > 0
\]

for sufficiently large \( r \).

Therefore, \( H \) has Property B.
Theorem (Lu) Let $H_1, H_2, \ldots, H_k$ be hypergraphs over a common vertex set $V$ with minimum edge cardinality $r$ satisfying

$$\sum_{F \in E(H_i)} \frac{1}{k|F|} \leq \left(\frac{k - 1}{4k^2} - o(1)\right) \frac{\ln r}{\ln \ln r}.$$

Then, there exists a $k$-coloring of $V$ such that $H_i$ contains no monochromatic edge in $i$-th color for all $1 \leq i \leq k$. 
Open Problems

Is it true $f(r) = \frac{m_2(r)}{2^r}$?
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- Find a better upper bound for \( f(r) \) using non-uniform hypergraph.

- Prove or disprove Erdős-Lovász’s stronger conjecture \( m_2(r) = \Theta(r2^r) \).