Coloring Non-Uniform Hypergraphs

Red and Blue

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Hypergraphs

Hypergraph $H$:
- $V(H)$: the set of vertices.
- $E(H)$: the set of edges.
  (A edge $F$ is a subset of $V(H)$.)
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  (A edge $F$ is a subset of $V(H)$.)

$H$ is $r$-uniform if $|F| = r$ for every edge $F$ of $H$. 
Property B

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History

Property \textbf{B} is first introduced by Miller in 1937.

\textbf{Bernstein} (1908) proved: Suppose an infinite hypergraph \( H \) has countable edges and each edge has infinite vertices. Then \( H \) has Property B.
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**Bernstein** (1908) proved: Suppose an infinite hypergraph $H$ has countable edges and each edge has infinite vertices. Then $H$ has Property B.

Erdős (1963) asked:

“What is the minimum edge number $m_2(r)$ of a $r$-uniform hypergraph not having property $B$?”
Edge cardinality matters!

- $m_2(1) = 1$: 

[Diagram of a single point enclosed in a circle]
Edge cardinality matters!

- $m_2(1) = 1$:

  ![Diagram showing a single vertex](image)

- $m_2(2) = 3$:

  ![Diagram showing a triangle](image)
Edge cardinality matters!

- $m_2(1) = 1$:

- $m_2(2) = 3$:

- $m_2(3) = 7$:

Fano plane
Perhaps $r^{2^r}$ is the correct order of magnitude of $m_2(r)$; it seems likely that

$$\frac{m_2(r)}{2^r} \to \infty.$$ 

A stronger conjecture would be: Let $E_{k=1}^m$ be a 3-chromatic (not necessarily uniform) hypergraph. Let

$$f(r) = \min \sum_{k=1}^m \frac{1}{2|E_k|},$$

where the minimum is extended over all hypergraphs with $\min |E_k| = r$. We conjecture that $f(r) \to \infty$ as $r \to \infty$. 

Erdős and Lovász (1975)
Erdős (1963)

\[ 2^{r-1} \leq m_2(r) \leq (1 + \epsilon)\frac{2 \ln 2}{4} r^2 2^r. \]
Previous results

- Erdős (1963)

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2^{r-1} \leq m_2(r) \leq (1 + \epsilon) \frac{2 \ln 2}{4} r^2 2^r.
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\[
m_2(r) > r^{\frac{1}{3} - o(1)} 2^r.
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Previous results

- **Erdős (1963)**

\[ 2^{r-1} \leq m_2(r) \leq (1 + \epsilon) \frac{2 \ln 2}{4} r^2 2^r. \]

- **Beck (1978), Spencer (1981)**

\[ m_2(r) > r^{\frac{1}{3} - o(1)} 2^r. \]


\[ m_2(r) > \left( \frac{\sqrt{2}}{2} - o(1) \right) \frac{\sqrt{r}}{\sqrt{\ln r}} 2^r. \]
Non-uniform hypergraphs

Let \( g_0(x) = x, \ g_k(x) = \log_2(g_{k-1}(x)) \) for \( k \geq 1 \). For all \( x > 0 \), let \( \log^*(x) = \min\{k: g_k(x) \leq 1\} \).

Beck (1978) proved

\[
f(r) \geq \frac{\log^*(r) - 100}{7}.
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The function \( \log^*(x) \) grows very slowly since it is the inverse function of the following tower function of height \( n \)

\[
n \rightarrow 2^{2^n}.
\]
Observation

- In Beck’s paper, the gap between the lower bound of $f(r)$ and the lower bound of $\frac{m_2(r)}{2^r}$ is huge.
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- Using probabilistic method, Spencer simplified Beck’s proof for the uniform case, but not for the non-uniform case.
Main result

Theorem (Lu) For any $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$, for all $r > r_0$, we have

$$f(r) \geq \left(\frac{1}{16} - \epsilon\right) \frac{\ln r}{\ln \ln r}.$$

Coloring Non-Uniform Hypergraphs Red and Blue – p.10/38
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$$f(r) \geq \left( \frac{1}{16} - \epsilon \right) \frac{\ln r}{\ln \ln r}.$$ 

An obvious upper bound:

$$f(r) \leq \frac{m_2(r)}{2r} \leq (1 + \epsilon) \frac{2 \ln 2}{4} r^2.$$
Theorem (Beck 1978) Any $r$-hypergraph $H$ with at most $r^{1/3-o(1)}2^r$ edges has Property B.
Recoloring method

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Spencer’s Proof:

- Randomly and independently color each vertex red and blue with probability $\frac{1}{2}$.

- With probability $p$, independently flip the color of vertices lying in monochromatic edges.
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Spencer’s Proof:

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Observation: With positive probability, the recoloring process destroys all monochromatic edges and does not create any new monochromatic edge.
Type I: a red edge survives.

Let \( h = |E(H)|2^{-r} \) be the expected number of red edges.

The probability of this event is

\[
|E(H)|2^{-r}(1 - p)^r \leq he^{-rp}.
\]
Type II: a blue edge is created.

\[
\sum_{i \geq 1} \sum_{|F \cap F'| = i} 2^{-2r+i} \sum_{s \geq 0} \binom{r-i}{s} p^{i+s}
\]

\[
= 2^{-2r} \sum_{i \geq 1} (2p)^i \sum_{|F \cap F'| = i} (1+p)^{r-i}
\]

\[
\leq 2^{-2r} (1+p)^r \frac{2p}{1+p} |E(H)|^2
\]

\[
\leq 2ph^2 e^{pr}.
\]
Put together

$H$ has Property B if

$$2he^{-rp} + 4ph^2e^{pr} < 1.$$ 

Choose $h = r^{(1-\epsilon)/3}$ and $p = \frac{(1+\epsilon) \ln h}{r}$. Done!
The difficulty

A critical case:

- \( S \) is red while \( F' \setminus S \) is blue.
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A critical case:

- $S$ is red while $F' \setminus S$ is blue.
- For any $v \in S$, there exists a red edge $F$ containing $v$.
- The size of $F'$ is unbounded.
- There are too many choices of $S$. 
Our approach

- Introduce a new concept “twin-hypergraph”.
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- Adapt the recoloring method to twin-hypergraphs.
- Reduce the problem using irreducible core.
- Carefully separate independence relations between random variables.
A *twin-hypergraph* is a pair of hypergraphs \((H_1, H_2)\) with the same vertex set \(V(H_1) = V(H_2)\).
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The twin-hypergraph \((H_1, H_2)\) is said to have *Property B* if there exists a red-blue vertex-coloring satisfying

- \(H_1\) has no red edge.
- \(H_2\) has no blue edge.
Residue twin-hypergraphs

Let $C$ be a coloring of $H = (H_1, H_2)$. The red residue $R_C(H)$ is a twin-hypergraph $(H'_1, H'_2)$ satisfying

- $V(H'_1) = V(H'_2) = R$: the set of vertices lying in red edges of $H_1$. 

Coloring Non-Uniform Hypergraphs Red and Blue – p.18/38
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$R_C(H)$
Recoloring Lemma

*Blue* residue $B_C(H)$ is defined similarly.

**Lemma 1** For any coloring $C$, the twin-hypergraph $H$ has Property B if both *red* residue $R_C(H)$ and *blue* residue $B_C(H)$ have Property B.
Blue residue $B_C(H)$ is defined similarly.

Lemma 1 For any coloring $C$, the twin-hypergraph $H$ has Property B if both red residue $R_C(H)$ and blue residue $B_C(H)$ have Property B.
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Properties of Recoloring Lemma

- Advantage: The number of edges of is reduced.
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Can not apply it recursively unless one of the residues is empty.
Irreducibility

A twin-hypergraph $H = (H_1, H_2)$ is called *irreducible* if

1. $\forall F_1 \in E(H_1)$ and $v \in F_1$, $\exists F_2 \in E(H_2)$ such that $F_1 \cap F_2 = \{v\}$.

2. $\forall F_2 \in E(H_2)$ and $v \in F_2$, $\exists F_1 \in E(H_1)$ such that $F_1 \cap F_2 = \{v\}$. 

[Diagram of a twin-hypergraph with two components connected by edges]
Reducibility

A twin-hypergraph $H = (H_1, H_2)$ is called reducible if there is an evidence $(F, v)$ satisfying

1. $v \in F$, and $F \in E(H_i)$ for $i = 1$ or $2$.
2. $\forall F' \in E(H_{3-i})$, if $v \in F'$ then $|F \cap F'| \geq 2$. 

Coloring Non-Uniform Hypergraphs Red and Blue – p.22/38
Reducing twin-hypergraphs

If $H$ is reducible, there is an evidence $(F, v)$. Removing $F$ from $H$ we get a twin-hypergraph with one edge less. Repeat this process until an irreducible twin-hypergraph is reached.

\[ H = H^{(0)} \supset H^{(1)} \supset \cdots \supset H^{(s)}. \]
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Irreducible core

**Lemma 2** In the above process, $H^{(s)}$ is unique and does not depend on the order of edges being removed.
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Sketch of Proof:

- The irreducibility is closed under union.
- The maximum irreducible sub-twin-hypergraph exists and is unique.
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- $H^{(s)}$ is the maximum irreducible sub-twin-hypergraph.
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- The maximum irreducible sub-twin-hypergraph exists and is unique.
- $H^{(s)}$ is the maximum irreducible sub-twin-hypergraph.

Such a unique $H^s$ is called the irreducible core of $H$. 
Lemma 3 A twin-hypergraph $H$ has Property B if and only if its irreducible core has Property B.
Lemma on irreducible core

**Lemma 3** A twin-hypergraph $H$ has Property B if and only if its irreducible core has Property B.

**Proof:** It suffices to add a removed edge $F$ back.

- If $F$ is not monochromatic, do nothing.
- Otherwise, flip the color of $v$. For any $F'$ containing $v$, $F'$ contains another vertex of $F$. Thus, $F'$ is not monochromatic.
Randomized testing algorithm

Let $C: V(H) \rightarrow \text{red, blue}$: (independently)

$$\Pr(C(v) = \text{red}) = \frac{1}{2}, \quad \Pr(C(v) = \text{blue}) = \frac{1}{2}$$
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Abort if an early termination condition is satisfied.
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- Compute red residue $R_C(H)$ and blue residue $B_C(H)$.
- Test $R_C(H)$. 
Randomized testing algorithm

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- Abort if an early termination condition is satisfied.
- Compute red residue $R_C(H)$ and blue residue $B_C(H)$.
- Test $R_C(H)$.
- Test $B_C(H)$. 
Procedure to test $R_C(H)$

- An edge $F$ has rank $i$ if $r2^{i-1} \leq |F| < r2^i$. 
Procedure to test $R_C(H)$

- An edge $F$ has rank $i$ if $r2^{i-1} \leq |F| < r2^i$.

- For each $v$ lying in edges of rank $i$, flip the color of $v$ independently with probability $\frac{q}{r2^{i-1}}$. At this stage, all red edges with rank $i$ should be destroyed.
Procedure to test $R_C(H)$

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- Always reduce it to the irreducible core after recoloring.
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- Abort the program if a red edge survives or a blue edge is created.
Claims

- If the program succeeds, then $H$ has Property B.
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- Suppose a twin-hypergraph $H = (H_1, H_2)$ with minimum edge-cardinality $r$ satisfies

$$h_i \overset{def}{=} \sum_{F \in E(H_i)} \frac{1}{2|F|} \leq \left( \frac{1}{16} - o(1) \right) \frac{\ln r}{\ln \ln r}$$

for $i = 1, 2$. Then the program succeeds with positive probability.
Random variables

$$X_{F'} = \#\{F \in E(H_1) \mid |F \cap F'| = 1, F \setminus F' \text{ is red.}\}$$
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- $X^{(i)} = \#\{(v, F) \mid v \in F, F \in E(H_1), |F| = i, F \setminus \{v\} \text{ is red.}\}$
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- \( X^{(i)} = \# \{(v, F) \mid v \in F, F \in E(H_1), |F| = i, F \setminus \{v\} \text{ is red} \} \)

- \( X = \sum_{i \geq r} \frac{X^{(i)}}{i} \).
Lemma 4 With probability at least $1 - \frac{1}{M}$, we have

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\forall F', \quad \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} \leq 2M h_1.
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Sketch of Proof:

- $\sum_{i \geq r} \frac{X^{(i)}_{F'}}{i} \leq \sum_{i \geq r} \frac{X^{(i)}}{i} = X.$

- $E(X^{(i)}) = \sum_{F \in E(H_1)} \frac{2i}{2^i}$.
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- $$E(X^{(i)}) = \sum_{F \in E(H_1), |F| = i} \frac{2i}{2^i}.$$  
- $$E(X) = \sum_{F \in E(H_1)} \frac{2}{2|F|} = 2h_1.$$
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- \[E(X^{(i)}) = \sum_{F \in E(H_1), |F| = i} \frac{2i}{2^i}.\]
- \[E(X) = \sum_{F \in E(H_1)} \frac{2}{2^{|F|}} = 2h_1.\]

Markov’s inequality.
Type I: a red edge survives.

The failure probability of type I event is at most

\[ \sum_{F \in E(H_1)} \frac{1}{2|F|} (1 - \frac{q}{r2^{\lceil \log_2 \frac{|F|}{r} \rceil}})|F| \]

\[ \leq \sum_{F \in E(H_1)} \frac{1}{2|F|} (1 - \frac{q}{|F|})|F| \]

\[ \leq \sum_{F \in E(H_1)} \frac{1}{2|F|} e^{-q} \]

\[ = h_1 e^{-q}. \]
Type II: a blue edge is created.

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S

F'
Type II: a blue edge is created.

- $S$ is red while $F' \setminus S$ is blue in $C$.
- For any $v \in S$, $\exists F_v$ such that $F_v \cap F = \{v\}$. Moreover, $F_v$ survives until $v$ is recolored into blue.
Type II: a blue edge is created.

- \( S \) is red while \( F' \setminus S \) is blue in \( C \).
- For any \( v \in S \), \( \exists F_v \) such that \( F_v \cap F = \{v\} \). Moreover, \( F_v \) survives until \( v \) is recolored into blue.

- All vertices in \( S \) are changed into blue eventually. Let \( x \) be the rank of \( F_v \). For any \( v \in S \),

\[
\Pr(v \text{ is recolored into blue}) \leq \sum_{s=x}^{\infty} \frac{q}{r^s 2^{s-1}}
\]

\[
= \frac{4q}{r 2^x} < \frac{4q}{|F_v|}.
\]
Random variable $Z$

Let $F_v = \{F \mid F \cap F' = \{v\}, F \setminus \{v\} \text{ is red}\}$.

\[
Z \overset{def}{=} \sum \prod \sum_{\substack{S \subseteq F' \\ v \in S}} \frac{4q}{|F|} \\
= \prod_{v \in F'} (1 + \sum_{F \in F_v} \frac{4q}{|F|}) - 1 \\
\leq e^{\sum_{v \in F'} \sum_{F \in F_v} \frac{4q}{|F|}} - 1 \\
= e^{4q \sum_{i \geq r} \frac{x_{F'}^{(i)}}{i}} - 1.
\]
Upper-bound $Z$ over $A_{F'} = \left( \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} \right) \leq 2Mh_1$

$$1_{A_{F'}} Z = 1_{A_{F'}} \frac{e^{4q} \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} - 1}{\sum_{i \geq r} \frac{X_{F'}^{(i)}}{i}} \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} \leq \frac{e^{8Mh_1q} - 1}{2Mh_1} \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} \leq \frac{e^{8Mh_1q}}{2Mh_1} \frac{1}{r} \sum_{i \geq r} \frac{X_{F'}^{(i)}}{i} = \frac{e^{8Mh_1q}}{2Mh_1r} X_{F'}.$$
\[ \sum_{F' \in E(H_2)} E(1_A Z) \frac{1}{2^{|F'|}} \leq \sum_{F' \in E(H_2)} \frac{1}{2^{|F'|}} E\left(\frac{e^{8Mh_1q}}{2Mh_1r} X_{F'}\right) \]

\[ = \sum_{F' \in E(H_2)} \frac{1}{2^{|F'|}} \frac{e^{8Mh_1q}}{2Mh_1r} E(X_{F'}) \]

\[ \leq h_2 \frac{e^{8Mh_1q}}{2Mh_1r} 2h_1 \]

\[ = h_2 e^{8Mh_1q} \frac{1}{Mr}. \]
The probability of success is at least

\[ 1 - \frac{2}{M} - 2he^{-q} - \frac{2he^{8Mhq}}{Mr} . \]

Choose \( M = 2(1 + \epsilon) \), \( q = \ln \ln r \), and \( h = \frac{1 - \epsilon}{16} \frac{\ln r}{\ln \ln r} \).

The above probability is

\[ \frac{\epsilon}{1 + \epsilon} - \frac{2h}{\ln r} - \frac{2h}{Mr^\epsilon^2} > 0 \]

for sufficiently large \( r \).

Therefore, \( H \) has Property B.
Open Problems

- Is it true $f(r) = \frac{m_2(r)}{2r}$?
Open Problems

- Is it true $f(r) = \frac{m_2(r)}{2^r}$?
- Find a better upper bound for $f(r)$ using non-uniform hypergraph.
Open Problems

- Is it true $f(r) = \frac{m_2(r)}{2^r}$?
- Find a better upper bound for $f(r)$ using non-uniform hypergraph.
- Prove or disprove Erdős-Lovász’s stronger conjecture $m_2(r) = \Theta(r2^r)$.